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Graph Coloring Parameters and Graph Codes

Ph.D. Dissertation

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Declaration of own work and references

I, Anna Gujgiczer, hereby declare that this dissertation, and all results claimed therein are my own work, and rely solely on the references given. All segments taken word-by-word, or in the same meaning from others have been clearly marked as citations and included in the references.

Nyilatkozat önálló munkáról, hivatkozások átvételéről

Alulírott Gujgiczer Anna kijelentem, hogy ezt a doktori értekezést magam készítettem és abban csak a megadott forrásokat használtam fel. Minden olyan részt, amelyet szó szerint, vagy azonos tartalomban, de átfogalmazva más forrásból átvettem, egyértelműen, a forrás megadásával megjelöltem.

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Summary

This dissertation considers problems from various fields in graph theory. A common theme between the first two theses is that the results are related to some graph parameters which are expressible with a homomorphism to some universal graph class. The third thesis addresses problems with closer relation to information theory.

We say that a graph G admits a homomorphism to a graph H if there exists an edge preserving map from the vertex set of G to the vertex set of H. A well-known example for a graph parameter which can be expressed via graph homomorphism is the chromatic number. A graph G has chromatic number at most c if and only if it has a homomorphism to K_c, the complete graph on c vertices. In this case we call the complete graphs universal graphs for the chromatic number. In 1966 Stephen Hedetniemi formulated a conjecture that the chromatic number of the so-called tensor product of two graphs is equal to the minimum of the chromatic numbers of the factors. However, it is easy to see, that the chromatic number of the product is at most the chromatic number of the factors. Therefore, the conjecture essentially asked whether the reverse inequality holds as well. In other words, the conjecture asked whether if the product has a homomorphism to some complete graph then one of the factors should have it as well. This conjecture was refuted after a long time and in the later counterexamples a special graph class played an important role. These graphs are the universal graphs for the so-called wide-colorings. Determining their multichromatic number became relevant in the search for even smaller counterexamples. My first thesis deals with this question.

Multichromatic numbers are other good examples of graph parameters expressible via homomorphisms. In this case the universal graphs for this parameter are the Kneser graphs, $KG(n, k)$ with parameters $n \geq 2k$. Kneser graphs form a famous graph class whose chromatic number was determined by Lovász in his celebrated paper, where he proved, using topological tools, that the upper bound $n - 2k + 2$, constructed and conjectured to be tight by Kneser, is indeed the correct value. Soon afterwards Schrijver found that a certain induced subgraph $SG(n, k)$ of $KG(n, k)$, now called Schrijver graph, still has chromatic number $n - 2k + 2$ and moreover, it is also vertex-critical for this property, that is, deleting any of its vertices the chromatic number becomes smaller. Kneser and Schrijver graphs (with the same parameters) share the value for another graph parameter, namely the fractional chromatic number, which is $\frac{n}{k}$ for both. However, none of them are vertex-critical for this parameter. This suggested the problem of finding critical subgraphs of Schrijver graphs for the fractional chromatic number, which is the subject of my second thesis. The found special subgraph turned out to be isomorphic to another known graph, the circular complete graph, $K_{n/k}$, which is the universal graph for yet another coloring parameter, the so-called circular chromatic number.

My third thesis is related to information theory. In the usual setting one considers binary sequences as codewords and asks how many codewords of a given length can be constructed in such a way that any two of them differs in at least d bits. However, the binary sequences could encode graphs on a labeled vertex set and this way more general "distance" requirements can be formulated. E.g. if the requirement is to contain a triangle in the intersection of the edge sets, then we get the famous conjecture of Simonovits and Sós which was proven by Ellis, Filmus and Friedgut. The role of the intersection can be replaced by, for example, the symmetric difference of the edge sets of the two graphs (which we can arrive to from the basic code distance problem), and apart from the containment of a triangle or other fixed graph it is also interesting to examine global conditions like connectedness or Hamiltonicity. The last chapter of my dissertation explores this generalization.

Összefoglaló

Ezen disszertáció a gráfelmélet különböző területeiről származó problémákat vizsgál. Az első két tézisben közös elem, hogy az eredmények néhány olyan gráfparaméterhez kapcsolódnak, amelyek valamilyen univerzális gráfosztályba vezető homomorfizmussal kifejezhetőek. A harmadik tézis az információelmélethez is kapcsolódó problémákkal foglalkozik.

Azt mondjuk, hogy létezik homomorfizmus egy G gráfból egy H gráfba, ha létezik egy éltartó leképezés G csúcshalmazából H csúcshalmazába. Egy jól ismert példa ilyen gráfhomomorfizmussal kifejezhető gráfparaméterre a kromatikus szám. Egy G gráf kromatikus száma akkor és csak akkor legfeljebb c, ha létezik homomorfizmus G-ből K_c-be, a c csúcsú teljes gráfba. Ebben az esetben a teljes gráfokat a kromatikus számhoz tartozó univerzális gráfoknak nevezzük. 1966-ban Stephen Hedetniemi megfogalmazta azt a sejtést, hogy két gráf úgynevezett tenzorszorzatának kromatikus száma egyenlő a faktorok kromatikus számának minimumával. Könnyen belátható azonban, hogy a szorzat kromatikus száma legfeljebb a faktorok kromatikus száma. Ezért a sejtés lényegében azt kérdezte, hogy az ellenirányú egyenlőtlenség is teljesül-e. Más szóval, igaz-e, hogy ha a szorzatból van homomorfizmus valamilyen teljes gráfba, akkor legalább az egyik tényezőből szintén van homomorfizmus ugyanebbe a teljes gráfba. Ezt a sejtést hosszú idő után megcáfolták, és a későbbi ellenpéldákban fontos szerepet játszott egy speciális gráfosztály. Ezek a gráfok az univerzális gráfok az úgynevezett széles színezéshez. Multikromatikus számuk meghatározása a még kisebb ellenpéldák keresésében vált fontossá. A disszertációm első tézise ezzel a kérdéssel foglalkozik.

A multikromatikus szám egy másik jó példája a homomorfizmussal kifejezhető gráfparamétereknek. Ebben az esetben a paraméter univerzális gráfjai a $KG(n,k)$ Kneser-gráfok, $n \geq 2k$ paraméterekkel. A Kneser-gráfok egy híres gráfosztályt alkotnak, amelynek kromatikus számának meghatározása Lovász áttörő eredménye. Bizonyításában topologikus módszereket használt (ezzel összekötve a matematika e két ágát), hogy belássa, hogy a korábban Kneser által felső korlátként már bizonyított és pontosnak sejtett $n-2k+2$ érték valóban pontos. Nem sokkal később Schrijver észrevette, hogy a $KG(n,k)$ egy bizonyos feszített $SG(n,k)$ részgráfja, amelyet ma már Schrijver-gráfnak nevezünk, még mindig $n-2k+2$ kromatikus, ráadásul e paraméterre nézve csúcskritikus is, azaz bármelyik csúcsát törölve a kromatikus száma csökken. A Kneser- és a Schrijver-gráfoknak (azonos paraméterekkel) egy másik gráfparamétere, nevezetesen a frakcionális kromatikus száma is megegyezik, mindkettőnek $\frac{n}{k}$. Azonban a Schrijver gráf sem csúcskritikus erre a paraméterre. Ez motiválta a Schrijver gráfok frakcionális kromatikus számra kritikus részgráfjainak keresését, ami a disszertációm második tézisének témája. A megtalált speciális részgráfokról kiderült, hogy izomorfak egy másik ismert gráfosztály tagjaival, a $K_{n/k}$ cirkuláris teljes gráfokkal, amelyek egy másik színezési paraméter, az úgynevezett cirkuláris kromatikus szám univerzális gráfjai.

A harmadik tézisem az információelmélethez is kapcsolódik. A témakörben szokásosan a kódszavak bináris sorozatok, és a központi kérdés az, hogy hány adott hosszúságú kódszó konstruálható úgy, hogy közülük bármelyik kettő legalább d bitben különbözzön. A bináris sorozatok azonban kódolhatnak gráfokat egy felcímkézett csúcshalmazon, ezzel lehetőséget adva arra, hogy általánosabb „távolság” követelmények is megfogalmazhatók legyenek. Ha például a követelmény az, hogy a gráfok élhalmazainak a metszetében legyen háromszög, akkor megkapjuk Simonovits és Sós híres sejtését, amelyet Ellis, Filmus és Friedgut bizonyított. A metszet szerepe helyettesíthető például a két gráf élhalmazainak szimmetrikus differenciájával is (ami a fent említett szokásos probléma közvetlen általánosítása), és a háromszög vagy egyéb rögzített gráf tartalmazása mellett érdekes globális feltételeket is vizsgálni, mint például az összefüggőség vagy a Hamiltonicitás. Disszertációm utolsó fejezete ezt az általánosítást vizsgálja.

Contents

Chapter 1

Introduction

In graph theory, a much-studied graph parameter is the chromatic number, which is used in practice for problems such as frequency or time allocation. In many cases, the behavior of the chromatic number is difficult to understand. An example of this is how this parameter behaves in graph products. In 1966 Stephen Hedetniemi formulated the conjecture that the chromatic number of the so-called tensor product of two graphs is equal to the minimum of the chromatic number of the factors. It is clear, however, that the chromatic number of the product is at most the chromatic number of the factors. Therefore, the conjecture essentially asked whether the reverse inequality holds as well. This question remained unanswered for a long time, but in 2019, it was disproved [Shi19]. The first counterexample found was very large both in terms of the vertex number of the factors and their chromatic number. Later, smaller counterexamples were found [Zhu21; Tar22a; Wro20; Tar23] and now the conjecture is fully settled, meaning that for any number c if both factors have chromatic numbers greater than c, we know whether their product can be c-colorable or not.

Other interesting, well-studied and closely related graph parameters are the so-called fractional chromatic number and multichromatic numbers. In the earlier counterexamples to Hedetniemi's conjecture the fractional chromatic number turned out to be an important parameter and in the later counterexamples the multichromatic numbers of some special graph classes came into play. My first group of theses addresses some questions within this topic. It is also worth mentioning that Hedetniemi-type problems in which we consider other parameters of the graphs involved in place of the chromatic number, were formulated as well. In the case of the fractional chromatic number it is known that the Hedetniemi-type conjecture is true [Zhu11].

Multichromatic numbers are closely related to Kneser graphs - as those parameters can be expressed with homomorphisms to corresponding Kneser graphs - a famous graph class whose chromatic number was determined by Lovász in his celebrated paper [Lov78], where he proved that the already known upper bound that was conjectured to be tight is tight indeed. However, in general, those graphs are not vertex critical for this parameter, meaning that after a vertex removal the chromatic number does not necessarily decrease. Schrijver observed that special induced subgraphs, now called Schrijver graphs, have the same chromatic number as the Kneser graph (with the same parameters), and they are vertex critical for that. Moreover, Kneser and Schrijver graphs (with the same parameters) share the same fractional chromatic number as well [Tal03; ST06], but even the Schrijver graph is not critical for that (except for some special cases). Results in my second group of theses focuses on finding induced subgraphs of Schrijver graphs with the same fractional chromatic number, which are also vertex-critical for that parameter.

A research direction different from the ones mentioned above is to investigate the maximum size of graph families where some relation of any two members of the family (considered as the codewords) satisfies some prescribed condition. An example of this is the famous conjecture of Simonovits and Sós [SS76] proven by Ellis, Filmus and Friedgut [EFF12], that determines the maximum possible cardinality of a family of graphs on n labeled vertices in which the intersection of any two members contains a triangle. The role of the intersection can be replaced, to get new interesting questions, e.g. by the symmetric difference of the edge sets of the two graphs. It is what we can arrive to if the basic code distance problem (how many binary sequences of a given length can be given at most if any two differ in at least a given number of coordinates) is modified so that we do not prescribe the minimum distance of any two codewords but require that they differ in some specific structure. Apart from the containment of a triangle it is also interesting to examine global conditions like connectedness or Hamiltonicity.

Organisation of the dissertation by theses

The first two thesis groups are related to special graphs classes. These graphs serve as universal graphs for some coloring parameters, meaning, that if a graph G has the required coloring parameter then it has a homomorphism to the corresponding special graph. We say that a graph G admits a homomorphism to a graph H if there exists an edge preserving map from the vertex set of G to the vertex set of H and we denote the existence of such a homomorphism by $G \to H$. One can easily see that, for example, the chromatic number can be expressed in such a way. A graph G has chromatic number at most c if and only if it has a homomorphism to K_c, the complete graph on c vertices. In the first and the second thesis groups the universal graphs (or their subgraphs) for the so-called s-wide coloring and multicolorings are explored.

The third thesis group is more directly related to information theory, codewords which can be defined on graphs are investigated there.

For clarity, the numbering of the theorems in the following summary sections corresponds to that used later in this dissertation. However, since not all theorems are included in the summary, some statements have been merged, and the order of presentation has occasionally been adjusted for conciseness, the numbering sometimes may appear discontinuous or unusual.

Chapter 2 – Multichromatic Numbers of Widely Colorable Graphs

As mentioned in the Introduction, related to the Hedetniemi conjecture, a certain multichromatic number of a special graph class became interesting. This graph class plays an important role in the theory of wide colorings. A vertex-coloring of a graph is called s-wide if the two endvertices of every walk of length $2s-1$ receive different colors in it. It is easy to see that this is one possible generalization of the term coloring in graph theory, as 1-wide coloring is equivalent to the proper graph coloring. It can be shown that a graph is s-widely colorable with t colors if and only if it admits a homomorphism into the following universal graph [ST06] denoted by $W(s,t)$ some special cases of which appeared in the related question.

$$V(W(s,t)) = \{(x_1 \dots x_t) : \ \forall i \; x_i \in \{0, 1, \dots, s\}, \exists! i \; x_i = 0, \ \exists j \; x_j = 1\},$$

$$E(W(s,t)) = \{\{(x_1 \dots x_t), (y_1 \dots y_t)\} : \forall i \; |x_i - y_i| = 1 \text{ or } x_i = y_i = s\}.$$

If we set $s = 1$, then we get $W(1,t) = K_t$ by the definition, which is in line with our earlier observation that the complete graphs are universal graphs for proper colorings.

Multicoloring is when we color the vertices of a graph G with n colors in such a way that every vertex receives k distinct colors and if two vertices u and v are adjacent then the set of colors received by u is disjoint from the set of colors received by v. Formally, it is a function $f : v \mapsto \{c_1, \dots, c_k\}$ where for $\forall i \in [k] \; c_i \in [n]$, such that if $\{u, v\} \in E(G)$ then $f(u) \cap f(v) = \emptyset$ (where $[k] = \{1, 2, \dots, k\}$ and similarly $[n] = \{1, 2, \dots, n\}$). Such colorings were first considered by Geller and Stahl, see [GS75; Sta76]. Stahl [Sta76] introduced the corresponding multichromatic number $\chi_k(G)$ as the minimum number of colors needed for such a coloring, called a k-fold coloring. (This graph parameter can also be expressed by the existence of a homomorphism into some universal graph as discussed in the next section.)

The fractional chromatic number $\chi_f(G)$ can be defined as

$$\chi_f(G) = \inf_k \left\{ \frac{\chi_k(G)}{k} \right\}.$$

With my advisor in [*j*1] we have determined the exact values for the k-th multichromatic numbers for the above mentioned $W(s,t)$ universal graphs in cases when $k \leq s$.

This work was motivated by a question of Tardif in [Tar22a], where he constructed a counterexample graph pair G, H to the Hedetniemi conjecture, where G and H had large chromatic numbers, more than 14, but their product was 14-colorable. In that counterexample G was $W(3,9)[K_4]$, the graph which is obtained by blowing up each vertex of $W(3,9)$ into a clique of size 4, fully connecting the cliques corresponding to originally adjacent vertices in $W(3,9)$. It is easy to see that the chromatic number of this graph is exactly the 4-th multichromatic number of $W(3,9)$. In hope for constructing smaller counterexamples in a similar way he asked whether $\chi(W(3,t)[K_3]) = \chi_3(W(3,t))$ is large, in particular, for $t = 8$ more than 12 and for $t = 7$ more than 11. He also observed that in general

$$\chi_k(W(s,t)) \geq t + 2(k-1)$$

holds. In other words, he asked if strict inequality is true in the special case when $s = k = 3$ and $t = 7$ or $t = 8$. We have answered his question in the negative and generalized the result to all t and $k \leq s$:

Theorem 2.2. and Corollary 2.4. If $k \leq s$, then

$$\chi_k(W(s,t)) = t + 2(k-1).$$

We also showed that this result cannot be generalized for arbitrarily large k (with respect to s).

Proposition 2.6. For all pairs of positive integers $t \geq 3$ and $s \geq 1$ there exists some threshold $k_0 = k_0(s,t) > s$ for which

$$\chi_k(W(s,t)) > t + 2(k-1)$$

whenever $k \geq k_0$.

We also managed to prove the following theorems about the fractional chromatic number of a $W(s,t)$ graph. For that we have used some previous results concerning Mycielski graphs s-wide colorability [BS05; SST24; GJS04; ST06]. The Mycielskian $M(G)$ of a graph G is a result of a graph operation, introduced by Mycielski [Myc55], which does not increase the clique number of the graph G, but it increases its chromatic number. The construction can be generalised (see Chapter 2 of the dissertation) to get h-level Mycielskians $M_h(G)$, where the original construction $M(G) = M_2(G)$. The effect of the original Mycielski construction, $M_2(G)$, on the fractional chromatic number were investigated in [LPU95], where a simple function was given:

$$\chi_f(M(G)) = \chi_f(G) + \frac{1}{\chi_f(G)}.$$

For a general h, the fractional chromatic number $\chi_f(M_h(G))$ was studied by Tardif in [Tar01]. He proved that the value of $\chi_f(G)$ also determines $\chi_f(M_h(G))$.

$$\chi_f(M_h(G)) = \chi_f(G) + \frac{1}{\sum_{i=0}^{h-1}(\chi_f(G)-1)^i}.$$

Using this result we managed to prove the following two theorems by showing the existence of homomorphisms from $M_{3s-2}(W(s,t))$ to $W(s,t+1)$ and from $W(s,t+1)$ to $M_s(W(s,t))$.
Corollary of Proposition 2.11.

$$\chi_f(W(s,t)) + \frac{\chi_f(W(s,t))-2}{(\chi_f(W(s,t))-1)^{3s-2}-1}$$
$$\leq \chi_f(W(s,t+1))$$
$$\leq \chi_f(W(s,t)) + \frac{\chi_f(W(s,t))-2}{(\chi_f(W(s,t))-1)^{s}-1}$$

Theorem 2.7. For any fixed positive integer s we have

$$\lim_{t\to\infty} \chi_f(W(s,t)) = \infty.$$

Chapter 3 – Critical Subgraphs of Schrijver Graphs for the Fractional Chromatic Number

As the $W(s,t)$ graphs were universal graphs for wide-colorings, Kneser graphs are the universal graphs for multicolorings, meaning that the k-th multichromatic number of a graph is at most n if and only if it admints a homomorphism to the Kneser graph $KG(n,k)$. For positive integers $n \geq 2k$ the Kneser graph $KG(n,k)$ is defined on the vertex set that consists of the $\binom{n}{k}$ k-element subsets of $[n]$ with two such subsets forming an edge if and only if they are disjoint:

$$V(KG(n,k)) = \binom{[n]}{k}$$

$$E(KG(n,k)) = \{\{A,B\} : A \cap B = \emptyset\}.$$

Kneser [Kne55] observed that the chromatic number of $KG(n,k)$ is at most $n-2k+2$ and conjectured that this upper bound is tight. This was proved many years later by Lovász in his celebrated paper [Lov78] using the Borsuk-Ulam theorem. Soon afterwards Schrijver [Sch78] found that a certain induced subgraph $SG(n,k)$ of $KG(n,k)$, now called Schrijver graph, still has chromatic number

$n-2k+2$ and moreover, it is also vertex-critical for this property, that is, deleting any of its vertices the chromatic number becomes smaller.

The fractional chromatic number of $KG(n,k)$ is $\frac{n}{k}$ (which is a simple consequence of the Erdős-Ko-Rado theorem [EKR61]). Schrijver graphs $SG(n,k)$ share this fractional chromatic value [Tal03; ST06], but most Schrijver graphs are not vertex-critical for this parameter (the only exceptions are the trivial cases) and this suggested the problem of finding critical subgraphs of Schrijver graphs for the fractional chromatic number.

In a joint paper [*j*3] with my advisor we worked on this problem. We defined a natural property for the sets representing the vertices and named the subgraph formed by the vertices satisfying this property $Q(n,k)$ (the formal definition of $Q(n,k)$ can be found in Chapter 3 of the dissertation). A basic property of these graphs is the following:

Proposition 3.6. Let $n \geq 2k$ and $\ell \geq 2$ be any positive integer. Then the graphs $Q(n,k)$ and $Q(\ell n, \ell k)$ are isomorphic.

Based on the above theorem, when studying the properties of $Q(n,k)$ graphs, we can always assume that $\gcd(n,k)=1$.

Theorem 3.7. Assume $n \geq 2k$, $\gcd(n,k)=1$ and let a and b be the smallest positive integers for which $ak = bn-1$. The graph $Q(n,k) \subseteq SG(n,k)$ satisfies the following properties.

- $\chi_f(Q(n,k)) = \frac{n}{k} = \chi_f(SG(n,k))$.
- $\forall U \in V(Q(n,k))\ \ \chi_f(Q(n,k) \setminus \{U\}) = \frac{a}{b} < \frac{n}{k}$, that is $Q(n,k)$ is vertex-critical for the fractional chromatic number.
- $Q(n,k)$ contains an induced subgraph isomorphic to $Q(a,b)$.

While proving this result we realised that the above theorem is true because the found special subgraph is isomorphic to another known graph, the circular complete graph, $K_{n/k}$, which is the universal graph for yet another coloring parameter, the circular chromatic number. The definitions of the circular complete graph $K_{n/k}$ for $n \geq 2k$ and the related circular chromatic number χ_c are the following:

$$V(K_{n/k}) = \{0,1,\dots,n-1\}$$

$$E(K_{n/k}) = \{\{i,j\} : k \leq |i-j| \leq n-k\},$$

$$\chi_c(G) = \min\left\{\frac{p}{q} : p \leq |V(G)|, G \to K_{p/q}\right\}.$$

Proposition 3.8. $Q(n,k)$ is isomorphic with the circular complete graph $K_{n/k}$ when $\gcd(n,k)=1$.

It was known for circular complete graphs that they are vertex-critical for the fractional chromatic number, but edge-criticality was not studied before (neither for the fractional nor for the circular chromatic number). We also investigated this question. For that we called an edge $\{i,j\} \in E(K_{n/k})$ a *shortest edge* if $|i-j| = k$ or $|i-j| = n-k$. (The name comes from the fact that these are the shortest edges when the vertices are arranged in order along a circle.)

Proposition 3.18. If $\gcd(n,k)=1$, $e \in E(K_{n/k})$ and a, b are defined as the smallest positive integers for which $ak = bn-1$ then

$$\chi_f(K_{n/k} \setminus \{e\}) = \chi_c(K_{n/k} \setminus \{e\}) = \begin{cases} \frac{a}{b} & \text{if } e \text{ is a shortest edge} \\ \frac{n}{k} & \text{otherwise.} \end{cases}$$

Finally, we proved that $SG(n,k)$ itself is vertex critical for the fractional chromatic number only in some trivial cases.

Corollary 3.16. $\forall U \in V(SG(n,k))\ \ \chi_f(SG(n,k) \setminus \{U\}) < \chi_f(SG(n,k))$ if and only if one of the following holds: $k = 1, n = 2k$, or $n = 2k+1$.

Chapter 4 – Graph Codes

In a joint work [j2] with Noga Alon, János Körner, Aleksa Milojević and Gábor Simonyi we investigated the maximum size of graph families on a common vertex set of cardinality n such that the symmetric difference of the edge sets of any two members of the family satisfies some prescribed condition. Note, that if the prescribed condition is just to contain at least d edges, then we get back the basic code distance problem: How many codewords of length $\binom{n}{2}$ can be given such that any two of them differ in at least d coordinates?

In this subsection I will list some of the results that we had (see Chapter 4 of the dissertation for more). We considered global properties like connectedness, Hamiltonicity as well as local properties like containment of a triangle and some more. Formally all these can be described by saying that the graph defined by the symmetric difference of the edge sets of any two of our graphs belong to a prescribed family of graphs (namely those that are connected, contain a Hamiltonian cycle, or contain a triangle, etc.)

Let $\mathcal{F}$ be a fixed class of graphs. A graph family $\mathcal{G}$ on n labeled vertices is called $\mathcal{F}$-good if for any pair $G, G' \in \mathcal{G}$ the graph $G \oplus G'$ defined by

$$V(G \oplus G') = V(G) = V(G') = [n],$$

$$E(G \oplus G') = \{e : e \in (E(G) \setminus E(G')) \cup (E(G') \setminus E(G))\}$$

belongs to $\mathcal{F}$.

Let $M_{\mathcal{F}}(n)$ denote the maximum possible size of an $\mathcal{F}$-good family on n vertices. We were interested in the value of $M_{\mathcal{F}}(n)$ for various classes $\mathcal{F}$. The followings theorems give this value in some cases we considered.

Theorem 4.2. and 4.3. Let $\mathcal{F}_c$ denote the class of connected graphs and $\mathcal{F}_{2c}$ the class of 2-connected graphs. Then

$$M_{\mathcal{F}_c}(n) = 2^{n-1},\ M_{\mathcal{F}_{2c}}(n) = 2^{n-2}.$$

Theorem 4.5. and 4.6. Let $\mathcal{F}_{Hp}$ denote the class of graphs containing a Hamiltonian path and $\mathcal{F}_{Hc}$ denote the class of graphs containing a Hamiltonian cycle. Then for infinitely many values of n we have

$$M_{\mathcal{F}_{Hp}}(n) = 2^{n-1},\ M_{\mathcal{F}_{Hc}}(n) = 2^{n-2}.$$

In the above listed theorems for proving the maximality of $M_{\mathcal{F}}(n)$ for the family $\mathcal{F}$ in question we used the following lemma.

Lemma 4.1. For any graph class $\mathcal{F}$ we have

$$M_{\mathcal{F}}(n) \cdot D_{\mathcal{F}}(n) \le 2^{\binom{n}{2}},$$

where $D_{\mathcal{F}}(n)$ denotes the "dual" of $M_{\mathcal{F}}(n)$, i.e. the maximum possible size of a graph family on n labeled vertices, the symmetric difference of no two members of which belongs to $\mathcal{F}$. Note that denoting by $\overline{\mathcal{F}}$ the class containing exactly those graphs that do not belong to $\mathcal{F}$ we actually have $D_{\mathcal{F}}(n) = M_{\overline{\mathcal{F}}}(n)$. In all of the proofs of the above mentioned theorems we cunstructed $\mathcal{F}$-good and $\overline{\mathcal{F}}$-good families, A and B respectively, of "matching sizes", meaning that $|A| \cdot |B| = 2^{\binom{n}{2}}$, proving that they are both maximal. However, this technique does not work for every class of graphs.

Theorem 4.7. and 4.8. Let $\mathcal{F}_S$ denote the class of graphs containing a spanning star, that is a vertex connected to all other vertices in the graph. Then we have

$$M_{\mathcal{F}_S}(n) = \begin{cases} n+1 & \text{if } n \textit{ is odd} \\ n & \text{if } n \textit{ is even.} \end{cases}$$

The dual family does not have a mathcing size, as

$$2^{\binom{n}{2} - \lceil \frac{n}{2} \rceil} \le D_{\mathcal{F}_S}(n) \le 2^{\binom{n}{2} - \frac{n}{2}}.$$

For local conditions we could also use Lemma 4.1.

Proposition 4.14.–4.16. Let $\mathcal{F}_{K_3}$ denote the class of graphs containing a triangle. Then we have

$$M_{\mathcal{F}_{K_3}}(n) \le 2^{\binom{n}{2} - \lceil \frac{n}{2} \rceil \lfloor \frac{n}{2} \rfloor}.$$

This upper bound is sharp when $n \le 6$.

The above theorem is just a special case of a more general one, which brings extremal graph theory in the picture. Let $ex(n, G)$ denote the maximum number of edges an n-vertex graph can have without containing a subgraph isomorphc to G and let $\mathcal{F}_G$ denote the class of graphs containing the graph G as a subgraph.

Proposition 4.9.

$$M_{\mathcal{F}_G}(n) \le 2^{\binom{n}{2} - ex(n,G)}.$$

It turns out that asymptotically this upper bound is tight. To state that formally, we also defined a capacity-type asymptotic invariant and we showed that this invariant is upper bounded by a simple function of the chromatic number. Let

$$R_{\mathcal{F}_G}(n) := \frac{2}{n(n-1)} \log_2 M_{\mathcal{F}_G}(n)$$

and call the following always-existing limit the distance capacity:

$$DC(\mathcal{F}_G) := \lim_{n \to \infty} R_{\mathcal{F}_G}(n).$$

Using the Erdős-Stone-Simonovits theorem [ES46; ES66], stating that

$$\lim_{n \to \infty} \frac{ex(n, G)}{\binom{n}{2}} = 1 - \frac{1}{\chi(G) - 1},$$

we get $DC(\mathcal{F}_G) \le \frac{1}{\chi(G)-1}$. Moreover, equality can also be proven.

Theorem 4.12. If $\chi(G) \ge 2$ then we have $DC(\mathcal{F}_G) = \frac{1}{\chi(G)-1}$.

1.3 Application

This dissertation mainly concerns theoretical results that are interesting on their own right and connected to various parts of graph theory. Nevertheless, in the next subsection the connections of the fractional chromatic number and the multichromatic numbers to information theory will be explained, providing a more application based point of view of the results of the first two thesis groups. For the last thesis group, as it was already mentioned, defining codewords as graphs is a generalization of the classical code distance problem, therefore no further explanation is needed for its relevance to information theory.

Shannon capacity

Several problems in information theory lead to the definition of special graph parameters and the most famous example of this is the Shannon capacity of graphs [Sha56], which is the tight upper bound on the rate at which information can be transmitted over a discrete, memoryless communication channel with zero error probability.

One can model the communication channel as a graph: the transmittable letters are the vertices and a pair of them are connected if and only if they are distinguishable by the receiver. We consider two t-length codewords distinguishable if they are distinguishable in at least one index. Generally, we are interested in the maximum number of pairwise distinguishable t-length codewords.

Definition 3.1.1. *For two graphs G and H their OR-product $G \cdot H$ is defined as follows*

$$V(G \cdot H) = V(G) \times V(H),$$

$$E(G \cdot H) = \{\{(g_1, h_1)(g_2, h_2)\} : g_1, g_2 \in V(G), h_1, h_2 \in V(H),$$
$$\{g_1, g_2\} \in E(G) \text{ or } \{h_1, h_2\} \in E(H)\}.$$

Let G^t denote the t-fold OR-product of G by itself. By definition, the pairwise distinguishable t-length messages form a clique in G^t for a channel modeled by a graph G, so the question is to determine the clique number $\omega(G^t)$.

One can easily see that this value is always at most $|V(G)|^t$. Furthermore, the clique number is super-multiplicative with respect to the OR-product, meaning that for every pair of graphs G and H, the inequality $\omega(G \cdot H) \geq \omega(G) \cdot \omega(H)$ holds. So it makes sense to normalize this value by taking the t^{th} root. In fact, we are interested in the asymptotics of this value. The formal definition of the Shannon capacity is given below. (In the literature it is sometimes defined differently, by the complementary graph.)

Definition 3.1.2. *The Shannon capacity of a graph G is defined as*

$$C(G) := \limsup_{t \to \infty} \sqrt[t]{\omega(G^t)}.$$

Lemma 3.1.1. *The Shannon capacity is a homomorphism-monotone graph parameter, that is, for all graphs G and H, if there exists a homomorphism $G \to H$, then $C(G) \leq C(H)$.*

Proof. Any clique in G^t is mapped into a clique of at least the same size in H^t. □

The value of the Shannon capacity is unknown even for graphs with a very simple structure, for example the exact value is not known for any odd cycle longer than 5 (the case of the 5-cycle is a famous result of László Lovász [Lov79]). From the work of Bohman and Holzman [BH03] we know that the Shannon capacity of odd cycles (or their complements in the different interpretation of the problem) is strictly greater than its trivial lower bound 2. This lower bound, given by Bohman and Holzman, was recently improved in [Zhu25a].

Due to the considerable difficulty to determine this parameter, even in smaller cases, it is already an interesting result if only some bound is given. It follows from the definition of Shannon capacity, that $\omega(G)$, the clique number of the graph G, is always a lower bound. And certain graph coloring parameters can serve as upper bounds.

Lemma 3.1.2. *Let $\varphi(G)$ be a graph parameter. If the following two conditions hold then $C(G) \leq \varphi(G)$.*

1. $\omega(G) \leq \varphi(G)$,
2. $\varphi(G \cdot H) \leq \varphi(G) \cdot \varphi(H)$ *holds for every pair of graphs G and H.*

The fractional chromatic number satisfies these two conditions, therefore, that as well as $\chi_k(G)/k$ for every k are all upper bounds for this difficult to determine parameter.

Corollary 3.1.3. *If a graph G is s-widely colorable with t colors, then for all $k \leq s$ $C(G) \leq \frac{t+2(k-1)}{k}$.*

Proof. The statement follows directly from the results on the multichromatic number of the $W(s,t)$ graphs (Theorem 2.2 and Corollary 2.4), together with the homomorphism-monotonicity of the Shannon capacity (Lemma 3.1.1). □

Remark. An interesting fact is that the chromatic number (as a special case of $\chi_k(G)/k$ where $k = 1$) also satisfies these conditions. Hence, for those graphs where $\omega(G) = \chi(G) = c$ the Shannon capacity is known, $C(G) = c$ as well. This was the original motivation of Claude Berge to investigate perfect graphs (cf. [Ber97]).

It is also worth noting that the fractional chromatic number of graphs can be interpreted as an information theoretic parameter. In the case where feedback is allowed on the channel, a single graph alone cannot fully model it. However, among the memoryless channels that can be modeled by a given graph it will be true for the worst one that the fractional chromatic number gives the theoretical upper bound on the rate at which information can be transmitted over that channel with zero error probability [Sha56]. Furthermore, this parameter is similar to the Shannon capacity in another way as well, as it can be expressed as the normalized value of the chromatic number of a corresponding power graph [BS74; MP71].

Remark. As it was mentioned in the Introduction, Hedetniemi-type conjectures can be formulated for other graph parameters as well. The question is interesting whenever the value of that parameter for the product is at most as large as the values of the parameter for the factors. The Shannon capacity satisfies this condition. However, we do not know if the analogous conjecture holds for the Shannon capacity or not. In [Sim21] a lower bound on the Shannon capacity of a product graph were given and some graphs are shown that may provide counterexamples.

Chapter **2**

Multichromatic Numbers of Widely Colorable Graphs

As mentioned in the Introduction, the first group of theses, motivated by the search for small counterexamples to Hedetniemi's Conjecture, focuses on a question about the multichromatic numbers of the universal graphs for wide-colorings. Before getting to the result, we first need to know what are wide colorings and multichromatic numbers.

2.1 Wide coloring

A vertex-coloring of a graph is called s-wide if the two endvertices of every walk of length $2s-1$ receives different colors in it. If every vertex gets a different color then the coloring is s-wide if and only if the graph does not contain any odd cycle shorter than $2s+1$. The interesting phenomenon is that some graphs have s-wide colorings that are also optimal colorings.

A 1-wide coloring is just a proper coloring. 2-wide colorings were first investigated by Gyárfás, Jensen, and Stiebitz [GJS04] who, answering a question of Harvey and Murty, showed the existence of a t-chromatic graph for every $t \geq 2$ with the additional property that it admits a t-coloring in which the neighborhood of every color class is an independent set. The analogous statement including more distant neighborhoods is also proved in [GJS04].

3-wide colorings (that are called simply wide colorings in [ST06]) turned out to be relevant concerning the local chromatic number of several graph families whose chromatic number can be determined by the topological method of Lovász [Lov78]. For example, using wide coloring, t-chromatic Schrijver graphs (that will be intruduced in the next chapter) can be shown to have local chromatic number about $t/2$. For more details and also for the relevance of s-wide colorability in the context of the circular chromatic number cf. [ST06].

A graph homomorphism from a graph F to a graph G is an edge-preserving map of the vertex set of F to the vertex set of G. The existence of such a map is denoted by $F \to G$. It is easy to see that $G \to K_t$ is equivalent to the t-colorability of the graph G, that is, to $\chi(G) \leq t$. We refer to the book [HN04] for a general treatment of the theory of graph homomorphisms.

Several other types of graph colorings can also be expressed by the existence of a graph homomorphism to some target graph and s-wide colorability is no exception. It is proved independently in [BS05] and [ST06] (and already in [GJS04] for the $s = 2$ case) that s-wide colorability with t colors is equivalent to the existence of a homomorphism to the following graph we denote by $W(s,t)$ as in [ST06] (see Figure 2.1 for a small example):

$$V(W(s,t)) = \{(x_1 \dots x_t) :\ \forall i\ x_i \in \{0,1,\dots,s\}, \exists! i\ x_i = 0,\ \exists j\ x_j = 1\},$$

$$E(W(s,t)) = \{\{(x_1 \dots x_t),(y_1 \dots y_t)\} : \forall i\ |x_i - y_i| = 1 \text{ or } x_i = y_i = s\}.$$

Proposition 2.1. *([BS05; GJS04; ST06]) A graph G admits an s-wide coloring using t colors if and only if $G \to W(s,t)$.*

A different incarnation of the graphs $W(s,t)$ appears in the papers [Haj09; Tar05; Wro19], where (following Wrochna's notation in [Wro19]) a graph operation Ω_k is given for every odd integer k and when applied to the complete graph K_t for $k = 2s-1$ then the resulting graph is isomorphic to $W(s,t)$. We will give and make use of this alternative definition in Section 2.3.

It is easy to see that $W(s,t)$ can be properly colored with t colors: set the color of vertex $(x_1 \dots x_t)$ to be the unique i for which $x_i = 0$ (see Figure 2.1 for an illustration).

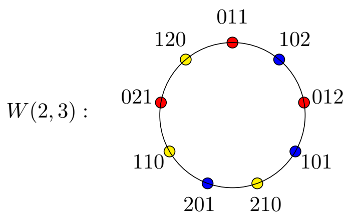

Figure 2.1: A 3-coloring of the graph $W(2,3)$. The first, second, and third indices in each sequence correspond to the colors red, blue, and yellow, respectively. A value of 0 in a given position indicates that the vertex is colored with the corresponding color. A value of 1 means that the color appears in the vertex's (first) neighborhood, while a value of 2 signifies that the color is far away from this vertex.

It is proved in [BS05; GJS04; ST06] (cf. also the chromatic properties of the more general Ω_k construction in [Haj09; Tar05; Wro19]) that this coloring is optimal, that is,

$$\chi(W(s,t)) = t. \tag{2.1}$$

This represents the surprising fact that there are t-chromatic graphs that can be optimally colored in such a way that the complete d-neighborhood of any color class is an independent set for every $d < s$. (By d-neighborhood of a color class we mean the set of vertices at distance exactly d from the closest element of the color class. In fact, if G is s-widely colored then not only the d-neighborhoods of color classes form independent sets for $d < s$ but all those vertices that can be attained via walks of length d from the given color class.) The proof of t-chromaticity of $W(s,t)$ goes via showing that some other graphs that are known to be t-chromatic admit a homomorphism into $W(s,t)$. These graphs include generalized Mycielski graphs, Schrijver graphs, and Borsuk graphs of appropriate parameters (for the definition of generalized Mycielski graphs see Section 2.4; cf. [Sch78; EH67; Mat07] for the definition of Schrijver graphs and Borsuk graphs and [ST06] for further details). This shows, in particular, that generalized Mycielski graphs, Schrijver graphs, and Borsuk graphs admit s-wide colorings. A common property of all these graphs is that their chromatic number can be determined by the already mentioned topological method introduced by Lovász in his celebrated paper [Lov78] proving Kneser's conjecture.

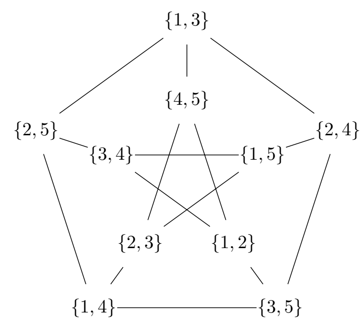

Figure 2.2: The graph $KG(5, 2)$, also known as the Petersen graph.

2.2 Multichromatic numbers

For n, k positive integers satisfying $n \geq 2k$ the Kneser graph $KG(n, k)$ is defined on $\binom{[n]}{k}$, the set of all k-element subsets of the n-element set $[n] = \{1, 2, \ldots, n\}$ as vertex set. Two vertices are adjacent if and only if the k-element subsets they represent are disjoint (see Figure 2.2 for a small example). It is not hard to show that $\chi(KG(n, k)) \leq n - 2k + 2$ (for all n, k satisfying $n \geq 2k$) and Kneser [Kne55] conjectured that this estimate is sharp. This was proved by Lovász [Lov78] thereby establishing the following result.

Theorem (Lovász-Kneser theorem).

$$\chi(KG(n, k)) = n - 2k + 2.$$

For more about the topological method we refer to the excellent book by Matoušek [Mat07].

The existence of a homomorphism to the Kneser graph $KG(n, k)$ can also be interpreted as a coloring property: $G \to KG(n, k)$ holds if and only if we can color the vertices of G with n colors in such a way that every vertex receives k distinct colors and if two vertices u and v are adjacent then the set of colors received by u is disjoint from the set of colors received by v. Such colorings were first considered by Geller and Stahl, see [GS75; Sta76]. Stahl [Sta76] introduced the corresponding chromatic number $\chi_k(G)$ as the minimum number of colors needed for such a coloring, called a k-fold coloring and $\chi_k(G)$ the k-fold chromatic number in [SU97] (or k-tuple chromatic number in [HN04]). The fractional chromatic number $\chi_f(G)$ can be defined as

$$\chi_f(G) = \inf_k \left\{ \frac{\chi_k(G)}{k} \right\} = \inf \left\{ \frac{n}{k} : G \to KG(n, k) \right\}.$$

Note the immediate consequence of this definition that if $G \to H$ then $\chi_f(G) \leq \chi_f(H)$.

As multichromatic numbers generalize the chromatic number (the latter being the special case for $k = 1$), determining their exact value (that is the value of the k-fold chromatic numbers for various k's) can be even more problematic than giving the value of the chromatic number. Indeed, while the chromatic number of Kneser graphs is already known by the Lovász-Kneser theorem, it is only a still open conjecture due to Stahl what homomorphisms exist and what do not between Kneser graphs. Stahl's conjecture asserts that there exists a homomorphism $KG(m, n) \to KG(m', n')$ in exactly the following cases and their compositions:

- $KG(m, n) \to KG(m + 1, n)$,
- $KG(m, n) \to KG(tm, tn)$, for every positive integer t,

- $KG(m,n) \to KG(m-2,n-1)$, for $n > 1$.

For further details, see Section 6.2 of [HN04]; see also [TZ19].

The starting point of our investigations was a question by Tardif [Tar22a] who observed that (2.1) combined with the Lovász-Kneser theorem implies that

$$\chi_r(W(s,t)) \geq t + 2(r-1) \tag{2.2}$$

and that equality holds for $r = s = 2$. (This is also true in the case of $r = s = 1$ when it simply means $\chi(K_t) = t$.) Tardif asked if there is equality also for $r = s = 3$. In particular, he was interested in whether $W(3,8) \not\to KG(12,3)$ and/or $W(3,7) \not\to KG(11,3)$ is true. Our main result will imply that this is actually not the case and equality does hold for $r = s = 3$. The motivation for Tardif's question, as already mentioned in the Introduction, came from recent developments concerning Hedetniemi's conjecture in which wide colorings also turned out to be relevant.

Hedetniemi's conjecture asked whether the so-called categorical (or tensor) product $G \times H$ satisfies $\chi(G \times H) = \min\{\chi(G), \chi(H)\}$. The conjecture is equivalent to say that $G \times H \to K_c$ implies that $G \to K_c$ or $H \to K_c$ must hold. (Although the latter directly only means $\chi(G \times H) \geq \min\{\chi(G), \chi(H)\}$, the reverse inequality is essentially trivial.) If this holds for K_c, then K_c is called multiplicative. Hedetniemi's conjecture formulated in 1966 thus stated that K_c is multiplicative for every positive integer c. This is trivial for $c = 1$, easy for $c = 2$. For $c = 3$, it is a far from trivial result by El-Zahar and Sauer [ES85]. For no other c it was decided (whether K_c is multiplicative or not) until 2019, when a breakthrough by Yaroslav Shitov took place who proved in [Shi19] that the conjecture is not true by constructing counterexamples for large enough c's. The smallest c for which Shitov's construction disproved the conjecture was extremely large (about 3^{95} according to an estimate in [Wro20]). This value was dramatically improved within a relatively short time. Using Shitov's ideas in a clever way, Zhu [Zhu21] first reduced c to 125. Then, developing the method further, Tardif [Tar22a] showed a counterexample for $c = 13$. He remarked that his construction would also work for $c = 12$ and 11, respectively, provided that $W(3,8) \not\to KG(12,3)$ and $W(3,7) \not\to KG(11,3)$. Our main result is the following that shows as a special case that these homomorphisms do exist.

Theorem 2.2.

$$\chi_s(W(s,t)) = t + 2(s-1).$$

Later Wrochna [Wro20] managed to improve on Tardif's result using the ideas in [Tar22a] in a different way and proving that K_c is not multiplicative for any $c \geq 5$ thus leaving $c = 4$ the only open case. Now, the $c = 4$ case is settled by Tardif [Tar23], K_4 is also not multiplicative. (For more details about Hedetniemi's conjecture see the recent papers cited above or Tardif's survey [Tar08] or a recent one [Zhu25b] by Zhu.)

The chapter is organized as follows. We present the proof of Theorem 2.2 in Section 2.3. In Section 2.4 we elaborate on the problem of what we can say about $\chi_r(W(s,t))$ for general r. It will be an immediate consequence of Theorem 2.2 combined with Tardif's observation (2.2) that $\chi_r(W(s,t)) = t + 2(r-1)$ whenever $r \leq s$. We will also observe that we cannot have equality in (2.2) for large enough r. We will also show that the fractional chromatic number of $W(s,t)$ goes to infinity when t grows and s remains fixed. The chapter concludes with some observations about the position of the graphs $W(s,t)$ in the homomorphism order of graphs.

2.3 Proof of the main result

First we give the alternative definition of the graphs $W(s,t)$ using the graph operation Ω_k. We put $k = 2s-1$ and give the definition of only $\Omega_{2s-1}(K_t)$ that we will use and refer to [Wro19] for the construction $\Omega_{2s-1}(G)$ for general graphs G.

Definition 2.1. *The graph $\Omega_{2s-1}(K_t)$ is defined as follows.*

$$V(\Omega_{2s-1}(K_t)) =$$

$$\{(A_0, A_1, \ldots, A_{s-1}) : \forall i\, A_i \subseteq [t], |A_0| = 1, A_1 \neq \emptyset, \forall i \in \{0, \ldots, s-3\}\, A_i \subseteq A_{i+2}, A_{s-2} \cap A_{s-1} = \emptyset\},$$

$$E(\Omega_{2s-1}(K_t)) = \{\{(A_0, A_1, \ldots, A_{s-1}), (B_0, B_1, \ldots, B_{s-1}) :$$
$$\forall i \in \{0, 1, \ldots, s-2\}\ A_i \subseteq B_{i+1}, B_i \subseteq A_{i+1} \text{ and } A_{s-1} \cap B_{s-1} = \emptyset\}.$$

Note that the above conditions also imply that $A_{i-1} \cap A_i = \emptyset$ for all $1 \leq i \leq s-1$ whenever $(A_0, A_1, \ldots, A_{s-1}) \in V(\Omega_{2s-1}(K_t))$.

It is straightforward and well-known (see e.g. [Wro19; Wro20]) that we have

$$W(s,t) \cong \Omega_{2s-1}(K_t).$$

Indeed, one can easily check that the following function $g : V(W(s,t)) \to V(\Omega_{2s-1}(K_t))$ provides an isomorphism between $W(s,t)$ and $\Omega_{2s-1}(K_t)$.

$$g : (x_1 \ldots x_t) \mapsto (A_0, A_1, \ldots, A_{s-1}),$$

where

$$\forall i \in \{0, 1, \ldots, s-1\} : A_i = \{j : x_j \leq i \text{ and } x_j \equiv i \pmod 2\}.$$

Remark 1. We gave both descriptions of the graphs $W(s,t)$, because we believe that both are useful. In particular, we will formulate the proof of Theorem 2.2 using the description of $\Omega_{2s-1}(K_t)$ as we believe that it makes the presentation of the proof easier to follow. Nevertheless, when we were thinking about the proof we felt we could understand the structure of these graphs better by considering its vertices as the sequences given in its definition as $W(s,t)$. (It is also remarked in [Wro20] that it is the $W(s,t)$ type description from which one easily sees that the number of vertices is $t(s^{t-1} - (s-1)^{t-1})$.)

Next we recall Tardif's observation (2.2) that we state as a lemma for further reference and also prove it for the sake of completeness.

Lemma 2.3 (Tardif [Tar22a]). *For all positive integers r and s*

$$\chi_r(W(s,t)) \geq t + 2(r-1).$$

Proof. We cannot have $W(s,t) \to KG(t+h, r)$ for $h < 2(r-1)$ as $\chi(KG(t+h,r)) = t+h-2r+2$ by the Lovász-Kneser theorem and this value is less than $t = \chi(W(s,t))$ whenever $h < 2(r-1)$. □

Proof of Theorem 2.2. We need to show

$$\chi_s(W(s,t)) = \chi_s(\Omega_{2s-1}(K_t)) = t + 2(s-1).$$

Lemma 2.3 already shows that the right hand side is a lower bound thus our task is to prove the reverse inequality which is equivalent to the existence of a graph homomorphism from $W(s,t) \cong \Omega_{2s-1}(K_t)$ to $KG(t+2(s-1), s)$. Below we give such a homomorphism

$$f : (A_0, A_1, \ldots, A_{s-1}) \mapsto \{z_0, \ldots, z_{s-1}\},$$

where $\{z_0, \ldots, z_{s-1}\} \in \binom{[t+2(s-1)]}{s} = V(KG(t+2(s-1), s))$. For $U = (A_0, A_1, \ldots, A_{s-1})$ we will use the notation $z_i = f_i(U)$ when $f((A_0, A_1, \ldots, A_{s-1})) = \{z_0, \ldots, z_{s-1}\}$. (Note that we do not assume that the z_i's are monotonically increasing with respect to their indices, we only need that all of them are distinct for a given $f(U) = \{z_0, \ldots, z_{s-1}\}$).

First assume that $s \geq 3$ is odd. (The $s = 1$ case is a trivial special case of (2.1).)

For every even $i \in \{2, \ldots, s-1\}$ we consider the three sets A_{i-2}, A_{i-1}, A_i and for each such triple we define two elements of $f(U)$, namely $f_{i-1}(U) = z_{i-1}$ and $f_i(U) = z_i$ as follows. According to the relative sizes of these three sets we will decide which of the elements $t+i-1, t+i, (t+s-1)+i-1 = t+s+i-2$, and $(t+s-1)+i = t+s+i-1$ will be put into the set $f(U)$. For every even i we will either put two of these elements into $f(U)$ or if not then we will find enough elements from $[t]$ to compensate this hiatus. This will give us $s-1$ distinct elements of $f(U)$. Finally we will define $f_0(U)$ as the missing s-th element of $f(U)$. The rules are as follows.

i) If $|A_{i-2}| > |A_{i-1}|$ then let $f_{i-1}(U) = t+i-1$ and $f_i(U) = t+i$. If $|A_{i-1}| > |A_i|$, then let $f_{i-1}(U) = t+s+i-2$ and $f_i(U) = t+s+i-1$. (Note that since $A_{i-2} \subseteq A_i$ at most one of the above two inequalities can hold so our definition is meaningful.)

ii) If $|A_{i-2}| < |A_{i-1}| < |A_i|$, then we must have $|A_i \setminus A_{i-2}| \geq 2$. In that case choose 2 distinct elements of $A_i \setminus A_{i-2}$ (these will be elements from $[t]$) to be $f_{i-1}(U)$ and $f_i(U)$.

iii) If $|A_{i-2}| < |A_{i-1}| = |A_i|$, then $|A_i \setminus A_{i-2}| \geq 1$. Let $f_{i-1}(U)$ be an arbitrary element of $A_i \setminus A_{i-2}$ and let

$$f_i(U) = \begin{cases} t+s+i-2 & \text{if } \min(A_{i-1} \cup A_i) \in A_{i-1} \\ t+s+i-1 & \text{if } \min(A_{i-1} \cup A_i) \in A_i. \end{cases}$$

Note that since $A_{i-1} \cap A_i = \emptyset$, $f_i(U)$ will be well defined.

iv) If $|A_{i-2}| = |A_{i-1}| < |A_i|$, then let

$$f_{i-1}(U) = \begin{cases} t+i-1 & \text{if } \min(A_{i-2} \cup A_{i-1}) \in A_{i-2} \\ t+i & \text{if } \min(A_{i-2} \cup A_{i-1}) \in A_{i-1}. \end{cases}$$

Since $A_{i-2} \cap A_{i-1} = \emptyset$, $f_{i-1}(U)$ is well defined. Let $f_i(U)$ be an arbitrary element of $A_i \setminus A_{i-2}$. Such a choice is possible as $A_i \setminus A_{i-2} \neq \emptyset$ in this case.

v) If $|A_{i-2}| = |A_{i-1}| = |A_i|$ (which means $A_i = A_{i-2}$) then let

$$\{f_{i-1}(U), f_i(U)\} = \begin{cases} \{t+i-1, t+s+i-1\} & \text{if } \min(A_{i-1} \cup A_i) \in A_i \\ \{t+i, t+s+i-2\} & \text{if } \min(A_{i-1} \cup A_i) \in A_{i-1}. \end{cases}$$

Note that since $A_i = A_{i-2}$, this formula is similar to the previous ones.

vi) Finally, let $f_0(U)$ be equal to the unique $h \in A_0$.

Note that by the above we have defined $f_j(U)$ for every $0 \leq j \leq s-1$ and if $j \neq j'$ then $f_j(U) \neq f_{j'}(U)$ thus we have $f(U) \in V(KG(t+2(s-1),s)$ as needed. We have to prove that f is indeed a graph homomorphism from $W(s,t) \cong \Omega_{2s-1}(K_t)$ to $KG(t+2(s-1),s)$. We do this first and consider the case of even s (that will be similar) afterwards.

Consider $U = (A_0, A_1, \ldots, A_{s-1})$ and $U' = (B_0, B_1, \ldots, B_{s-1})$. We have to show that if $f(U) \cap f(U') \neq \emptyset$, then $\{U, U'\} \notin E(\Omega_{2s-1}(K_t))$.

Assume that $f(U) \cap f(U') \neq \emptyset$ and we have $h \in f(U) \cap f(U')$ for some $h \in [t]$. Then we have h appearing in some A_j and some B_k, where both j and k are even. In particular, $h \in A_{s-1} \cap B_{s-1}$, thus $A_{s-1} \cap B_{s-1} \neq \emptyset$, therefore U and U' cannot be adjacent.

Now assume that $f(U) \cap f(U') \neq \emptyset$ but the intersection is disjoint from $[t]$ thus we have $t+d \in f(U) \cap f(U')$ for some $1 \leq d \leq 2s-2$.

If d is odd and $d \leq s-1$, then $d = i-1$ for some even $2 \leq i \leq s-1$, thus $t+d \in f(U)$ means $t+d = t+i-1 = f_{i-1}(U)$. If this happens then either $|A_{i-2}| > |A_{i-1}|$ or $|A_{i-2}| = |A_{i-1}|$ and $\min(A_{i-2} \cup A_{i-1}) \in A_{i-2}$. Similarly, $t+d = t+i-1 \in f(U')$ implies that either $|B_{i-2}| > |B_{i-1}|$ or $|B_{i-2}| = |B_{i-1}|$ and $\min(B_{i-2} \cup B_{i-1}) \in B_{i-2}$. Assume for contradiction that $\{U, U'\}$ is an edge of our graph $\Omega_{2s-1}(K_t)$. Then we must have $A_{i-2} \subseteq B_{i-1}$ and $B_{i-2} \subseteq A_{i-1}$ implying

$$|A_{i-2}| \leq |B_{i-1}| \leq |B_{i-2}| \leq |A_{i-1}| \leq |A_{i-2}|,$$

therefore we must have equality everywhere. By $A_{i-2} \subseteq B_{i-1}$ and $B_{i-2} \subseteq A_{i-1}$ (that follows from $\{U, U'\} \in E(\Omega_{2s-1}(K_t)))$ this implies $A_{i-2} = B_{i-1}$ and $B_{i-2} = A_{i-1}$ and therefore $j := \min(A_{i-2} \cup A_{i-1}) = \min(B_{i-2} \cup B_{i-1})$. Our assumption on d then implies both $j \in A_{i-2}$ and $j \in B_{i-2} = A_{i-1}$ which is impossible by $A_{i-2} \cap A_{i-1} = \emptyset$.

The situation is similar for the other possible values of d. If $d = i \leq s-1$ is even, then $t+d = t+i \in f(U) \cap f(U')$ for some adjacent vertices U, U' would again imply

$$|A_{i-2}| = |B_{i-1}| = |B_{i-2}| = |A_{i-1}|$$

and thus $A_{i-2} = B_{i-1}, B_{i-2} = A_{i-1}$ as above. Our assumption on d now would imply for $j = \min(A_{i-2} \cup A_{i-1}) = \min(B_{i-2} \cup B_{i-1})$ that it must be both in A_{i-1} and in $B_{i-1} = A_{i-2}$ leading to the same contradiction as in the previous paragraph.

For $s-1 < d$ and $t+d \in f(U) \cap f(U')$ for adjacent vertices U, U' we get a similar contradiction. In particular, this assumption implies $|A_{i-1}| \geq |A_i|$ and $|B_{i-1}| \geq |B_i|$ that by the adjacency of U and U' (meaning, in particular, $A_{i-1} \subseteq B_i$ and $B_{i-1} \subseteq A_i$) would imply

$$|A_{i-1}| = |B_i| = |B_{i-1}| = |A_i|$$

and thus $A_{i-1} = B_i$ and $B_{i-1} = A_i$. Then we obtain that $k := \min(A_{i-1} \cup A_i) = \min(B_i \cup B_{i-1})$ should belong (depending on the parity of d) to both A_{i-1} and $B_{i-1} = A_i$ or to both A_i and $B_i = A_{i-1}$ leading to the same contradiction that $A_{i-1} \cap A_i \neq \emptyset$. This finishes the proof for odd s.

Now assume that s is even. We need only some minor modifications compared to the odd s case. Let us now for every odd $i \in \{3, \ldots, s-1\}$ define $f_{i-1}(U)$ and $f_i(U)$ almost the same way as in points i)– v) above. (The only difference will be that the values $t+i-1$ and $t+i$ are shifted by 1 to become $t+i$ and $t+i+1$. In case of $s = 2$ the modified rules (i')-(v') will not apply, only those will that we denote by (vi') and (vii') below.) This gives the last $s-2$ values of the set $f(U) = \{f_0(U), f_1(U), \ldots, f_{s-1}(U)\}$, what is left is to define $f_0(U)$ and $f_1(U)$ by a modified version of the sixth point above that has now two parts. The modified rules are as follows.

i') If $|A_{i-2}| > |A_{i-1}|$ then let $f_{i-1}(U) = t+i$ and $f_i(U) = t+i+1$. If $|A_{i-1}| > |A_i|$, then let $f_{i-1}(U) = t+s+i-2$ and $f_i(U) = t+s+i-1$.

ii') and iii') are identical to ii) and iii), respectively.

iv') If $|A_{i-2}| = |A_{i-1}| < |A_i|$, then let

$$f_{i-1}(U) = \begin{cases} t+i & \text{if } \min(A_{i-2} \cup A_{i-1}) \in A_{i-2} \\ t+i+1 & \text{if } \min(A_{i-2} \cup A_{i-1}) \in A_{i-1}. \end{cases}$$

Let $f_i(U)$ be an arbitrary element of $A_i \setminus A_{i-2}$.

v') If $|A_{i-2}| = |A_{i-1}| = |A_i|$ then let

$$\{f_{i-1}(U), f_i(U)\} = \begin{cases} \{t+i, t+s+i-1\} & \text{if } \min(A_{i-1} \cup A_i) \in A_i \\ \{t+i+1, t+s+i-2\} & \text{if } \min(A_{i-1} \cup A_i) \in A_{i-1}. \end{cases}$$

Note again that since $A_i = A_{i-2}$, this formula is similar to the ones in cases iii') and iv').

vi') If $|A_1| = |A_0|$, then let

$$f_0(U) = \begin{cases} t+1 & \text{if } \min(A_0 \cup A_1) \in A_0 \\ t+2 & \text{if } \min(A_0 \cup A_1) \in A_1. \end{cases}$$

Note that in this case both A_0 and A_1 contains only one element and the value of $f_0(U)$ is $t+1$ or $t+2$ depending on which of the two is smaller. At the same time let

$$f_1(U) = h \text{ where } A_1 = \{h\},$$

that is, $h \in [t]$ is the unique element of A_1.

vii') If $|A_1| > |A_0|$, then since $|A_0| = 1$ we have $|A_1| \geq 2$. Now choose two arbitrary distinct elements of A_1 for $f_0(U)$ and $f_1(U)$.

Note that we have $|A_1| \geq 1 = |A_0|$ by the definition of $\Omega_{2s-1}(K_t)$, so we do not have to consider the possibility that $|A_0| > |A_1|$, it never occurs.

With this definition of $f(U)$ the proof that f is a graph homomorphism is essentially identical to that we presented in the odd s case. The main difference is that now those $j \in [t]$ that appear as elements of the sets $f(U)$ are all elements of some A_i where i is odd, while the corresponding i's were all even in the case of odd s. The rest of the arguments work the same way as in the case of odd s.

This completes the proof. □

We remark that by the composition of homomorphisms Theorem 2.2 determines the s-fold chromatic number of every s-widely colorable t-chromatic graph.

2.4 On other multichromatic numbers of $W(s,t)$

An immediate consequence of Theorem 2.2 is that we can give the multichromatic numbers $\chi_r(W(s,t))$ for all $r \le s$.

Corollary 2.4. *If $r \le s$, then*
$$\chi_r(W(s,t)) = t + 2(r-1).$$

The proof follows from the following simple lemma (which is essentially Lemma 2.3.(iv) of [Wro19]) combined with Tardif's observation given in Lemma 2.3.

Lemma 2.5 ([Wro19]). *For all $1 \le r \le s$ we have*
$$W(s,t) \to W(r,t)$$

Proof. Define the following function for all $0 \le a \le s$.
$$\varphi(a) = \begin{cases} a & \text{if } 0 \le a \le r \\ r & \text{if } r < a \le s. \end{cases}$$

It is straightforward to check that the mapping $g : (x_1 \dots x_t) \mapsto (\varphi(x_1) \dots \varphi(x_t))$ is a homomorphism from $W(s,t)$ to $W(r,t)$ for all $1 \le r \le s$.

Proof of Corollary 2.4. In view of Lemma 2.3 it is enough to prove that $\chi_r(W(s,t))$ is at most the claimed value if $r \le s$. Applying Lemma 2.5 and Theorem 2.2 to $r \le s$ we have
$$W(s,t) \to W(r,t) \to KG(t + 2(r-1), r)$$
implying
$$\chi_r(W(s,t)) \le t + 2(r-1)$$
as needed. □

For $r > s$ we do not know the value of $\chi_r(W(s,t))$. We know from Lemma 2.3 though that $\chi_r(W(s,t)) \ge t + 2(r-1)$ so the question naturally arises whether we could have equality here for every r. Below we show that this is not the case.

Proposition 2.6. *For all pairs of positive integers $t \ge 3$ and $s \ge 1$ there exists some threshold $r_0 = r_0(s,t) > s$ for which*
$$\chi_r(W(s,t)) > t + 2(r-1) \tag{2.3}$$
whenever $r \ge r_0$.

Proof. Assume for the sake of contradiction that for some fixed s and t we have $\chi_r(W(s,t)) = t + 2(r-1)$ for arbitrarily large r. That would imply that $\chi_f(W(s,t)) \le \lim_{r\to\infty} \frac{t+2(r-1)}{r} = 2$. However, this cannot be true since $W(s,t)$ is not bipartite for $t \ge 3$ and thus it contains an odd cycle C_{2b+1} for some positive integer b. Thus we must have $\chi_f(W(s,t)) \ge \chi_f(C_{2b+1}) = \frac{2b+1}{b}$, a number larger than 2 with the constant value $\frac{1}{b}$. □

The problem of determining the smallest possible r for which (2.3) holds is left as an open problem. It is frustrating that we were not able to decide even whether this value is just $s+1$ as the proof of Theorem 2.2 might suggest or larger.

Remark 2. The previous proof does not specify b as its value is not essential there. Nevertheless one can easily see that $W(s,3) \cong C_{6s-3}$. It is also easy to see that $g_o(W(s,t))$, the odd girth of $W(s,t)$ must be at least $2s+1$ and we have equality here for $t \geq 2s+1$ since a cycle C_{2s+1} is formed in $W(s, 2s+1)$ by the vertices given by the sequence $(0,1,2,\ldots,s,s,s-1,\ldots,2,1)$ and its cyclic permutations. (For larger t these sequences can be extended by an arbitrary number of coordinates equal to s.) As one of the referees noted $g_o(W(s,t)) = 2s+1$ for $t \geq 2s+1$ is also immediately implied by Proposition 1 and the fact that C_{2s+1} admits an s-wide coloring with at most t colors. In fact, the unpublished paper by Baum and Stiebitz [BS05] gives the general formula $2s-1+2\left\lceil \frac{2s-1}{t-2} \right\rceil$ for the odd girth of $W(s,t)$.

The previous proof raises the question what we can say about the fractional chromatic number of the graphs $W(s,t)$. As a consequence of Theorem 2.2 we know $\chi_f(W(s,t)) \leq \frac{t+2(s-1)}{s}$ and the previous simple proof implies that it is at least $2+\frac{1}{3s-2}$ for $t \geq 3$. Unfortunately we were not able to prove matching lower and upper bounds. But we can at least show that for any fixed s the fractional chromatic number of $W(s,t)$ gets arbitrarily large as t tends to infinity.

Theorem 2.7. *For any fixed positive integer s we have*

$$\lim_{t\to\infty} \chi_f(W(s,t)) = \infty.$$

The proof will be a simple consequence of the (already known) fact that certain generalized Mycielski graphs admit s-wide colorings. To give more details we introduce generalized Mycielski graphs below.

Definition 2.2. *The h-level generalized Mycielskian $M_h(G)$ of a graph G is defined as follows.*

$$V(M_h(G)) = \{(v,j) : v \in V(G), 0 \leq j \leq h-1\} \cup \{z\}.$$

$$E(M_h(G)) = \{\{(u,i),(v,j)\} : uv \in E(G) \text{ and } (|i-j| = 1 \text{ or } i = j = 0\} \cup \{\{z,(v,(h-1))\}.$$

The d times iterated h-level generalized Mycielskian $M_h(M_h(\ldots M_h(G)\ldots))$ of a graph G will be denoted by $M_h^{(d)}(G)$.

The term Mycielskian of a graph G usually refers to $M(G) = M_2(G)$ and Mycielski graphs are the iterated Mycielkians of K_2 introduced by Mycielski [Myc55] as triangle-free graphs whose chromatic number grows by one at every iteration. The property $\chi(M(G)) = \chi(G)+1$ is well-known to hold for any G but the analogous equality is not always true for h-level Mycielskians if $h > 2$, cf. Tardif [Tar01]. Nevertheless Stiebitz [Sti85] showed that $\chi(M_h(G)) = \chi(G)+1$ is also true if G is a complete graph or an odd cycle. (More generally one can say that this is the case whenever G is a graph for which the topological lower bound on the chromatic number by Lovász [Lov78] is sharp, cf. [GJS04; Mat07; Sti85] or [ST06] for more details.) So by Stiebitz's result we have

$$\chi(M_h^{(d)}(K_2)) = d+2$$

for all positive integers d and h.

The t-chromaticity of $W(s,t)$ is proved in [BS05; GJS04; ST06] by showing the existence of t-chromatic graphs that admit a homomorphism into $W(s,t)$. In case of [BS05; GJS04] these are generalized Mycielski graphs $M_h^{(t-2)}(K_2)$ for appropriately large h. (Since [BS05] is unpublished and [GJS04] gives this explicitly only for $s=2$, we give some more details for the sake of completeness. Nevertheless, this is a straightforward generalization of the construction given in [GJS04] as already noted in [ST06] where the case $s=3$ is made explicit. So the following is a straightforward extension of Lemma 4.3 from [ST06] also attributed to [GJS04] there.)

Lemma 2.8 ([GJS04]). *If G has an s-wide coloring with t colors, then $M_{3s-2}(G)$ has an s-wide coloring with $t+1$ colors.*

Proof. Fix an s-wide coloring $c_0 : V(G) \to [t]$ of G. Let $c : V(M_{3s-2}(G)) \to [t] \cup \{\gamma\}$ be the following coloring using the additional color γ. Set $c(z) = \gamma$ and

$$c((v,j)) = \begin{cases} \gamma & \text{if } j \in \{s, s+2, \ldots, 3s-4\} \\ c_0(v) & \text{otherwise.} \end{cases}$$

If we have a walk of odd length between vertices (u,i) and (v,j) with $c(u,i) = c(v,j) \in [t]$ that walk must either traverse the vertex z or use an edge of the form $\{(a,0),(b,0)\}$. In the latter case the walk projects down to a walk of the same length between u and v in G with $c_0(u) = c_0(v)$ so its length must be at least $2s+1$ by c_0 being s-wide. In case the walk traverses z we can assume that we have $i \not\equiv j \bmod 2$ and thus without loss of generality $j \equiv s \bmod 2$ implying that $j \le s-2$. But then the distance between (v,j) and z is already at least $2s$, so the length of our walk is at least $2s+1$.

Since deleting the set of vertices $\{(v,0)\}_{v \in V(G)}$ from $M_{3s-2}(G)$ the remaining induced subgraph is bipartite and γ appears only on one side of this bipartite graph, any odd length walk between two vertices colored γ must use an edge of the form $\{(u,0),(v,0)\}$. But the distance of any γ-colored vertex from such vertices is at least s, so such a walk also cannot be shorter than $2s+1$. Thus c is indeed an s-wide coloring. □

For $M(G) = M_2(G)$ Larsen, Propp and Ullman [LPU95] made the very nice observation that $\chi_f(M(G))$ can be given by a simple function of $\chi_f(G)$, namely

$$\chi_f(M(G)) = \chi_f(G) + \frac{1}{\chi_f(G)}.$$

This was later generalized by Tardif for generalized Mycielskians.

Theorem 2.9 (Tardif [Tar01]).

$$\chi_f(M_h(G)) = \chi_f(G) + \frac{1}{\sum_{i=0}^{h-1} (\chi_f(G)-1)^i}.$$

Note that for non-bipartite graphs Tardif's theorem gives

$$\chi_f(M_h(G)) = \chi_f(G) + \frac{\chi_f(G)-2}{(\chi_f(G)-1)^h - 1}.$$

and it implies that $\chi_f(M_h^{(d)}(G))$ tends to infinity as d goes to infinity for any fixed finite h.

Proof of Theorem 2.7. The proof follows immediately from the preceding results. Lemma 2.8 and Tardif's Theorem 2.9 together imply that

$$\chi_f(W(s,t+1)) \geq \chi_f(M_{3s-2}(W(s,t))) = \chi_f(W(s,t)) + \frac{1}{\sum_{i=0}^{3s-3}(\chi_f(W(s,t))-1)^i},$$

and this implies the statement.

In view of Lemma 2.8 it may be interesting to note that while a generalized Mycielskian of $W(s,t)$ admits a homomorphism into $W(s,t+1)$, the latter also admits a (very natural) homomorphism into another generalized Mycielksian of $W(s,t)$.

Proposition 2.10.

$$W(s,t+1) \to M_s(W(s,t)).$$

Proof. We explicitly give the homomorphism. Let

$$g((x_1 \dots x_{t+1})) = \begin{cases} ((x_1 \dots x_t), s - x_{t+1}) & \text{if } x_{t+1} > 0 \text{ and } (x_1 \dots x_t) \in V(W(s,t)) \\ ((01 \dots 1), s-1) & \text{if } \{i : x_i = 1\} = \{t+1\} \\ z & \text{if } x_{t+1} = 0. \end{cases}$$

(In fact, in the second case $((01 \dots 1), s-1)$ could be substituted by an arbitrarily chosen $((y_1 \dots y_t), s-1)$ for which $(y_1 \dots y_t) \in V(W(s,t))$.)

It is straightforward to check that the given function is indeed a graph homomorphism. □

Thus we obtained that in the homomorphism order of graphs (cf. [HN04]) in which $F \preceq G$ if and only if $F \to G$ we have $W(s,t+1)$ sandwiched between two different generalized Mycialskians of $W(s,t)$, in particular,

$$M_{3s-2}(W(s,t)) \preceq W(s,t+1) \preceq M_s(W(s,t)). \tag{2.4}$$

This excludes the possibility that our upper bound $\frac{t+2(s-1)}{s}$ on $\chi_f(W(s,t))$ provided by Theorem 2.2 would be tight at least for all sufficiently large t, because then the difference $\chi_f(W(s,t+1)) - \chi_f(W(s,t))$ would be equal to $\frac{1}{s}$ for large t contradicting Tardif's Theorem 2.9. Note that (2.4) implies

$$\begin{aligned} \chi_f(W(s,t)) + \frac{\chi_f(W(s,t))-2}{(\chi_f(W(s,t))-1)^{3s-2}-1} \\ \leq \chi_f(W(s,t+1)) \\ \leq \chi_f(W(s,t)) + \frac{\chi_f(W(s,t))-2}{(\chi_f(W(s,t))-1)^{s}-1} \end{aligned}$$

.

With a little more considerations we can also show that $W(s,t+1)$ is actually *strictly* sandwiched between the above two generalized Mycielskians of $W(s,t)$ if $s>1$ and $t>2$.

Proposition 2.11. *If $s \geq 2, t \geq 3$ then*

$$M_{3s-2}(W(s,t)) \prec W(s,t+1) \prec M_s(W(s,t)). \tag{2.5}$$

For $s=1$ all three graphs are isomorphic to K_{t+1}. For $s>1, t=2$ we have

$$M_{3s-2}(W(s,2)) \cong C_{6s-3} \cong W(s,3) \prec M_s(W(s,2)) \cong C_{2s+1}.$$

Proof. It is well-known and easy to prove that if G is a vertex-color-critical graph (that is, one from which deleting any vertex its chromatic number decreases) and $\chi(M_h(G)) = \chi(G)+1$, then $M_h(G)$ is also vertex-color-critical (see this e.g. as Problem 9.18 in the book [Lov93] for $h = 2$). It is shown independently both in [BS05] and [ST06] that $W(s,t)$ is edge-color-critical for every $s \geq 1, t \geq 2$. Thus all three graphs appearing in (2.5) are vertex-color-critical. Since they all have the same chromatic number this implies that any homomorphism that exists between any two of them should be onto. This also means that if any two of them would be homomorphically equivalent, then those two should have the same number of vertices, in particular, any homomorphism between them is a one-to-one mapping between their vertex sets. This is clearly not the case for the homomorphism given in the proof of Proposition 2.10 since several distinct vertices (their exact number is $s^t-(s-1)^t$) are mapped to the vertex z unless $s = 1$.

If a homomorphism between $M_{3s-2}(W(s,t))$ and $W(s,t+1)$ was one-to-one then by the edge-color-criticality of $W(s,t+1)$ it cannot happen that we map two non-adjacent vertices of $M_{3s-2}(W(s,t))$ to two adjacent ones of $W(s,t+1)$, since then deleting the latter adjacency we would still have a homomorphism but into a graph of smaller chromatic number. Thus such a homomorphism would then be an isomorphism, that is the two graphs would be isomorphic which is clearly not the case if $s > 1$ and $t > 2$. (A quick way to see this is the following. The maximum degree of $W(s,t+1)$ is 2^{t-1} attained by vertices $(x_1 \dots x_{t+1})$ for which $|\{i : x_i = 1\}|$ is equal to 1 or 2. The maximum degree of $M_{3s-2}(W(s,t))$ is $|V(W(s,t))| = t(s^{t-1}-(s-1)^{t-1})$ that cannot be a power of 2 for $s > 1$ unless $t = 2$.) The remaining cases in the statement are straightforward to check. □

Chapter 3

Critical Subgraphs for the Fractional Chromatic Number

Some definitions (such as that of Kneser graphs, homomorphism, etc.) were given in the previous chapters; however, for the sake of completeness, we include them in this chapter as well. Kneser graphs $KG(n,k)$ are defined for every pair of positive integers n, k satisfying $n \geq 2k$. Kneser [Kne55] observed (using different terminology) that their chromatic number is not more than $n-2k+2$ and conjectured that this upper bound is tight. This was proved by Lovász in his celebrated paper [Lov78] using the Borsuk-Ulam theorem. Soon afterwards Schrijver [Sch78] found that a certain induced subgraph $SG(n,k)$ of $KG(n,k)$, now called Schrijver graph, still has chromatic number $n-2k+2$ and is also vertex-critical for this property, that is, deleting any of its vertices the chromatic number becomes smaller. It is also well-known that the fractional chromatic number of $KG(n,k)$ is $\frac{n}{k}$, a consequence of the vertex-transitivity of these graphs and the Erdős–Ko–Rado theorem. Proving a conjecture of Holroyd and Johnson [Hol99] Talbot [Tal03] gave the exact value of the independence number of Schrijver graphs that easily implies, as already observed in [ST06], that their fractional chromatic number is also $\frac{n}{k}$. Most Schrijver graphs are not vertex-critical for this property (the only exceptions are the trivial cases when $k=1$, $n=2k$, or $n=2k+1$, cf. Corollary 3.16 in Section 3.2) and this suggests the problem of finding critical subgraphs of Schrijver graphs for the fractional chromatic number. In this chapter we present such a subgraph for all values of n and k with $n \geq 2k$. These subgraphs, that turn out to be isomorphic to the circular (also called rational) complete graphs $K_{n'/k'}$ for $n' = \frac{n}{\gcd(n,k)}$, $k' = \frac{k}{\gcd(n,k)}$, are vertex-transitive, so deleting any of their vertices the value of the fractional chromatic number drops to the same smaller value. We also locate the edges of these special subgraphs that are critical for the fractional chromatic number and show that their deletion already results in the same decrease of the fractional chromatic number as the deletion of a vertex.

In the next section we give the necessary definitions to define the above mentioned vertex-critical subgraph and state our main theorem. A proposition is also given there claiming the relation to circular complete graphs. From the latter the theorem will easily follow via known results about circular complete graphs. Section 3.2 contains the proof of the mentioned proposition thus completing the proof of our main result. The last section is devoted to characterizing the critical edges of circular complete graphs for the fractional chromatic number.

3.1 Well-spread subsets and the subgraph $Q(n,k)$

Definition 3.1. *For positive integers $n \geq 2k$ the Kneser graph $KG(n,k)$ is defined on the vertex set that consists of the $\binom{n}{k}$ k-element subsets of $[n] = \{1, \dots n\}$ with two such subsets forming an edge if and only if they are disjoint. A k-subset X of $[n]$ is called r-separated if for any two of its elements x, y we have $r \leq |x - y| \leq n - r$. The Schrijver graph $SG(n,k)$ is the subgraph of $KG(n,k)$ induced by vertices representing 2-separated sets.*

Notice that arranging the elements of the basic set $[n]$ around a cycle, the r-separated sets are exactly those any two elements of which have at least $r-1$ elements on both of the two arcs between them on this cycle (see Figure 3.1).

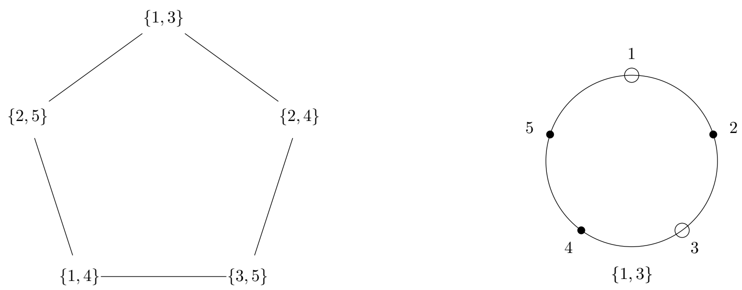

Figure 3.1: On the left, we see the Schrijver graph $SG(5,2)$. On the right, the elements of [5] are arranged around a cycle, illustrating a subset corresponding to a vertex of this graph. As shown, the set $\{1,3\}$ is 2-separated.

The following theorem is a condensed version of the well-known results in [Lov78; Sch78].

Theorem 3.1 (Lovász–Kneser and Schrijver theorem [Lov78; Sch78]). *For every $n \geq 2k$ we have*

$$\chi(SG(n,k)) = \chi(KG(n,k)) = n - 2k + 2.$$

Moreover, $SG(n,k)$ is vertex-color-critical, i.e.,

$$\forall X \in V(SG(n,k)): \ \chi(SG(n,k) \setminus \{X\}) = n - 2k + 1.$$

The graphs $KG(n,k)$ and $SG(n,k)$ are widely investigated, cf. e.g. [Bra10; Bra11; BL03; Che11; KS17; KS20; KS22; BV18; Meu05; ST20] to mention just a few more of the results related to them.

Recall that a graph homomorphism from graph F to graph G is an edge-preserving map $f: V(F) \to V(G)$, that is one for which $\{u,v\} \in E(F)$ implies $\{f(u), f(v)\} \in E(G)$. The existence of a graph homomorphism from F to G is denoted by $F \to G$.

Definition 3.2. *The fractional chromatic number $\chi_f(G)$ of a graph G can be defined as*

$$\chi_f(G) = \min\left\{\frac{n}{k} : G \to KG(n,k)\right\}.$$

It follows from the definition that $F \to G$ implies $\chi_f(F) \leq \chi_f(G)$, in particular this is always the case if F is a subgraph of G.

It is well-known that, denoting the independence number of graph G by $\alpha(G)$, one always has

$$\chi_f(G) \geq \frac{|V(G)|}{\alpha(G)}$$

and equality holds whenever the graph is vertex-transitive, see e.g. [SU97] for this and other basic facts about the fractional chromatic number.

The independence number of Kneser graphs is given by the famous Erdős–Ko–Rado theorem. This theorem concerns the maximum number of pairwise intersecting k-element subsets of $[n]$. Note that, by the definition of Kneser graphs, this quantity is precisely the independence number of $KG(n, k)$.

Theorem 3.2 (Erdős–Ko–Rado [EKR61]).

$$\alpha(KG(n, k)) = \binom{n-1}{k-1}.$$

Moreover, for $n > 2k$ the only independent sets of this size are the ones whose vertices represent k-element subsets that all contain a fixed element $i \in [n]$.

Corollary 3.3 (cf. e.g. [SU97]).

$$\chi_f(KG(n, k)) = \frac{n}{k}.$$

Holroyd and Johnson [Hol99] conjectured that a similar phenomenon to the one expressed by the Erdős–Ko–Rado theorem is also true for Schrijver graphs and more generally, for families of r-separated sets. Here we state the result only for $r = 2$.

Theorem 3.4 (Talbot [Tal03]).

$$\alpha(SG(n, k)) = \binom{n-k-1}{k-1}.$$

Moreover, for $n > 2k, n \neq 2k + 2$ the only independent sets of this size in $SG(n, k)$ are the ones whose vertices represent k-element subsets that all contain a fixed element $i \in [n]$. For $n = 2k + 2$ other independent sets of this size exist, too.

Since $|V(SG(n, k))| = \frac{n}{k}\binom{n-k-1}{k-1}$ and obviously $\chi_f(SG(n, k)) \leq \chi_f(KG(n, k))$ the above theorem has the following immediate consequence already noted in [ST06].

Corollary 3.5.

$$\chi_f(SG(n, k)) = \frac{n}{k}.$$

Let C_n denote the cycle on vertex set $[n]$ where the edges are formed by the pairs of vertices $\{i, i+1\}$ for $i \in \{1, \ldots, n-1\}$ and $\{1, n\}$. In particular, the vertices of $SG(n, k)$ are exactly the independent sets of size k in C_n. (We will refer to this cycle as the *defining cycle* for $SG(n, k)$.)

Definition 3.3. *We call a subset U of $V(C_n)$* well-spread *if for any two sets $A, B \subseteq [n]$ with $|A| = |B| \leq n-1$ satisfying that both induce a (connected) path in C_n we have*

$$||A \cap U| - |B \cap U|| \leq 1.$$

The induced subgraph of $SG(n,k)$ on all well-spread k-subsets will be denoted by $Q(n,k)$.

Example 1. For $n = 11$ the set $U = \{1,4,8\}$ is well-spread but the set $U' = \{1,4,9\}$ is not as the size of its intersection with the 4-element sets $\{1,2,3,4\}$ and $\{5,6,7,8\}$ of consecutive vertices of C_n differs by 2.

Now we state a basic property of the graphs $Q(n,k)$.

Proposition 3.6. *Let $n \geq 2k$ and $\ell \geq 2$ be any positive integer. Then the graphs $Q(n,k)$ and $Q(\ell n, \ell k)$ are isomorphic.*

Proof. Let $U \subseteq V(C_{\ell n}) = [\ell n]$ be a well-spread set of size ℓk. We will show that rotating the set U n times along the cycle $C_{\ell n}$ it will map to itself and that will easily imply the statement.

Consider the n-element sets $A_i \subseteq [\ell n], i \in [\ell n]$ defined by

$$A_i := \{i, i+1, \ldots, i+n-1\},$$

where the addition is intended modulo ℓn (and 0 is represented by ℓn), that is the sets A_i are exactly those subsets of $[\ell n]$ that induce a path of length $n-1$ in $C_{\ell n}$. First we show that the number of pairs in the set

$$\{(j, A_i) \colon i \in [\ell n], j \in A_i \cap U\},$$

where $j \in [\ell n]$ and A_i is one of the sets just defined is $\ell k n$. Indeed, since each $j \in U$ will appear in exactly n distinct A_i's and $|U| = \ell k$, this claim follows. Since there are ℓn distinct A_i's, this means that if any A_i would contain less than k elements of U, then some other $A_{i'}$ should contain more than k elements of U. However, this would imply that these two sets, A_i and $A_{i'}$ are of the same size, both induce a path of $C_{\ell n}$ and the size of their intersection with U differs by at least 2. This would contradict the well-spread property of U, so this is impossible. The situation is similar if any A_i would contain more than k elements of U, therefore we have

$$\forall i \colon |A_i \cap U| = k.$$

This implies that we have $j \in U$ if and only if $j + n \pmod{\ell n} \in U$ for every $j \in V(C_{\ell n})$ (otherwise $|A_j \cap U| = |A_{j+1} \cap U|$ would not be satisfied). Hence, if we have $X \in V(Q(\ell n, \ell k))$, that is X is a well-spread (ℓk)-subset of $[\ell n]$, and we rotate the defining cycle $C_{\ell n}$ exactly n times, then we get a vertex $Y \in Q(\ell n, \ell k)$, that is identical to X.

Let $g : V(C_{\ell n}) \to V(C_n)$ be defined by

$$g : i \mapsto i - n \left\lfloor \frac{i-1}{n} \right\rfloor$$

and for a subset $X = \{x_1, \ldots, x_{\ell k}\} \subseteq \binom{[\ell n]}{\ell k}$ we let $\hat{g}(X)$ denote the set $\{g(x_1), \ldots, g(x_{\ell k})\} \subseteq V(C_n)$. The foregoing implies that if $X \in V(Q(\ell n, \ell k))$ then $\hat{g}(X) \in V(Q(n,k))$. It also follows that for $X, Y \in V(Q(n,k))$ we have $\hat{g}(X) \cap \hat{g}(Y) = \emptyset \Leftrightarrow X \cap Y = \emptyset$. The latter means that $Q(n,k) \cong Q(\ell n, \ell k)$ and this proves the statement. □

Example 2. Let $n = 7, k = 3$ and $\ell = 2$. Then the statement of Proposition 3.6 is that $Q(14,6)$ is isomorphic to $Q(7,3)$. The vertices of $Q(7,3)$ are the 3-element sets

$$\{1,3,5\}, \{2,4,6\}, \{3,5,7\}, \{4,6,1\}, \{5,7,2\}, \{6,1,3\}, \{7,2,4\}.$$

The vertices of $Q(14,6)$ are

$$\{1,3,5,8,10,12\}, \{2,4,6,9,11,13\}, \{3,5,7,10,12,14\},$$

$$\{4,6,8,11,13,1\}, \{5,7,9,12,14,2\}, \{6,8,10,13,1,3\}, \{7,9,11,14,2,4\}.$$

Note that the latter seven sets have the form $\{i, i+2, i+4, i+7, i+9, i+11\}$. Thus if we identify i and $i+7$ for every $i \in \{1, 2, \dots, 7\}$ (the mapping g defined in the proof of Proposition 3.6 does essentially this by mapping both to i), then the seven vertices of $Q(14,6)$ become identical to the seven vertices of $Q(7,3)$.

Note that Proposition 3.6 implies that $Q(n,k) \cong Q\left(\frac{n}{\gcd(n,k)}, \frac{k}{\gcd(n,k)}\right)$, therefore when discussing the properties of $Q(n,k)$ we may assume that $\gcd(n,k) = 1$.

Now we can already state our result on the vertex-criticality of $Q(n,k)$ for the fractional chromatic number.

Theorem 3.7. *Assume $n \geq 2k$, $\gcd(n,k) = 1$ and let a and b be the smallest positive integers for which $ak = bn - 1$. The graph $Q(n,k) \subseteq SG(n,k)$ satisfies the following properties.*
- $\chi_f(Q(n,k)) = \frac{n}{k} = \chi_f(SG(n,k))$.
- $\forall U \in V(Q(n,k))\ \chi_f(Q(n,k) \setminus \{U\}) = \frac{a}{b} < \frac{n}{k}$, *that is $Q(n,k)$ is vertex-critical for the fractional chromatic number.*
- *$Q(n,k)$ contains an induced subgraph isomorphic to $Q(a,b)$.*

For an example see Example 3 after Proposition 3.8.

For proving Theorem 3.7 it will be enough to show that if $\gcd(n,k) = 1$ then the $Q(n,k)$ subgraph is isomorphic to the circular (also called rational) complete graph $K_{n/k}$ that we define next.

Definition 3.4. *The circular complete graph $K_{n/k}$ is defined as follows:*

$$V(K_{n/k}) = \{0, 1, \dots, n-1\}$$

$$E(K_{n/k}) = \{\{i,j\} : k \leq |i-j| \leq n-k.\}$$

The name *circular complete graph* refers to the central role of $K_{n/k}$ in the following definition.

Definition 3.5. *The circular chromatic number $\chi_c(G)$ of a graph G can be defined as*

$$\chi_c(G) = \min\left\{\frac{p}{q} : p \leq |V(G)|, G \to K_{p/q}\right\}.$$

For detailed accounts on the circular chromatic number see the survey articles [Zhu01; Zhu06] or Section 6.1 of the book [HN04].

Some important properties of the graphs $K_{n/k}$ are that they are vertex-transitive, that $K_{n/k}$ is homomorphically equivalent to $K_{n'/k'}$ whenever $\frac{n}{k} = \frac{n'}{k'}$ and that $\chi(K_{n/k}) = \lceil \frac{n}{k} \rceil$ (for these and further properties, see [HN04]). Note that the just stated homomorph equivalence cannot be an isomorphism if $n \neq n'$ since then $|V(K_{n/k})| = n \neq n' = |V(K_{n'/k'})|$. This is a crucial difference between the graphs $K_{n/k}$ and $Q(n,k)$ and shows that the condition $\gcd(n,k) = 1$ cannot be dropped in the following statement from which Theorem 3.7 already easily follows.

Proposition 3.8. *The graph $Q(n,k)$ is isomorphic with the circular complete graph $K_{n/k}$ whenever $\gcd(n,k) = 1$.*

Proof of Theorem 3.7 from Proposition 3.8. It is known that the fractional chromatic number of $K_{n/k}$ is n/k since it is vertex transitive and has n vertices, while its independence number is k (cf. [HN04]). This already implies the first statement of Theorem 3.7. It is also known that removing any vertex x from $K_{n/k}$, the remaining graph $K_{n/k} - \{x\}$ is homomorphically equivalent to $K_{a/b}$, where a and b are the unique solution for the equation $nb - ka = 1$, see Lemma 6.6 in [HN04], where a retract of $K_{n/k} - \{x\}$ which is isomorphic to $K_{a/b}$ is shown. This implies the second and third statement of Theorem 3.7. □

Example 3. Let $n = 8, k = 3$. Figure 3.5 (see it at the end of Section 3.2) illustrates the vertices of $Q(8,3)$ and its isomorphism with $K_{8/3}$. The values of a and b as defined in Theorem 3.7 will be $a = 5, b = 2$. Deleting, say vertex $X_0 = \{1,3,6\}$ (cf. Figure 3.5 for the labeling of the vertices as X_i's) the remaining graph admits a homomorphism to its subgraph induced by the vertices $X_2 = \{3,5,8\}$, $X_3 = \{1,4,6\}$, $X_4 = \{2,5,7\}$, $X_5 = \{3,6,8\}$, $X_6 = \{1,4,7\}$ which is isomorphic to $Q(5,2) \cong K_{5/2} \cong C_5$ having fractional chromatic number $5/2$.

Thus our main task is to prove Proposition 3.8. This is done in the next section.

3.2 $Q(n,k)$ and $K_{n/k}$

Our argument will need the following alternative characterization of well-spread k-subsets.

Lemma 3.9. *Let $U \subseteq V(C_n)$ be fixed and let $A, B \subseteq V(C_n)$ be any two sets inducing a path in the graph C_n both starting and ending with vertices of C_n that belong to U. The subset $U \subseteq V(C_n)$ is well-spread if and only if for any two such sets A, B that also satisfy $|A \cap U| = |B \cap U|$ we have*

$$||A| - |B|| \leq 1.$$

Proof. Assume to the contrary that for two sets A, B as in the statement $||A| - |B|| \geq 2$ and w.l.o.g. assume that $|A| - 2 \geq |B|$. Then, we can modify the subset A by removing its two extremal (that is starting and ending) vertices and $|A| - |B| - 2$ more vertices from one end. This way we obtain a path A' for which $|A'| = |B|$ but $||A' \cap U| - |B \cap U|| \geq 2$ which means that U is not well-spread by Definition 3.3.

For the other direction suppose that U is not well-spread. Then there exist $A, B \subseteq V(C_n)$ both inducing a path in C_n for which $|A| = |B|$ but $||A \cap U| - |B \cap U|| \geq 2$. W.l.o.g. assume, that $|A \cap U| \geq |B \cap U| + 2$. We may assume that A induces a path in C_n that both starts and ends with elements of U because otherwise we can make both A and B shorter so that $|A \cap U|$ does not change while $|B \cap U|$ may only become smaller, so the relations $|A \cap U| \geq |B \cap U| + 2$ and $|A| = |B|$ remain valid. Now extend B at both of its ends until it will contain a new element of U at both ends, that is we obtain a B' which induces a path of C_n that both starts and ends with elements of U and intersects U in $|B \cap U| + 2$ elements. If this number is still less than $s := |A \cap U|$ then extend B' further (on one end) to make it a similar path containing exactly s elements of U. Since in the first step we extended B at both ends we certainly have $|B'| \geq |A| + 2$, so A and B' are two sets satisfying the conditions in the statement for which $||A| - |B'|| \leq 1$ does not hold. This completes the proof. □

Example 4. Let $n = 11$ and $U = \{1, 4, 8\}$ which is easy to check to be well-spread according to Definition 3.3. Also, if $A = \{1, 2, 3, 4\}$ and $B = \{4, 5, 6, 7, 8\}$, then they satisfy the conditions in Lemma 3.9 and also satisfy $||A| - |B|| \leq 1$. On the other hand, $U' = \{1, 4, 9\}$ is not well-spread as we already have seen in Example 1 as its intersection with the 4-element sets $\{1, 2, 3, 4\}$ and $\{5, 6, 7, 8\}$ differs by 2. Accordingly, the sets $A = \{1, 2, 3, 4\}$ and $B' := \{4, 5, 6, 7, 8, 9\}$ satisfy $|A \cap U'| = |B' \cap U'|$, they both start and end with elements of U, but do not satisfy $||A| - |B'|| \leq 1$. (See Figure 3.2 for an illustration.)

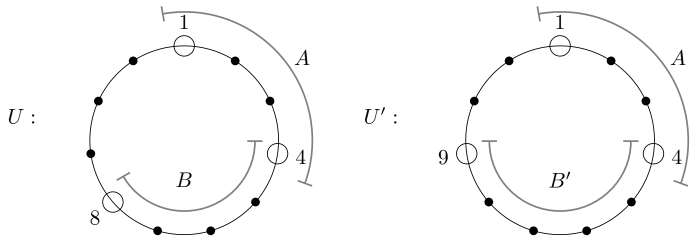

Figure 3.2: This figure shows sets U and U' along with the sets A, B and B' described in Example 4.

Note that a set $U \subseteq V(C_n)$ need not be 2-separated for being well-spread. Moreover, the following observations hold for U and $\overline{U} := V(C_n) \setminus U$.

Observation 3.10. *U is well-spread if and only if $\overline{U}$ is well-spread.*

Proof. If $A, B \subseteq V(C_n)$, $|A| = |B|$ and both of them induce a path, then

$$||A \cap \overline{U}| - |B \cap \overline{U}|| = ||(|A| - |A \cap U|)| - |(|B| - |B \cap U|)|| = ||B \cap U| - |A \cap U||,$$

so $||A \cap \overline{U}| - |B \cap \overline{U}|| \leq 1$ is equivalent to $||A \cap U| - |B \cap U|| \leq 1$.

Observation 3.11. *If U is well-spread and* $\gcd(n, |U|) = 1$ *(and $n > 2$) then exactly one of U and $\overline{U}$ is a 2-separated set.*

Proof. Assume U is well-spread, then so is $\overline{U}$ as well by Observation 3.10. If $|U| = |\overline{U}| = n/2$ (in which case both $|U|$ and $|\overline{U}|$ are 2-separated, alternatingly containing the vertices of C_n), then $\gcd(n, |U|) = n/2 \neq 1$. So w.l.o.g. U has less than $n/2$ elements. Then $\overline{U}$ must contain two adjacent vertices of the cycle C_n, say $\overline{u}_1$ and $\overline{u}_2$. If U would also contain two adjacent vertices of C_n, say u_1 and u_2 then taking $A = \{u_1, u_2\}$ and $B = \{\overline{u}_1, \overline{u}_2\}$ we would have two sets with $|A| = |B| = 2$ for which $|A \cap U| = 2$ and $|B \cap U| = 0$, so U cannot be well-spread, a contradiction. □

In what follows we denote by f_i the i-fold clockwise rotation of the defining cycle. In particular, for $j \in V(C_n)$ we let $f_i(j) = i + j$, where addition is meant modulo n and 0 is represented by n. For a set $X = \{x_1, \ldots, x_h\} \subseteq V(C_n)$ $f_i(X) = \{f_i(x_1), \ldots, f_i(x_h)\}$.

Lemma 3.12. *Let $U, W \subseteq V(C_n)$ be two well-spread sets of the same size k. Then there is a bijection between the elements of U and W that is given by a rotation of the cycle C_n. The graph $Q(n, k)$ is vertex-transitive for any n and k and if $\gcd(n, k) = 1$ then $|V(Q(n, k))| = n$.*

Proof. Let U be a well-spread set of size k on the cycle C_n. If $gcd(n, k) \neq 1$ then we have already seen in the proof of Proposition 3.6 that every well-spread set maps to itself when we rotate the defining cycle C_n by $\frac{n}{gcd(n,k)}$ elements. In particular, there are only $|V(Q(\frac{n}{gcd(n,k)}, \frac{k}{gcd(n,k)}))|$ distinct well-spread subsets of $V(C_n)$ and they behave exactly as the well-spread sets we obtain on $V(C_{\frac{n}{gcd(n,k)}})$ when identifying every element of $V(C_n)$ that are $\frac{n}{gcd(n,k)}$ distance apart. This means that we may assume, that we do from now on, that $gcd(n, k) = 1$.

Call $x, y \in U$ U*-consecutive* in U if one of the arcs between them does not contain any other $z \in U$. By Lemma 3.9 if $x, y \in U$ are U-consecutive, then x and y should be $q_0 := \lfloor \frac{n}{k} \rfloor$ or $\lfloor \frac{n}{k} \rfloor + 1$ distance apart on $V(C_n)$, that is, they are separated by $q_0 - 1$ or q_0 other elements of the cycle. If $n = q_0 k + r_1$ then we have exactly r_1 U-consecutive pairs whose distance is $q_0 + 1$ and $k - r_1$ U-consecutive pairs that have distance q_0. Let $U = \{x_1, x_2, \ldots, x_k\}$, where the indices are chosen so that each x_{i+1} is the U-consecutive element of x_i as we go along the cycle C_n in the clockwise direction. Let $(a_1, a_2, , \ldots, a_k)$ be the sequence of numbers that denote the distances of U-consecutive elements, that is, a_i is the distance of x_{i+1} from x_i (in the clockwise direction) for each $i \in \{1, \ldots, k - 1\}$ and a_k is the (also clockwise) distance of x_1 from x_k. We identify two sequences $(a_1, \ldots, a_k)$ and $(b_1, \ldots, b_k)$ if one can be obtained from the other by cyclically permuting its elements, that is, if $(a_1, a_2, \ldots, a_k) = (b_{i+1}, b_{i+2} \ldots, b_k, b_1, \ldots, b_i)$ for some i and call it the *placement pattern* of U. In case U has only 1 element, we consider its placement pattern to be (n). Notice that if two k-element subsets U and W of $V(C_n)$ have the same placement pattern then they must be rotations of each other, so to prove the first statement of the Lemma it is enough to prove that any two well-spread k-subsets of $V(C_n)$ should have the same placement pattern. This is what we do next.

Remove $q_0 - 1$ vertices of C_n from the arcs between every pair of U-consecutive elements. This way we obtain a shorter cycle $C_{n-(q_0-1)k}$ on which U is still well-spread and $\overline{U} = V(C_{n-(q_0-1)k}) \setminus U$ is also well-spread by Observation 3.10. On this shorter cycle U is not 2-separated any more (since there were U-consecutive elements in U separated by exactly $q_0 - 1$ other elements that are now removed), so $\overline{U}$ is a 2-separated set by Observation 3.11. Using the notation $n_1 := n - (q_0 - 1)k = k + r_1$ we have $|\overline{U}| = n_1 - |U| = r_1$ and the $\overline{U}$-consecutive elements of $\overline{U}$ are separated by $q_1 := \left\lfloor \frac{k+r_1}{r_1} \right\rfloor$ or by $q_1 - 1$ elements of U. Now performing the previous removal process with C_{n_1} in the place of C_n and its r_1-element subset $U_1 := \overline{U}$ in place of U is essentially performing a second step of the Euclidean algorithm with $k + r_1$ and r_1 (instead of k and r_1 but this is not an essential difference since $gcd(k + r_1, r_1) = gcd(k, r_1) = gcd(n, k) = 1$). This means that now we remove $q_1 - 1$ elements of the current cycle between any two U_1-consecutive elements of U_1. That results in a cycle C_{n_2} of length $n_2 := n_1 - |U_1|(q_1 - 1) = n_1 - (q_1 - 1)r_1$ and we have U_1 as its subset that is not 2-separated any more (since it did have U_1-consecutive elements separated by exactly $q_1 - 1$ other elements). Thus by Observation 3.11 $U_2 := V(C_{n_2}) \setminus U_1$ is 2-separated. It has size $n_2 - |U_1| = n_1 - q_1 r_1 =: r_2$, that is, $n_2 = r_1 + r_2$ and we clearly have $gcd(r_1 + r_2, r_2) = gcd(r_1, r_2) = 1$. We can go on iterating this process. Let U_i be a 2-separated well-spread set on C_{n_i} with U_i-consecutive elements having distance q_i and q_i+1 on $V(C_{n_i})$. We remove q_i-1 elements not belonging to U_i between any two U_i-consecutive elements of U_i. This way we obtain the cycle $C_{n_{i+1}}$ with $n_{i+1}=n_i-|U_i|(q_i-1)$, and assuming $gcd(n_i,|U_i|)=1$ we will have $gcd(n_{i+1},|U_i|)=1$. Define $U_{i+1}:=V(C_{n_{i+1}})\setminus U_i$. Then U_{i+1} is 2-separated on $C_{n_{i+1}}$ and $gcd(n_{i+1},|U_{i+1}|)=1$ also holds, so we can continue until we will arrive to a situation where we have a cycle C_m for some $m=n_h$ and our current 2-separated set U_h will have only $\gcd(n,k)=1$ element. This process is illutrated on Figure 3.3 for $n=14,k=5$.

We can place the remaining 1-element set on our final cycle C_m into m different points, but the m different sets we can get this way are obviously just rotations of each other. In other words, their placement pattern is the same for every possible choice. Now observe that our removal process was completely deterministic, thus so is also its reversed process. This means that if at every step we make a note of how many elements were removed between two U_i-consecutive elements of the current U_i on C_{n_i} (these are simply the numbers q_i-1) to obtain the shorter cycle $C_{n_{i+1}}$ and $U_{i+1}=[n_{i+1}]\setminus U_i$, then getting back U_i on C_{n_i} from U_{i+1} is also determined. It simply means that we should put back the appropriate number of removed elements between any pair of U_i-consecutive elements of U_i on $C_{n_{i+1}}$. (This is also illustrated on Figure 3.3 if we follow the three pictures from right to left.)

The foregoing implies that if the placement pattern of U_{i+1} is uniquely determined, then so is the placement pattern of U_i. As we have seen the placement pattern of our final 1-element set U_h is uniquely determined, thus the placement pattern of U itself on the original C_n is also uniquely determined. This proves the first statement in the Lemma and implies $|V(Q(n,k))|\leq n$.

We still have to prove the two statements in the last sentence of the Lemma, that is that $\gcd(n,k)=1$ also implies $|V(Q(n,k))|=n$ from which vertex-transitivity follows also for the other cases via Proposition 3.6.

If U is well-spread, then so is $f_i(U)$, so the latter is also a vertex of $Q(n,k)$. Let t be the smallest positive integer i for which $f_i(U)=U$ for some vertex $U\in V(Q(n,k))$. Since we have $t\leq n$, it is enough to prove that if $gcd(n,k)=1$, then t cannot be smaller than n. Thus we assume $gcd(n,k)=1$ and first we show that t is a divisor of n. Indeed, let $n=\ell t+r$, where $r<t$. Then for some vertex U we have $f_{\ell t}(U)=U$ and $f_r(U)=f_r(f_{\ell t}(U))=f_n(U)=U$ implying $r=0$ by the minimality of t. Thus t divides n.

Now we show that $\ell=\frac{n}{t}$ also divides k. Assume $f_t(u_1)=u_s$. Then we must have $f_t(u_i)=u_{i+s-1}$ for every $i\in\{1,\dots,k\}$ (addition in the indices intended modulo k with k identified to 0) otherwise we could not have $f_t(U)=U$. Therefore $u_1=f_n(u_1)=f_{\ell t}(u_1)=u_{1+\ell(s-1)}$ showing $\ell(s-1)=k$ meaning that ℓ divides k. (Here we used that f_n "winds around" C_n exactly once.) Since ℓ also divides n, it should be 1, therefore $t=n$. ☐

The following Corollary is essentially implicit already in the proof of the previous Lemma, yet we state it separately for further reference.

Corollary 3.13. *If* $\gcd(n,k)=1$ *then for every* $X,Y\in V(Q(n,k))$ *there is a unique rotation of* C_n *that maps* X *to* Y.

Proof. We have $|V(Q(n,k))|=n$, where the vertices can only be different by some rotation and we have exactly n possible rotations for each vertex. ☐

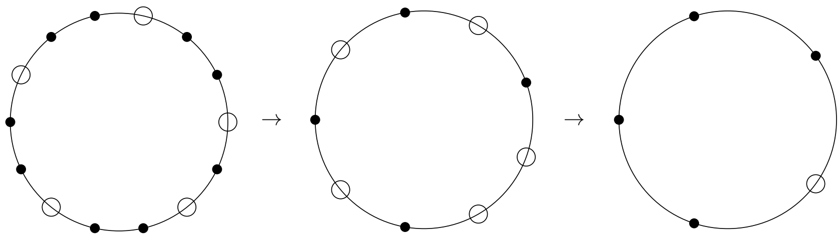

Figure 3.3: The process in the proof of Lemma 3.12 performed for $n = 14, k = 5$. In the first picture we see the defining cycle C_{14} where the elements of a well-spread 5-subset U are illustrated by empty circles. The second picture shows the situation after removing one of the one or two elements we have between any pair of U-consecutive elements of U. This results in the cycle C_9 of the second picture where the empty circles still denote the elements of U, while the elements of $U_1 = [9] \setminus U$ are shown by the remaining 4 black dots. Then we remove one element of the original set U from between any pair of U_1-consecutive elements of U_1 to obtain the third picture with $C_{n_2} = C_5$ and the 1-element set U_2.

Lemma 3.14. *Let* $\gcd(n, k) = 1$ *and* $X, Y \in V(Q(n, k))$ *be such that* $XY \notin E(Q(n, k))$*, that is,* $X \cap Y \neq \emptyset$*. Let* $f \colon V(C_n) \to V(C_n)$ *be the unique clockwise rotation moving* X *to* Y *and let* i *be an element of* $X \cap Y$*. Then the number of elements of* Y *on the arc of* C_n *between* i *and* $f(i)$ *(moving from* i *to* $f(i)$ *in the clockwise direction) is independent of the choice of* $i \in X \cap Y$*.*

Proof. Let $i, j \in X \cap Y$ and let A and B be the arcs of C_n between i and $f(i)$ and between j and $f(j)$, respectively ($i, f(i)$ and j, $f(j)$ included). We obviously have $|A| = |B|$. Assume to the contrary of the statement that w.l.o.g. $|A \cap Y| + 1 \leq |B \cap Y|$. Add the minimal number of consecutive vertices to A from C_n in the same (clockwise) direction to get A', such that $|A' \cap Y| = |B \cap Y|$. As $Y \in V(SG(n, k))$, we have that Y is a 2-separated set. So, since A ended with $f(i) \in Y$, $|A'| \geq |A| + 2 = |B| + 2$. Since A' and B are arcs starting and ending with elements of Y and also containing the same number of elements of Y, this gives a contradiction by Lemma 3.9 with the well-spreadness of Y. □

Definition 3.6. *Under the conditions of Lemma 3.14 we call vertex* $Y \in V(Q(n, k))$ *a right* j*-jumper of vertex* $X \in V(Q(n, k))$ *if the number of elements of* Y *on the arc of* C_n *strictly between* i *and* $f(i)$ *for some* $i \in X \cap Y$ *(moving from* i *to* $f(i)$ *in the clockwise direction) is* $j - 1$*.*

Note that by Lemma 3.14 the previous definition is meaningful as it does not depend on the choice of $i \in X \cap Y$.

Example 5. Let $n = 24, k = 7$. Then $X = \{1, 4, 8, 11, 15, 18, 22\}$ and $Y = \{1, 5, 8, 12, 15, 18, 22\}$ are two intersecting well-spread subsets of [24], therefore two non-adjacent vertices of $Q(24, 7)$. The unique rotation that moves X to Y is f_{14}, the 14-fold clockwise rotation of the defining cycle. In particular, it maps element 1 to 15 and Y has three other elements on the arc between these two, so Y is a 4-jumper of X. There is one more 4-jumper of X, namely $Z = \{4, 7, 11, 14, 17, 21, 24\}$. We have $Z = f_{13}(X)$, in particular, element 4 is moved to 17 and there are three other elements of Z on the clockwise arc connecting 4 to 17. (See Figure 3.4 for an illustration.)

Corollary 3.15. *If* $gcd(n, k) = 1$ *then the degree of every vertex in* $Q(n, k)$ *is* $n - 2k + 1$*.*

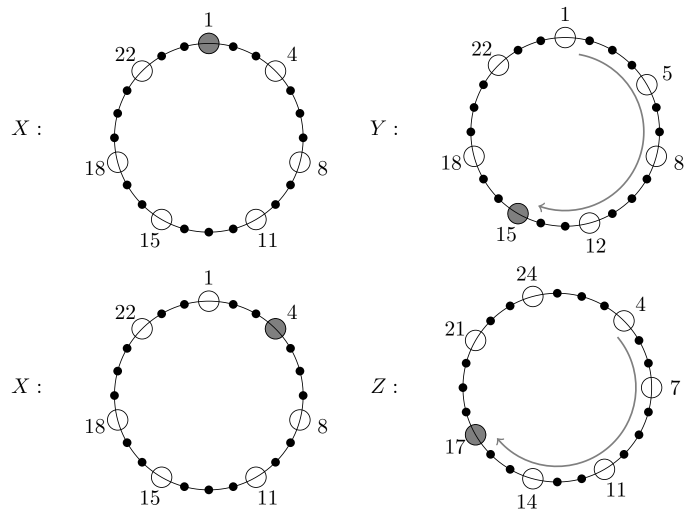

Figure 3.4: This figure shows the well-spread set $X = \{1, 4, 8, 11, 15, 18, 22\}$ in the defining cycle C_{24} together with its two 4-jumpers Y and Z as discussed in Example 5. The elements $1 \in X$ and $15 \in Y$ are darkened on the first pair of pictures to emphasize that 1 will be mapped to 15 by the unique rotation moving X to Y. Similarly, $4 \in X$ and $17 \in Z$ are darkened in the second pair of pictures, because the unique rotation moving X to Z maps 4 to 17.

Proof. We show that each vertex is non-adjacent to exactly $2k - 2$ vertices different from itself from which the statement follows. By vertex-transitivity it is enough to show this to an arbitrary vertex $X \in V(Q(n, k))$.

If Y is another vertex for which $\{X, Y\} \notin E(Q(n, k))$, then there is some $u \in X \cap Y$, so Y is a j-jumper of X for some j. Since any two vertices of $Q(n, k)$ are rotations of each other, we know that $Y = f_i(X)$ for some i. We claim that if $gcd(n, k) = 1$ and j is fixed then there are exactly two distinct values i can take in the set $\{1, 2, \ldots, n-1\}$. Indeed, by Lemma 3.9 the length of the clockwise arc from u to $f_i(u)$ can take only two different values (differing by 1) and if $gcd(n, k) = 1$ then two such distinct values exist indeed. (Otherwise for some $0 < i < n$ and $U \in V(Q(n, k))$ we would have $f_i(u) \in U$ for every $u \in U$ implying $f_i(U) = U$. But we have already seen in the proof of Lemma 3.12 that this is impossible if $gcd(n, k) = 1$.) Lemma 3.14 implies that j will not depend on the choice of $u \in X \cap Y$ which also implies that we cannot get the same f_i for two different j's. This means that the number of non-neighbors of an $X \in V(Q(n, k))$ different from itself is exactly twice the number of possible values of j, that is $2(k - 1)$ as claimed. □

Now we show that $SG(n, k)$ itself is critical for the fractional chromatic number only in the cases already mentioned in the Introduction.

Corollary 3.16. *We have $Q(n, k) = SG(n, k)$ if and only if $k = 1, n = 2k$, or $n = 2k + 1$. In particular, $SG(n, k)$ is vertex-critical for the fractional chromatic number in exactly these cases.*

Proof. We know from Schrijver's theorem, that $\chi(SG(n, k)) = n - 2k + 2$. By Corollary 3.15 this is exactly one more than the (maximum) degree of $Q(n, k)$. Thus, since $SG(n, k)$ is connected, Brooks' theorem implies that in case $SG(n, k) = Q(n, k)$ we must have that $SG(n, k)$ is a complete graph or an odd cycle. This happens only in the cases listed in the statement and in those cases we indeed have $Q(n, k) = SG(n, k)$. □

Now we have all the necessary lemmas to prove that our $Q(n, k)$ graph is isomorphic to the circular complete graph $K_{n/k}$ whenever $\gcd(n, k) = 1$.

Proof of Proposition 3.8. As $|V(Q(n, k))| = |V(K_{n/k})| = n$ and in both graphs each vertex has degree $n - 2k + 1$ it is enough to show a bijection between the vertex sets that maps non-adjacent vertices to non-adjacent vertices.

Fix a vertex $X_0 \in V(Q(n, k))$ and let for every $i \in \{1, \ldots, n-1\}$ $X_i = f_i(X_0)$. Let $\varphi : V(Q(n, k)) \to V(K_{n/k})$ be defined by

$$\varphi : X_u \mapsto uk \pmod{n}$$

This is a one-to-one function since $\gcd(n, k) = 1$. Now look at $X_u \neq X_v$ arbitrary non-adjacent vertices in $Q(n, k)$. Let $\ell := |u - v|$ be their distance measured in rotations. If they are not adjacent, then one of them must be a right j-jumper of the other for some $j \in \{1, \ldots, k-1\}$. Since all j-jumpers in $Q(n, k)$ have to be either ℓ or $\ell - 1$ rotations apart, or they all have to be ℓ or $\ell + 1$ rotations apart one of the equations $(k - x)\ell + x(\ell + 1) = jn$ or $(k - x)\ell + x(\ell - 1) = jn$ has an integral solution with $0 < x < k$. (This is because if we consider the clockwise arc from each $z \in X_u$ to the $z' \in X_u$ for which this arc contains j elements of X_u including z' but excluding z, then we cover C_n exactly j times.) That means that $k\ell$ must belong to the same congruent class modulo n as x or $-x$, meaning that in the image the vertices $uk \pmod{n}$ and $vk \pmod{n}$, whose distance is $|u - v|k = \ell k$, should be either less than k, or more than $n - k$ apart, i.e., they are indeed non-adjacent in $K_{n/k}$. □

Example 6. Let $n = 8, k = 3$. The vertices of $Q(n, k)$ are the sets $\{1, 3, 6\}$, $\{2, 4, 7\}$, $\{3, 5, 8\}$, $\{1, 4, 6\}$, $\{2, 5, 7\}$, $\{3, 6, 8\}$, $\{1, 4, 7\}$, $\{2, 5, 8\}$. Choosing X_0 to be $\{1, 3, 6\}$ the mapping given in the proof of Proposition 3.8 above sends the above vertices into vertices $0, 3, 6, 1, 4, 7, 2, 5$ of $K_{8/3}$, respectively. Vertices belonging to disjoint sets in $V(Q(8, 3))$ are mapped to adjacent vertices of $K_{8/3}$. Since both graphs are 3-uniform, this shows that they are isomorphic. (For an illustration see Figure 3.5.)

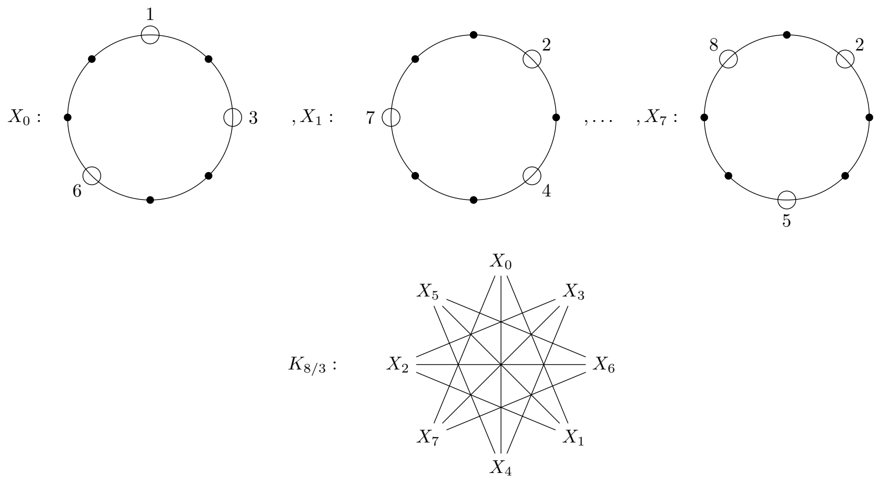

Figure 3.5: This figure shows how the sets X_i are mapped to the vertices of $K_{8/3}$ as described in Example 6.

With the above we have completed the proof of Theorem 3.7. The following is an easy consequence of Proposition 3.8.

Corollary 3.17. *For all $n \geq 2k$ we have*

$$\chi(Q(n,k)) = \left\lceil \frac{n}{k} \right\rceil.$$

Proof. From Proposition 3.8 and the properties of the circular complete graphs it follows that

$$\chi(Q(n,k)) = \chi(K_{n'/k'}) = \left\lceil \frac{n'}{k'} \right\rceil = \left\lceil \frac{n}{k} \right\rceil,$$

where $n' = \frac{n}{\gcd(n,k)}, k' = \frac{k}{\gcd(n,k)}$. □

Note that Corollary 3.17 gives a second proof for Corollary 3.16 as $Q(n,k) = SG(n,k)$ implies the equality of their chromatic number and $n - 2k + 2 = \left\lceil \frac{n}{k} \right\rceil$ also implies that we must have $n = 2k, n = 2k+1$ or $k = 1$.

3.3 Critical edges

Here we are going to prove a strengthening of the second statement of Theorem 3.7.

Definition 3.7. *An edge $\{i,j\} \in E(K_{n/k})$ of the circular complete graph $K_{n/k}$ is called a* shortest edge *if $|i-j| = k$ or $|i-j| = n-k$.*

We remark that in terms of $Q(n,k)$ a shortest edge of $K_{n/k}$ (when $gcd(n,k) = 1$) belongs to one that connects a vertex $X \in V(Q(n,k))$ to a vertex that can be obtained by one rotation along the defining cycle. This can be read out from the proof of Proposition 3.8.

Theorem 3.18. *An edge of $K_{n/k}$ is critical for the fractional chromatic number if and only if $gcd(n,k) = 1$ and e is a shortest-edge. The same statement holds also if we exchange the word "fractional" to "circular" in the previous sentence. More precisely, if $\gcd(n,k) = 1$, $e \in E(K_{n/k})$ and a, b are defined as in Theorem 3.7 then*

$$\chi_f(K_{n/k} \setminus \{e\}) = \chi_c(K_{n/k} \setminus \{e\}) = \begin{cases} \frac{a}{b} & \textit{if } e \textit{ is a shortest edge} \\ \frac{n}{k} & \textit{otherwise.} \end{cases}$$

Proof. For both parameters $\chi_f(K_{n/k}) = \chi_c(K_{n/k}) = \frac{n}{k}$ is a trivial upper bound and $\chi_f(K_{a/b}) = \chi_c(K_{a/b}) = \frac{a}{b}$ is a lower bound, because $K_{a/b}$ is a subgraph of $K_{n/k}$ (see Lemma 6.6 in [HN04]). It is well-known that $\chi_f(G) \leq \chi_c(G)$ holds for any graph G (cf. [HN04]), so it is enough to prove that if $gcd(n,k) > 1$ or e is not a shortest edge then $\chi_f(K_{n/k} \setminus \{e\}) \geq \frac{n}{k}$, while if $gcd(n,k) = 1$ and e is a shortest edge then $\chi_c(K_{n/k} \setminus \{e\}) \leq \frac{a}{b}$.

If $gcd(n,k) > 1$ then $K_{n/k}$ is homomorphically equivalent to $K_{n'/k'}$ for $n' = \frac{n}{gcd(n,k)}, k' = \frac{k}{gcd(n,k)}$ and since $|V(K_{n'/k'})| = n' < n = |V(K_{n/k})|$ in this case, $K_{n/k}$ cannot have any critical edges. Thus from now on we assume $gcd(n,k) = 1$.

It is well-known that the independence number $\alpha(K_{n/k}) = k$ (see this as a Claim within the proof of Theorem 6.3 in [HN04]). One can also easily show that if $n > 2k$ (and for $n \geq 2k, gcd(n,k) = 1$ this is always the case) the only independent sets of $K_{n/k}$ with size exactly k consist of k cyclically consecutive elements. That is, a largest independent set must have the form $\{i, i+1, \ldots, i+k-1\}$, where addition is intended modulo n. Indeed, if S is an independent set in $K_{n/k}$ having size k and $j \in S$ then $S \subseteq \{j-k+1, j-k+2, \ldots, j, j+1, \ldots, j+k-1\}$ and since $\forall h \in \{1, \ldots, k-1\}$: $\{j-h, j-h+k\} \in E(K_{n/k})$, $|S| = k$ implies that exactly one of the vertices $j-h$ and $j-h+k$ must belong to S for every $h \in \{1, \ldots, k-1\}$. If S was not a set of cyclically consecutive vertices, then we must have a $h \in \{2, \ldots, k-1\}$ for which $j-h \in S$ and $j-h+1 \notin S$. Then $|S| = k, j-h+1 \notin S$ implies $j-h+k+1 \in S$ by the above. However $j-h$ and $j-h+k+1$ are adjacent in $K_{n/k}$ (whenever $n > 2k$) contradicting that S is an independent set.

Since $\chi_f(G) \geq \frac{|V(G)|}{\alpha(G)}$ and for $K_{n/k}$ we have equality because $K_{n/k}$ is vertex-transitive, $\chi_f(K_{n/k} \setminus \{e\}) < \chi_f(K_{n/k})$ is possible only if $\alpha(K_{n/k} \setminus \{e\}) > \alpha(K_{n/k}) = k$. This requires that $e = \{x, y\}$ for two vertices x, y for which there exists a set $U \subseteq V(K_{n/k})$ of size $|U| = k-1$ for which $x, y \notin U$ and both $U \cup \{x\}$ and $U \cup \{y\}$ are k-element independent sets of $K_{n/k}$. Since k-element independent sets are formed by cyclically consecutive elements, this means that w.l.o.g. we must have $U = \{x+1, \ldots, x+k-1\}$ and $y = x+k$, in which case $\{x, y\}$ is a shortest edge.

What is left to prove is that if $gcd(n, k) = 1$ and $e = \{x, x+k\}$ is a shortest edge then we have $\chi_c(K_{n/k} \setminus \{e\}) \leq \frac{a}{b}$. To show this we give a homomorphism from $K_{n/k} \setminus \{e\}$ to $K_{n/k} \setminus \{x\}$. By Lemma 6.6 in [HN04] we know that a rectract of $K_{n/k} \setminus \{x\}$ is isomorphic to $K_{a/b}$, so by transitivity of the existence of homomorphisms we get that $K_{n/k} \setminus \{e\} \to K_{a/b}$. Let $f : V(K_{n/k} \setminus \{e\}) \to V(K_{n/k} \setminus \{x\})$ be the function $f(x) = x+1$ and $f(i) = i \, \forall i \in [n] \setminus \{x\}$. Since the neighborhood of x in $V(K_{n/k} \setminus \{e\})$ is $\{x+k+1, \ldots, x+n-k\}$ which is a subset of $\{x+k+1, \ldots, x+n-k+1\}$, the neighborhood of $x+1$ in $V(K_{n/k} \setminus \{e\})$ and also in $V(K_{n/k} \setminus \{x\})$, f is indeed a homomorphism. □

Chapter 4

Graph Codes

Celebrated problems of extremal combinatorics may get an exciting new flavour when the presence of some special structure is imposed in the condition. A prominent example is the famous Simonovits–Sós conjecture [SS76] proven by Ellis, Filmus and Friedgut [EFF12], which determines the maximum possible cardinality of a family of graphs on n labeled vertices in which the intersection of any two members contains a triangle. (The result of [EFF12] shows, along with several far reaching generalizations, that the best is to take all graphs containing a given triangle, just as it was conjectured in [SS76]. This is clearly reminiscent of the Erdős–Ko–Rado theorem [EKR61].) As another example we can also mention the Ramsey type problem investigated in [KS95] that was also initiated by a question of Sós and can be considered as a graph version of the first unsolved case of the so-called perfect hashing problem. (For details we refer to [KS95]).

In this chapter we study several problems we arrive to if the basic code distance problem (how many binary sequences of a given length can be given at most if any two differ in at least a given number of coordinates) is modified so that we do not prescribe the minimum distance of any two codewords but require that they differ in some specific structure. In particular, just as in the Simonovits–Sós problem we seek the largest family of (not necessarily induced) subgraphs of a complete graph such that the symmetric difference of the edge sets of any two graphs in the family has some required property. We will consider properties like connectedness, Hamiltonicity, containment of a triangle and some more. Formally all these can be described by saying that the graph defined by the symmetric difference of the edge sets of any two of our graphs belongs to a prescribed family of graphs (namely those that are connected, contain a Hamiltonian cycle, or contain a triangle, etc.)

Let $\mathcal{F}$ be a fixed class of graphs. A graph family $\mathcal{G}$ on n labeled vertices is called $\mathcal{F}$-good if for any pair of distinct $G, G' \in \mathcal{G}$ the graph $G \oplus G'$ defined by

$$V(G \oplus G') = V(G) = V(G') = [n],$$

where $[n] = \{1, \dots, n\}$ and

$$E(G \oplus G') = \{e : e \in (E(G) \setminus E(G')) \cup (E(G') \setminus E(G))\}$$

belongs to $\mathcal{F}$.

Let $M_{\mathcal{F}}(n)$ denote the maximum possible size of an $\mathcal{F}$-good family on n vertices. We are interested in the value of $M_{\mathcal{F}}(n)$ for various classes $\mathcal{F}$. We will give exact answers or both lower and upper bounds in several cases.

We mention that codes where the codewords are described by graphs already appear in the literature. In [Ton02], for example, Tonchev looked at the usual code distance problem restricted to codes whose codewords are characteristic vectors of edge sets of graphs. Gray codes on graphs are also considered, see [Müt22], where the graphs representing the codewords should have some similarity properties if they are consecutive in a certain listing. Problems analogous to the present ones, though restricted to special classes of graphs, were also considered in [KMS12] and [CFK14]. A very interesting result along these lines is the one in [KS18], where the authors investigate the maximum number of pairwise triangle-different Hamiltonian paths — that is, paths whose union contains a triangle — motivated by a question of Körner, Messuti, and Simonyi. They showed that this number equals the number of balanced bipartitions of the ground set.

The chapter is organised as follows. In Section 4.1 we give a general upper bound that will turn out to be sharp in several of the cases we consider. In Section 4.2 we consider classes $\mathcal{F}$ defined by some global criterion as connectivity or 2-connectivity, Hamiltonicity or containing a full star, that is, a vertex of degree $n-1$. We determine $M_{\mathcal{F}}(n)$ for infinitely many values of n and for all n in the first and the last case. In most of the cases when we give sharp bounds it is via also solving the problem we call dual: we give the largest possible size of a graph family for which the symmetric difference of no two of its members satisfies the original requirement. The case of the full star is an exception in this sense, nevertheless we also solve the dual problem in that case for all even n by using a celebrated lemma of Shearer. In Section 4.3 we consider classes $\mathcal{F}$ defined by local conditions. This means that for certifying the condition it is enough to see just a special part of the graph pair at hand. A capacity-type asymptotic invariant is natural to define in these cases. It turns out that when the requirement is that the pairwise symmetric differences contain a certain subgraph then this asymptotic invariant depends only on the chromatic number of the graph to be contained. The final section contains a collection of open problems.

4.1 A general upper bound

To bound $M_{\mathcal{F}}(n)$ for various graph classes $\mathcal{F}$ it will often be useful to also consider the related problem of constructing large graph families in which no pair satisfies the condition prescribed by $\mathcal{F}$.

Definition 4.1. *For a class of graphs $\mathcal{F}$ let $D_{\mathcal{F}}(n)$ denote the maximum possible size of a graph family on n labeled vertices (that is, each member of the family has $[n] = \{1, \dots, n\}$ as vertex set), the symmetric difference of no two members of which belongs to $\mathcal{F}$. Determining $D_{\mathcal{F}}(n)$ will be referred to as the dual problem of determining $M_{\mathcal{F}}(n)$.*

Note that denoting by $\overline{\mathcal{F}}$ the class containing exactly those graphs that do not belong to $\mathcal{F}$ we actually have

$$D_{\mathcal{F}}(n) = M_{\overline{\mathcal{F}}}(n),$$

that is the requirement of having no symmetric difference in $\mathcal{F}$ is clearly the same as saying that all symmetric differences belong to the complementary family $\overline{\mathcal{F}}$. Nevertheless, we will use the $D_{\mathcal{F}}(n)$ notation to emphasize the dual nature of the problem in those cases.

Lemma 4.1. *For any graph class $\mathcal{F}$ we have*

$$M_{\mathcal{F}}(n) \cdot D_{\mathcal{F}}(n) \leq 2^{\binom{n}{2}}.$$

Proof. Let us define a graph $H_{\mathcal{F}}$ whose vertices are all the possible (simple) graphs on the vertex set $[n]$. Connect two such vertices if and only if the corresponding pair of graphs have their symmetric difference belonging to $\mathcal{F}$. Then by definition we have

$$M_{\mathcal{F}}(n) = \omega(H_{\mathcal{F}}) \text{ and } D_{\mathcal{F}}(n) = \alpha(H_{\mathcal{F}}),$$

where $\omega(H)$ and $\alpha(H)$ denote the clique number and the independence number of the graph H, respectively. Observe that $H_{\mathcal{F}}$ is vertex-transitive, (in fact it is a Cayley graph of the group $Z_2^{\binom{n}{2}}$). Indeed, if G_1 and G_2 are two graphs forming vertices of $H_{\mathcal{F}}$ then taking the symmetric difference of all n-vertex graphs forming vertices of $H_{\mathcal{F}}$ with the graph $G_1 \oplus G_2$ is an automorphism of $H_{\mathcal{F}}$ that maps G_1 to G_2. Since a vertex-transitive graph H always satisfies $\alpha(H)\omega(H) \leq |V(H)|$ (this can be seen by using that the fractional chromatic number $\chi_f(H)$ always satisfies $\omega(H) \leq \chi_f(H)$, while if H is a vertex-transitive graph we also have $\chi_f(H) = \frac{|V(H)|}{\alpha(H)}$, cf. [SU97]), the statement follows. □

The above lemma makes it possible to bound $M_{\mathcal{F}}(n)$ from above by bounding $D_{\mathcal{F}}(n)$ from below. In particular, whenever we construct two families of graphs $\mathcal{A}$ and $\mathcal{B}$ on $[n]$ such that $A, A' \in \mathcal{A}$ implies $A \oplus A' \in \mathcal{F}$ and $B, B' \in \mathcal{B}$ implies $B \oplus B' \notin \mathcal{F}$, while $|\mathcal{A}||\mathcal{B}| = 2^{\binom{n}{2}}$, then we know that $|\mathcal{A}|$ and $|\mathcal{B}|$ realize the optimal values $M_{\mathcal{F}}(n)$ and $D_{\mathcal{F}}(n)$ for such families. Below we will see several cases when this simple technique can indeed be used to obtain these optimal values. An exception to this phenomenon is also presented by Theorems 4.7 and 4.8.

Remark 1. It is worth noting that Lemma 4.1 can be proven in a different way, with no reference to the fractional chromatic number. Indeed, if $G_1, \ldots, G_k$ is an $\mathcal{F}$-good family, while $T_1, \ldots, T_m$ is a family satisfying the conditions of the dual problem, then all the symmetric differences of the form $G_i \oplus T_j$ are different, implying $km \leq 2^{\binom{n}{2}}$. This is true because if $G_i \oplus T_j$ and $G_r \oplus T_s$ would be the same for some $\{i, j\} \neq \{r, s\}$, then $(G_i \oplus T_j) \oplus (G_r \oplus T_s)$ would be the empty graph that could also be written (by commutativity and associativity of the symmetric difference) as $(G_i \oplus G_r) \oplus (T_j \oplus T_s)$. This would mean that $G_i \oplus G_r$ and $T_j \oplus T_s$ are two identical graphs. But if one of them is the empty graph (that is, $G_i = G_r$ or $T_j = T_s$), then the other cannot be empty and if both are nonempty, then one of them belongs to $\mathcal{F}$ while the other one does not, so this is impossible.

4.2 Global conditions

Connectivity

When we speak about the class of connected graphs in the following theorem, we mean graphs with a single connected component, and hence without isolated vertices.

Theorem 4.2. *Let $\mathcal{F}_c$ denote the class of connected graphs. Then*

$$M_{\mathcal{F}_c}(n) = 2^{n-1}.$$

Proof. First we give a very simple dual family $\mathcal{B}_c$. Let it consist of all graphs on $[n]$ in which the vertex labeled n is isolated. Clearly $|\mathcal{B}_c| = 2^{\binom{n-1}{2}}$ (that is the number of all graphs on $[n-1]$) and n is also isolated in the symmetric difference of any two of them, so no such symmetric difference is connected, This gives $D_{\mathcal{F}_c}(n) \geq 2^{\binom{n-1}{2}}$ and thus by Lemma 4.1 we have

$$M_{\mathcal{F}_c}(n) \leq 2^{\binom{n}{2} - \binom{n-1}{2}} = 2^{n-1}.$$

Now we show that this upper bound can be attained. Let the family $\mathcal{A}_c$ consist of all those graphs on $[n]$ that are the vertex-disjoint union of two complete graphs (where each vertex belongs to one of them) including the case when one of the two is on the empty set. Clearly, the number of these graphs is just half the number of subsets of $[n]$, that is exactly 2^{n-1}. All we have to show is that the symmetric difference of any two of these graphs is connected. Choose two arbitrary graphs G and G' from our family. Let G be the union of complete graphs on the complementary vertex sets K and L, while G' be the same on K' and L'. Let $A = K \cap L'$, $B = L' \cap L$, $C = L \cap K'$ and $D = K' \cap K$. It is possible that one, but only one of A, B, C, D is empty. The edges of $G \oplus G'$ are all the edges of the complete bipartite graph with partite classes $A \cup C$ and $B \cup D$, so it must be connected. □

With just a little more consideration one can also treat the case of 2-connectedness at least for even n.

Theorem 4.3. *Let $\mathcal{F}_{2c}$ denote the class of 2-connected graphs. Then if n is even, we have*

$$M_{\mathcal{F}_{2c}}(n) = 2^{n-2}.$$

Proof. The proof is a modification of the previous one, therefore we use the notation introduced there. The construction given there may result in symmetric differences that are not 2-connected only if $A \cup C$ or $B \cup D$ contains only one element. For even n this can be avoided if we consider only such graphs in our construction where the bipartition of $[n]$ defining the individual graphs has an even number of elements in both partite classes K and L. This proves the lower bound.

For the upper bound we consider all graphs in which the vertex n is either isolated or it has one fixed neighbor, say $n-1$. The symmetric difference of any two such graphs is not 2-connected, since n has at most one neighbor in it. The number of such graphs is just twice the number of graphs in which n is an isolated point, that is, $2^{\binom{n-1}{2}+1}$ proving the matching upper bound by Lemma 4.1. □

Remark 2. The upper bound proven in Theorem 4.3 clearly holds also for odd n but we have not found a matching construction in general. For $n = 3$ a triangle and the empty graph would do, still achieving the upper bound. But for larger odd n the best we could do is to take only those graphs from our construction for which in the corresponding bipartition the smaller partition class has an odd number of elements if $n \equiv 1 \pmod 4$ and it has an even number of elements if $n \equiv 3 \pmod 4$. The number of graphs obtained this way is $2^{n-2} - \binom{n-2}{(n-3)/2}$.

Remark 3. Changing the graphs to their complements in the proofs of Theorems 4.2 and 4.3 makes these graph families vector spaces over the 2-element field, while they still satisfy the conditions as the symmetric differences do not change by complementation (or by taking the symmetric difference of all elements with any fixed graph which is the complete graph in case of complementation).

It does not sound surprising that if we step further on to k-connectedness for $k > 2$ then the problem becomes rather more complicated. Nevertheless, if we insist on linear codes, that is graph families closed under the symmetric difference operation then for $k = 3$ we can still determine the largest possible cardinality for infinitely many values on n using Hamming codes.

Theorem 4.4. *Let $\mathcal{F}_{3c}$ be the class of 3-connected graphs and let $M^{(\ell)}_{\mathcal{F}_{3c}}(n)$ denote the size of a largest possible linear graph family on vertex set $[n]$ any two members of which have a 3-connected symmetric difference. If $n = 2^k - 1$ for some integer $k \geq 2$, then*

$$M^{(\ell)}_{\mathcal{F}_{3c}}(n) = 2^{n-k-1}.$$

Proof. First we prove that $D_{\mathcal{F}_{3c}}(n) \geq n2^{\binom{n-1}{2}}$ holds in general. Consider the family of all graphs on vertex set $[n]$ in which the degree of vertex n is at most 1. There are exactly $n2^{\binom{n-1}{2}}$ such graphs. The symmetric difference of any two of these graphs is at most 2-connected, since the vertex n has degree at most 2 in all these symmetric differences. This proves the claimed inequality and by Lemma 4.1 this implies $M_{\mathcal{F}_{3c}}(n) \leq 2^{n-1}/n$.

It is well-known that if a family of subsets of a finite set contains the empty set and is closed under the symmetric difference operation then the cardinality of this set must be a power of 2. This follows immediately from linear algebra and the fact that such a family forms a vector space over $GF(2)$, cf. e.g. Lemma 3.1 in Kozlov's book [Koz08] where a simple combinatorial proof of this fact is also presented. Since a linear graph family code on $[n]$ can be viewed as a collection of subsets of $E(K_n)$, this implies that $M^{(\ell)}_{\mathcal{F}_{3c}}(n)$ is a power of 2. Since we obviously have $M^{(\ell)}_{\mathcal{F}_{3c}}(n) \leq M_{\mathcal{F}_{3c}}(n)$, the upper bound proved above implies $M^{(\ell)}_{\mathcal{F}_{3c}}(n) \leq 2^d$ with $d = \lfloor \log_2 \frac{1}{n} 2^{n-1} \rfloor$ giving

$$M^{(\ell)}_{\mathcal{F}_{3c}}(n) \leq 2^{n-k-1}$$

for $n = 2^k - 1, k \geq 2$, which proves the required upper bound.

For the lower bound consider the Hamming code $\mathcal{C}_H(n)$ with length $n = 2^k - 1$ that exists for every $k \geq 2$. (For a nice quick account on Hamming codes see e.g. [Ber15].) It is a linear code with minimum distance 3 that consists of 2^{n-k} binary codewords having the property that if $\boldsymbol{c} = (c_1, \ldots, c_n)$ belongs to the code then so does also $\bar{\mathbf{c}} = (\bar{c}_1, \ldots, \bar{c}_n)$ where $\bar{c}_i = 1 - c_i$. For each codeword $\boldsymbol{c} \in \mathcal{C}_H(n)$ consider the bipartition of $[n]$ into the subsets $K_{\boldsymbol{c}}, L_{\boldsymbol{c}}$, where $K_{\boldsymbol{c}} = \{i : c_i = 0\}, L_{\boldsymbol{c}} = \{i : c_i = 1\}$ and the complete bipartite graph $G_{K_{\boldsymbol{c}}, L_{\boldsymbol{c}}}$ with partite classes $K_{\boldsymbol{c}}, L_{\boldsymbol{c}}$. Note that by the above mentioned property of Hamming codes we have $\boldsymbol{c} \in \mathcal{C}_H(n)$ if and only if $\bar{\boldsymbol{c}} \in \mathcal{C}_H(n)$ and thus since $G_{K_{\boldsymbol{c}}, L_{\boldsymbol{c}}} = G_{K_{\bar{\boldsymbol{c}}}, L_{\bar{\boldsymbol{c}}}}$, we get $\frac{1}{2}|\mathcal{C}_H(n)| = 2^{n-k-1}$ different complete bipartite graphs this way. All we have to prove is that the symmetric difference of any two of our graphs is 3-connected. This is equivalent to show that if $\mathbf{c}' \neq \mathbf{c}, \bar{\mathbf{c}}$, then the cardinality of both partite classes of $G_{K_{\boldsymbol{c}}, L_{\boldsymbol{c}}} \oplus G_{K_{\boldsymbol{c}'}, L_{\boldsymbol{c}'}}$, that is of $(K_{\boldsymbol{c}} \cap K_{\boldsymbol{c}'}) \cup (L_{\boldsymbol{c}} \cap L_{\boldsymbol{c}'})$ and $(K_{\boldsymbol{c}} \cap L_{\boldsymbol{c}'}) \cup (K_{\boldsymbol{c}'} \cap L_{\boldsymbol{c}})$ is at least 3. However, this immediately follows from the fact that the codeword $\mathbf{c}'$ must differ from both $\mathbf{c}$ and $\bar{\mathbf{c}}$ in at least 3 coordinates. This completes the proof. □

Hamiltonicity

A graph is connected if and only if it contains a spanning tree. Next we consider what happens if we require the containment of specific spanning trees: a path in this subsection and a star in the next one.

Theorem 4.5. *Let $\mathcal{F}_{Hp}$ denote the class of graphs containing a Hamiltonian path. Then for infinitely many values of n we have*

$$M_{\mathcal{F}_{Hp}}(n) = 2^{n-1}.$$

In particular, this holds whenever $n = p$ or $n = 2p - 1$ for some odd prime p.

To prove the above theorem we will refer to the following old conjecture that is known to be true in several special cases. To state it we need the notion of perfect 1-factorization. It means the partition of the edge set of a graph into perfect matchings such that the union of any two of them is a Hamiltonian cycle.

Perfect 1-factorization conjecture (P1FC)(Kotzig [Kot64]). *The complete graph K_n has a perfect 1-factorization for all even $n > 2$.*

This conjecture is still open in general, however it is known to hold in several special cases, for example, whenever $n = p + 1$ (Kotzig [Kot64]) or $n = 2p$ for some odd prime p (Anderson [And73] and Nakamura [Nak75], cf. also Kobayashi [Kob89]). For a recent survey, see Rosa [Ros19], according to which the smallest open case of the conjecture is $n = 64$.

Proof of Theorem 4.5. Since Hamiltonian paths are connected, it follows from the proof of Theorem 4.2 that 2^{n-1} is again an upper bound. Now we show that it is also a lower bound whenever the Perfect 1-factorization conjecture holds for $n + 1$. (Note that if the conjecture is true, then this means that our statement holds for all odd numbers at least 3, while for 1 it is void.)

Let n be an odd number for which K_{n+1} has a perfect 1-factorization $\mathcal{M}$ and v a fixed vertex of K_{n+1}. Note that deleting the edge incident to v from all matchings belonging to $\mathcal{M}$ we obtain n matchings of K_n such that the union of any two of them is a Hamiltonian path in $K_n := K_{n+1} \setminus \{v\}$. Now consider all those subgraphs of K_n that can be obtained as the union of an even number of these n matchings. Clearly, the symmetric difference of any two of them is also the union of at least two of these matchings and thus contains a Hamiltonian path. The number of graphs obtained this way is 2^{n-1}, matching the upper bound. □

The case of Hamiltonian cycles can be treated essentially the same way.

Theorem 4.6. *Let $\mathcal{F}_{Hc}$ denote the class of graphs containing a Hamiltonian cycle. For all even values of n for which the P1FC holds, we have*

$$M_{\mathcal{F}_{Hc}}(n) = 2^{n-2}.$$

In particular, this is the case if $n = p + 1$ or $n = 2p$ for some odd prime p.

Proof. Since Hamiltonian cycles are 2-connected, it follows from the proof of Theorem 4.3 that 2^{n-2} is again an upper bound.

Let n be an even number for which the P1FC holds and let $\mathcal{M}$ be a perfect 1-factorization of K_n. Note that $\mathcal{M}$ contains $n - 1$ matchings (indeed the edge-chromatic number of K_n for even n is $n - 1$). Now consider the 2^{n-2} graphs we can obtain as the union of an even number of matchings from $\mathcal{M}$. Clearly, the symmetric difference of any two of them contains a Hamiltonian cycle. □

Remark 4. Since Hamiltonian cycles are 2-connected graphs the proof of Theorem 4.6 obviously gives an alternative proof of Theorem 4.3 for those values of n for which the Perfect 1-factorization conjecture is known to hold. (The situation is similar for Theorems 4.5 versus 4.2.) On the other hand, the construction in the proof of Theorem 4.3 utterly fails to give a good lower bound for the value of $M_{\mathcal{F}_{Hc}}(n)$ investigated in Theorem 4.6. Indeed, the symmetric difference of two graphs in the construction given in the proof of Theorem 4.3 contains a Hamiltonian cycle if and only if the sets denoted by $A \cup C$ and $B \cup D$ in that proof both have cardinality $\frac{n}{2}$ and this happens exactly when the partition classes of the partitions (K, L) and (K', L') are orthogonal in the sense that representing these bipartitions by characteristic vectors consisting of $+1$ and -1 coordinates in the obvious way, we get a collection of vectors that are pairwise orthogonal. So their number cannot be more than just n and we can give exactly n such vectors if and only if an $n \times n$ Hadamard matrix exists.

Containing a spanning star

We have seen in the previous subsection that if we want every symmetric difference to contain a spanning tree which is a path, then for infinitely many values of n our family can be just as large as if we did not want more than just the connectedness of these symmetric differences. In this subsection we show that if the required spanning tree is a star, then the largest possible family is drastically smaller.

Theorem 4.7. *Let $\mathcal{F}_S$ denote the class of graphs containing a spanning star, that is a vertex connected to all other vertices in the graph. Then we have*

$$M_{\mathcal{F}_S}(n) = \begin{cases} n+1 & \text{if } n \text{ is odd} \\ n & \text{if } n \text{ is even.} \end{cases}$$

Proof. First we prove the upper bound. Let $G_1, \ldots, G_m$ be an $\mathcal{F}_S$-good family on the vertex set $[n]$. Consider the complete graph K_m whose vertices are labeled with the graphs $G_1, \ldots, G_m$. For each edge $\{G_i, G_j\}$ of this graph assign an element $h \in [n]$ for which h is adjacent to all other elements of $[n]$ in the graph $G_i \oplus G_j$. By the definition of $\mathcal{F}_S$-goodness such an h exists for every pair of our graphs. Now observe that if an element $a \in [n]$ is assigned to two distinct edges e and f of our graph K_m, then e and f must be independent edges. Indeed, if that was not the case then we would have $e = \{G_i, G_j\}, f = \{G_i, G_k\}$ for some $i, j, k \in [n]$ and a would be a full-degree vertex (that is one, connected to all other vertices) in both of the graphs $G_i \oplus G_j$ and $G_i \oplus G_k$. But since $G_j \oplus G_k = (G_i \oplus G_j) \oplus (G_i \oplus G_k)$, that would mean that a is an isolated vertex in $G_j \oplus G_k$, so no vertex of this latter graph can have full degree contradicting the $\mathcal{F}_S$-goodness of our family. Thus our assignment of vertices from $[n]$ to the edges of our K_m partitions the edge set of K_m into sets of independent edges (every partition class consisting of the edges with the same assigned label), in other words, it defines a proper edge-coloring of K_m. This means that the number of possible labels, which is n, should be at least as large as the edge-chromatic number $\chi_e(K_m)$ of K_m. Since the latter is $m-1$ for even m and m for odd m, turning it around we obtain that for odd n we must have $m \leq n+1$ and for even n we must have $m \leq n$.

Now we show that the upper bound we proved is sharp. First assume that n is odd and consider a complete graph K_{n+1} on the vertices $v_1, \ldots, v_{n+1}$ along with an optimal edge-coloring $c\colon E(K_{n+1}) \to [n]$ of this graph. This edge-coloring partitions $E(K_{n+1})$ into n disjoint matchings $M_1, \ldots, M_n$, where M_j consists of the edges colored j for every $j \in [n]$. Now we construct the graphs $G_1, \ldots, G_{n+1}$ by telling for each potential edge ij of the complete graph on $[n]$ which G_k's will contain it and which ones will not. Consider the edge ij and the union of the matchings M_i and M_j (note that these matchings are in the "other" complete graph on $n+1$ vertices). This union is a bipartite graph on the vertex set $\{v_1, \ldots, v_{n+1}\}$ with two equal size partite classes A and B. Let ij be an edge of the graph G_k if and only if $v_k \in A$. (So ij will be an edge of exactly half of our graphs $G_1, \ldots, G_{n+1}$.) Do this similarly for all edges of K_n, the complete graph on vertex set $[n]$. This way we defined our $n+1$ graphs. We have to show that they form an $\mathcal{F}_S$-good family.

To this end consider two of our graphs, say G_h and G_k. The edge $\{v_h, v_k\}$ has got some color in our coloring c, call this color j. This means that $\{v_h, v_k\}$ belongs to the matching M_j. We claim this means that $j \in [n]$ is a full-degree vertex of $G_h \oplus G_k$. The latter is equivalent to the statement that every edge ji incident to the point j appears in exactly one of the graphs G_h and G_k. But this follows from the way we constructed our graphs: when we decided about the edge ji we considered the matchings M_i and M_j and the bipartite graph their union defines. Since $\{v_h, v_k\} \in M_j$, the points v_h and v_k are always in different partite classes of this bipartite graph, so whichever was called A, exactly one of v_h and v_k belonged to it. Thus the edge ij was declared to be an edge of exactly one of G_h and G_k. Since this is so for every $i \neq j$, j is indeed a full-degree vertex in $G_h \oplus G_k$.

Assume now that n is even. Then $n-1$ is odd and we can construct graphs $G_1, \ldots, G_n$ on vertex set $[n-1] = \{1, \ldots, n-1\}$ as given in the previous paragraph. These are not yet good, however, since we have an nth vertex that does not appear yet in any of the graphs. Note that we have $n-1$ matchings $M_1, \ldots, M_{n-1}$ involved in the construction so far whose indices are just the first $n-1$ vertices of our graphs. Think about the additional vertex n as the index of an additional "matching" M_n that has no edges at all. We decide about the involvement of the edges ni ($i < n$) in our graphs analogously as we did for the earlier edges: Consider the bipartite graph $M_i \cup M_n$, that consists of just the edges of M_i, so it is a perfect matching on the vertex set $\{v_1, \ldots, v_n\}$. Let the two partite classes defined by this perfect matching be A and B and add the edge ni to the graph G_h if and only if v_h belongs to A. Now we can prove analogously to the odd case that the symmetric difference of any two of our graphs contains a vertex of degree $n-1$. Consider G_h and G_k. The edge between v_h and v_k in the auxiliary complete graph belongs to exactly one of the matchings M_j and every edge ij is in exactly one of the graphs G_h and G_k if $i \in \{1, \ldots, j-1, j+1, \ldots, n\}$. This completes the proof. □

The following remark is due to Gábor Tardos [Tar22b].

Remark 5. The statement and proof of the above theorem can also be presented in a more compact form as follows. There exists m graphs on vertex set $[n]$ forming an $\mathcal{F}_S$-good family if and only if $\chi_e(K_m) \leq n$. The proof is essentially the same what was shown above but in the second part we do not have to distinguish between odd and even n, rather just say that $M_1, \ldots, M_n$ are the color classes of a proper edge-coloring of K_m (some of which may be empty) and then define the graphs $G_1, \ldots, G_m$ the same way as above.

The only graph family code proven to be optimal and nonlinear (or not the coset of a linear code) in this chapter is the one appearing in the above Theorem 4.7. This is also the first case so far when the upper bound is proven without the use of Lemma 4.1. This suggests the question of what could be said about the dual problem in this case. The next theorem solves this dual problem for even values of n also showing that Lemma 4.1 would not give a sharp upper bound for $M_{\mathcal{F}_S}(n)$.

Theorem 4.8. *If n is even, then*

$$D_{\mathcal{F}_S}(n) = 2^{\binom{n}{2} - \frac{n}{2}}.$$

When n is odd, then we have

$$2^{\binom{n}{2} - \frac{n+1}{2}} \leq D_{\mathcal{F}_S}(n) \leq 2^{\binom{n}{2} - \frac{n}{2}}.$$

For the proof we will need the following celebrated result from [Chu+86] (see also Corollary 15.7.7 in [AS16]).

Shearer's Lemma([Chu+86]). *Let S be a finite set and $A_1, \ldots, A_m$ be subsets of S such that every element of S is contained in at least k of the sets $A_1, \ldots, A_m$. Let $\mathcal{M}$ be a collection of subsets of S and let $\mathcal{M}_i = \{T \cap A_i : T \in \mathcal{M}\}$ for $1 \leq i \leq m$. Then*

$$|\mathcal{M}|^k \leq \prod_{i=1}^{m} |\mathcal{M}_i|.$$

Proof of Theorem 4.8. We will prove

$$2^{\binom{n}{2}-\lceil \frac{n}{2} \rceil} \le D_{\mathcal{F}_S}(n) \le 2^{\binom{n}{2}-\frac{n}{2}}$$

that implies both the even and the odd case. For the lower bound fix a subgraph T of K_n with the minimum number $\lceil \frac{n}{2} \rceil$ of edges such that no vertex is isolated and take all possible subgraphs of K_n that contain none of the edges of T. The number of such subgraphs is $2^{\binom{n}{2}-\lceil \frac{n}{2} \rceil}$ and no two of them has a symmetric difference that contains all edges incident to any fixed vertex. This proves the lower bound.

For the upper bound consider a graph family $\mathcal{M}$ that satisfies the condition that no two of its elements have a symmetric difference with a vertex of degree $n-1$. For $i = 1, \ldots, n$ let S_i be the set of $n-1$ edges (of K_n) incident to vertex i. Then for any $T, T' \in \mathcal{M}$ we cannot have $E(T') \cap S_i = S_i \setminus (E(T) \cap S_i)$, that is, $E(T)$ and $E(T')$ cannot be complementary on any S_i. So if $\mathcal{M}_i$ denotes the family of graphs obtained by taking the projection of all graphs from $\mathcal{M}$ to the edge set S_i, then $|\mathcal{M}_i| \le 2^{n-2}$. Since each edge of K_n appears in exactly two of the sets S_i, we can apply Shearer's Lemma to these sets with $k = 2$. This gives

$$|\mathcal{M}|^2 \le \prod_{i=1}^{n} |\mathcal{M}_i| \le 2^{n(n-2)}.$$

Taking square roots we get the upper bound. □

Note that if we restrict attention to linear graph families for the dual problem treated in Theorem 4.8, then using again that the cardinality of such a family should be a power of 2 (cf. the similar argument in the proof of Theorem 4.4) we get that our lower bound is also sharp for odd values of n.

4.3 Local conditions

In the previous section we investigated $M_{\mathcal{F}}(n)$ in cases when the required symmetric differences contain specific spanning subgraphs, therefore to check whether these conditions are satisfied we have to consider our graphs on the whole vertex set. Now we turn to families $\mathcal{F}$ defined by containing some fixed small finite graphs, so the nature of these conditions will be local.

General local conditions

Definition 4.2. *A graph class $\mathcal{L}$ defines a* local condition *if it has the property that whenever H_1 is an induced subgraph of H_2 and H_1 belongs to $\mathcal{L}$ then so does also H_2. In short, we will refer to such an $\mathcal{L}$ as a* local graph class*.*

Note that the above definition implies that whenever two graphs F and G are in the $\mathcal{L}$-good relation (that is, $F \oplus G \in \mathcal{L}$) then any F' with $F'[U] \cong F$ and G' with $G'[U] \cong G$ for some $U \subseteq V(F') = V(G')$ (that is, F' and G' induce subgraphs isomorphic to F and G, respectively, on the same subset U of their vertex set) are also in the $\mathcal{L}$-good relation. This means that if two graphs are in this relation then there is always some local certificate for this.

Here are some examples of local graph classes that we considered.

1. $\mathcal{L} = \{H : L \subseteq H\}$ for some fixed finite simple graph L. That is $\mathcal{L}$ contains all graphs that contain a (not necessarily induced) subgraph isomorphic to L. When $\mathcal{L}$ is such a family we will use the simplified notation $M_L(n)$ for $M_{\mathcal{L}}(n)$.
2. $\mathcal{L} = \mathcal{C}_{\text{odd}} := \{H : C_{2k+1} \subseteq H$ for some integer $1 \leq k\}$, that is, $\mathcal{C}_{\text{odd}}$ contains all graphs that contain an odd cycle.

In the following we prove some general results related to $M_{\mathcal{L}}(n)$ for local graph classes $\mathcal{L}$ and will further investigate the special case belonging to our first example above in the next subsection. Later, we will focus on $M_{K_3}(n)$ and $M_{\mathcal{C}_{\text{odd}}}(n)$.

The next proposition gives a straightforward upper bound on the value of $M_{\mathcal{L}}(n)$. It is in terms of $ex(n, \mathcal{L})$ that, as usually in extremal graph theory, denotes the maximum number of edges a graph on n vertices can have without containing any $L \in \mathcal{L}$ as a subgraph.

Proposition 4.9. *For any local graph class $\mathcal{L}$*

$$M_{\mathcal{L}}(n) \leq 2^{\binom{n}{2} - ex(n,\mathcal{L})}.$$

Proof. Consider an n-vertex graph H satisfying $|E(H)| = ex(n, \mathcal{L})$ and containing no subgraph isomorphic to any $L \in \mathcal{L}$. The family of all subgraphs of H clearly satisfies the requirements of the dual problem of $M_{\mathcal{L}}(n)$. This is because no subgraph of H can have a subgraph belonging to $\mathcal{L}$ and the symmetric difference of two such subgraphs is also a subgraph of H so such a symmetric difference can also not contain any $L \in \mathcal{L}$. This family has size $2^{ex(n,\mathcal{L})}$, thus the claimed upper bound follows from Lemma 4.1. □

Proposition 4.9 and our following results will justify the relevance of the following notion in our current setting.

Definition 4.3. *The rate $R_{\mathcal{L}}(n)$ of an optimal graph family code on n vertices satisfying the requirement prescribed by the local graph class $\mathcal{L}$ is defined as*

$$R_{\mathcal{L}}(n) := \frac{2}{n(n-1)} \log_2 M_{\mathcal{L}}(n).$$

We will soon see that the value $\limsup_{n\to\infty} R_{\mathcal{L}}(n)$ is strictly positive for any $\mathcal{L}$ belonging to this section. We will use the following theorem due to Wilson to show that the limit actually exists for all local graph classes.

Wilson's Theorem([Wil76]). *For every finite simple graph T there exists a threshold $n_0(T)$ such that if $n > n_0(T)$ and the following two conditions hold then the edge set of the complete graph K_n can be partitioned into subgraphs each of which is isomorphic to T. The two conditions are:*
1. $\binom{n}{2}$ is divisible by $|E(T)|$;
2. $n - 1$ is divisible by the greatest common divisor of the degrees of vertices in T.

Note that the two conditions in the above theorem are obviously necessary. The decomposition of K_n in the conclusion of the theorem is called a T-design when it exists, cf. [ABB08].

Theorem 4.10. *Let $\mathcal{L}$ be an arbitrary fixed local graph class. Then the value $\lim_{n\to\infty} R_{\mathcal{L}}(n)$ exists and is bounded from below by $R_{\mathcal{L}}(n)$ for every n.*

Proof. Let n be an arbitrary natural number and let $\mathcal{G} = \{G_1, \dots, G_m\}$ be an optimal graph family code for $\mathcal{L}$ with $V(G_i) = [n], i \in \{1, \dots, m\}$, that is one with $m = M_{\mathcal{L}}(n)$. By Wilson's theorem a K_n-design exists for K_N, whenever N is large enough and both $n-1$ divides $N-1$ and $\binom{n}{2}$ divides $\binom{n}{2}$. Take such an N and consider the K_n-design on K_N consisting of the subgraphs $K^{(1)}, \dots, K^{(r)}$, where $r = \frac{N(N-1)}{n(n-1)}$ and each $K^{(i)}$ is isomorphic to K_n. Now let $\mathcal{G}_j := \{G_1^{(j)}, \dots, G_m^{(j)}\}$ be an optimal graph family code for $\mathcal{L}$ on $V(K^{(j)})$ for every $j \in \{1, \dots, r\}$. (Obviously, we can choose each $\mathcal{G}_j$ to be isomorphic to $\mathcal{G}$.) Now define a graph family code on K_N for $\mathcal{L}$ as the collection of graphs that can be written in the form of $G_{\boldsymbol{a}} := \cup_{j=1}^r G_{a_j}^{(j)}$ where $\boldsymbol{a} = (a_1, \dots, a_r)$ runs through all possible sequences satisfying $a_i \in \{1, \dots, m\}$ for every i. Since there are m^r such sequences $\boldsymbol{a}$, this way we have m^r different graphs in our family. They form indeed a graph family code for $\mathcal{L}$ since for any two of them, $G_{\boldsymbol{a}}$ and $G_{\boldsymbol{b}}$ there is some j for which $a_j \neq b_j$ and thus $G_{\boldsymbol{a}} \oplus G_{\boldsymbol{b}} \supseteq_{ind} G_{a_j} \oplus G_{b_j} \supseteq_{ind} L$ for some $L \in \mathcal{L}$. This implies $M_{\mathcal{L}}(N) \geq m^r$ and thus

$$R_{\mathcal{L}}(N) \geq \frac{2}{N(N-1)} \log_2 m^r = \frac{2}{n(n-1)} \log_2 M_{\mathcal{L}}(n) = R_{\mathcal{L}}(n).$$

The requirements for N are satisfied if $N = kn(n-1)+1$ and k is large enough. (Also for $N = kn(n-1)+n$ and large enough k but considering the former is enough for our argument.) Since $M_{\mathcal{L}}(n)$ is clearly monotone nondecreasing in n (as we can always ignore some vertices and consider a graph family code only on the rest), we can write that for any $kn(n-1)+1 \leq i \leq (k+1)n(n-1)$ we have $M_{\mathcal{L}}(i) \geq m^r$ for $r = \frac{\binom{kn(n-1)+1}{2}}{\binom{n}{2}}$. Introducing the sequence $b_i := m^r$ for $r = \frac{\binom{kn(n-1)+1}{2}}{\binom{n}{2}}$ whenever $kn(n-1)+1 \leq i \leq (k+1)n(n-1)$ we can write

$$\liminf_{i\to\infty} \frac{2}{i(i-1)} \log_2 M_{\mathcal{L}}(i) \geq \liminf_{i\to\infty} \frac{2}{i(i-1)} \log_2 b_i \geq$$

$$\liminf_{k\to\infty} \frac{1}{\binom{(k+1)n(n-1)}{2}} \log_2 m^{\frac{\binom{kn(n-1)+1}{2}}{\binom{n}{2}}} =$$

$$\liminf_{k\to\infty} \frac{\binom{kn(n-1)+1}{2}}{\binom{(k+1)n(n-1)}{2}} \frac{2}{n(n-1)} \log_2 m = R_{\mathcal{L}}(n).$$

This proves that $\lim_{n\to\infty} R_{\mathcal{L}}(n)$ exists and is equal to $\sup_n R_{\mathcal{L}}(n)$. □

Remark 6. The above proof is similar to proving that the limit defining the Shannon capacity of graphs exists which is usually done using Fekete's Lemma. Here, however, there are some technical subtleties (because of the divisibility requirements for N) that made it simpler to present a full proof than to refer simply to Fekete's Lemma.

In view of Theorem 4.10 the following definition is meaningful.

Definition 4.4. *The distance capacity (or* distancity *for short) of a local graph class $\mathcal{L}$ is defined as*

$$DC(\mathcal{L}) := \lim_{n\to\infty} R_{\mathcal{L}}(n).$$

Based on Turán's celebrated theorem [Tur41] (see its statement below; cf. also e.g. in [Die17]) and the famous theorem of Erdős and Stone [ES46], Erdős and Simonovits [ES66] proved the following result.

Erdős-Stone-Simonovits Theorem([ES46; ES66]). Let $\mathcal{L}$ be an arbitrary family of at least 2-chromatic graphs, then

$$\lim_{n\to\infty} \frac{ex(n,\mathcal{L})}{\binom{n}{2}} = 1 - \frac{1}{\chi_{\min}(\mathcal{L}) - 1}, \tag{4.1}$$

where $\chi_{\min}(\mathcal{L}) = \min_{L\in\mathcal{L}} \chi(L)$.

Note that Proposition 4.9 and the above result determining the order of magnitude of $ex(n,\mathcal{L})$ has the following immediate consequence for the distancity.

Corollary 4.11. *For any local graph class $\mathcal{L}$ with $\chi_{min}(\mathcal{L}) \geq 2$ we have*

$$DC(\mathcal{L}) \leq \frac{1}{\chi_{min}(\mathcal{L}) - 1}.$$

Containing a prescribed subgraph

Now we focus on local graph classes mentioned in our first example after Definition 4.2: we have some fixed finite simple graph L and consider $\mathcal{L} = \{H : L \subseteq H\}$. As said above in this case we will use the notation $M_L(n)$ for $M_{\mathcal{L}}(n)$ and similarly, we will also denote $R_{\mathcal{L}}(n)$ and $DC(\mathcal{L})$ by $R_L(n)$ and $DC(L)$, respectively. We prove that in this case the upper bound of Corollary 4.11 is always sharp.

Theorem 4.12. *For any fixed graph L we have*

$$DC(L) = \frac{1}{\chi(L) - 1}.$$

For the proof we will use a result by Erdős, Frankl and Rödl [EFR86] about the number $F_n(L)$ of graphs on n labeled vertices containing no subgraph isomorphic to L.

Erdős–Frankl–Rödl Theorem([EFR86]). *Suppose $\chi(L) = r \geq 3$. Then*

$$F_n(L) = 2^{ex(n,K_r)(1+o(1))}.$$

Note that this gives

$$F_n(L) = 2^{\binom{n}{2}\left(1-\frac{1}{\chi(L)-1}+o(1)\right)}$$

by (4.1). In fact, it already follows from Turán's theorem.

Turán's Theorem([Tur41]). Let $r \geq 3$ be an integer. Among all graphs with n vertices that contain no complete subgraph on r vertices, the maximum number of edges,

$$ex(n,K_r) = \left(1 - \frac{1}{r-1} + o(1)\right)\frac{n^2}{2},$$

and this maximum is uniquely attained by the Turán graph $T_{r-1}(n)$, the complete $(r-1)$-partite graph with parts as equal in size as possible.

While the proof of the Erdős-Frankl-Rödl theorem is based on Szemerédi's Regularity Lemma, a similar result for bipartite L easily follows from (4.1) (or from the Kővári–Sós–Turán Theorem [KST54]).

Indeed, it implies that if L is bipartite then $F_n(L) < \binom{\binom{n}{2}}{\varepsilon\binom{n}{2}}$ for any $\varepsilon > 0$ provided $n > n_0(\varepsilon)$, and that implies the claimed statement. (To see the latter one can use the well-known fact, cf. e.g. Lemma 2.3 in [CK11], that

$$\binom{t}{\alpha t} = 2^{t(h(\alpha)+o(1))},$$

where $h(x) = -x\log_2 x - (1-x)\log_2(1-x)$ is the binary entropy function and $0 \le \alpha \le 1$ is meant to be such that αt is an integer. Applying this for $t := \binom{n}{2}$ and $\alpha = \varepsilon$ we obtain that for any $0 < \varepsilon < 1$ the number $\binom{\binom{n}{2}}{\varepsilon\binom{n}{2}}$ is more than $2^{\delta\binom{n}{2}}$ for some positive δ.)

Proof of Theorem 4.12. It follows immediately from Corollary 4.11 that the right hand side is an upper bound on the left hand side so we only have to prove the reverse inequality.

To this end let G_L denote the graph whose vertices are all possible graphs on n labeled vertices and two are connected if and only if their symmetric difference does not contain L as a subgraph. (Note that this is just the complementary graph of $H_{\mathcal{F}}$ used in the proof of Lemma 4.1 when $\mathcal{F}$ is set to be the local graph class $\mathcal{L}$ belonging to our problem.) Then $M_L(n)$ is equal to the independence number $\alpha(G_L)$ of G_L. Clearly, G_L is vertex-transitive (cf. the argument in the proof of Lemma 4.1 for $H_{\mathcal{F}}$), in particular, it is regular. Since the degree of its vertex representing the edgeless graph is just $F_n(L)$, we get (denoting the maximum degree of a graph G by $\Delta(G)$) that

$$M_L(n) = \alpha(G_L) \ge \frac{|V(G_L)|}{\Delta(G_L)+1} = \frac{|V(G_L)|}{F_n(L)+1} =$$

$$\frac{2^{\binom{n}{2}}}{2^{\binom{n}{2}\left(1-\frac{1}{\chi(L)-1}+o(1)\right)}} = 2^{\binom{n}{2}\left(\frac{1}{\chi(L)-1}+o(1)\right)}$$

by the Erdős–Frankl–Rödl theorem (and by the above discussion also for bipartite graphs). Putting this inequality into the definition of $DC(L)$ the required result follows. □

Corollary 4.13. *Let $\mathcal{G}$ be a set of graphs, each containing at least one edge, and let $\mathcal{L}_{\mathcal{G}}$ be the local graph class containing all graphs that contain at least one $G \in \mathcal{G}$ as a subgraph. Then*

$$DC(\mathcal{L}_{\mathcal{G}}) = \frac{1}{\chi_{min}(\mathcal{L}_{\mathcal{G}})-1} = \frac{1}{\chi_{min}(\mathcal{G})-1}.$$

In particular,

$$DC(\mathcal{C}_{odd}) = DC(K_3) = \frac{1}{2}.$$

Proof. The second statement is clearly a special case of the first one, so it is enough to prove the latter. It is a straightforward consequence of Corollary 4.11 that the left hand side is bounded from above by the right hand side. For the reverse inequality note the trivial fact that $DC(\mathcal{L}_{\mathcal{G}}) \ge DC(G)$ for any $G \in \mathcal{G}$. Applying this for some $G \in \mathcal{G}$ that satisfies $\chi(G) = \min_{G\in\mathcal{G}} \chi(G) = \chi_{\min}(\mathcal{L}_{\mathcal{G}})$ the statement follows from Theorem 4.12. □

Remark 8. It is straightforward from the foregoing that the above results also determine for any graph family $\mathcal{G}$ the asymptotic behaviour of the value $D_{\mathcal{L}_{\mathcal{G}}}(n)$ belonging to the dual problem. Indeed, by Lemma 4.1 and Corollary 4.13 we have that $\lim_{n\to\infty} \frac{1}{\binom{n}{2}} \log D_{\mathcal{L}_{\mathcal{G}}}(n) \le 1 - DC(\mathcal{L}_{\mathcal{G}}) = 1 - \frac{1}{\chi_{\min}(\mathcal{G})-1}$

while a matching lower bound follows from the argument in the proof of Proposition 4.9. Thus we have

$$\lim_{n\to\infty} \frac{2}{n(n-1)} \log D_{\mathcal{L}_{\mathcal{G}}}(n) = 1 - \frac{1}{\chi_{\min}(\mathcal{G}) - 1}$$

for any graph family $\mathcal{G}$. This means that by taking all subgraphs of a graph with the largest possible number of edges without containing a subgraph from $\mathcal{G}$ we obtain asymptotically a largest family of graphs no two of which have any $G \in \mathcal{G}$ in their symmetric difference.

Containing a triangle or an odd cycle

In this subsection we are investigating $M_L(n)$ for small values of n and the simplest 3-chromatic graph, which is the triangle K_3. We will also look at the analogous problem when K_3, the cycle of length 3 is replaced by the family of all odd cycles.

For $L = K_3$ the bound of Proposition 4.9 gives us $M_{K_3}(n) \leq 2^{\binom{n}{2} - \lceil \frac{n}{2} \rceil \lfloor \frac{n}{2} \rfloor}$. Below we show that this upper bound is tight whenever n is at most 6.

The first part of the following Proposition is very simple and we present it only for the sake of completeness.

Proposition 4.14. *We have* $M_{K_3}(3) = 2$ *and* $M_{K_3}(4) = 4$.

Proof. For $n = 3$ the statement is trivial: take the empty graph and a triangle on three vertices, this 2-element family already achieves the value of the upper bound which is 2 for $n = 3$.

For $n = 4$ we give the following four graphs on the vertex set $\{1, 2, 3, 4\}$ by their edge sets. Let

$$E(G_0) = \emptyset, E(G_1) = \{12, 23, 13, 34\},$$

$$E(G_2) = \{23, 34, 24, 14\}, E(G_3) = \{12, 13, 24, 14\}.$$

It takes an easy checking that the symmetric difference of any two of these graphs contains a triangle. Since the upper bound in Proposition 4.9 is also 4 in this case, this proves that $M_{K_3}(4) = 4$. □

Remark 9. Note that both of the above simple constructions are closed under the symmetric difference operation, that is they form a linear space over $GF(2)$ when the graphs are represented by the characteristic vectors of their edge sets. In fact, the second construction could also be presented as the vector space generated in this sense by any two of the graphs G_1, G_2, G_3.

Proposition 4.15.

$$M_{K_3}(5) = 16.$$

Proof. The value of the upper bound in Proposition 4.9 gives 16 for $n = 5$, so we only have to prove that 16 is also a lower bound. To this end we will give a set of graphs forming a vector space in the sense of Remark 9. We will give this vector space by a set of generators, although in a somewhat redundant way. (Our reason to keep this redundancy is that the construction has more symmetry this way.)

Think about the vertices $\{1, 2, 3, 4, 5\}$ as if they were given on a circle at the vertices of a regular pentagon in their natural order. Consider the graph with edge set

$$E(G_1) := \{12, 23, 13, 35\}.$$

Let G_2, G_3, G_4, G_5 be the four graphs we obtain from G_1 by rotating it along the circle containing the vertices so that vertex 1 moves to 2, 2 to 3, etc. Thus we have

$$E(G_2) = \{23, 34, 24, 41\}, E(G_3) = \{34, 45, 35, 52),$$

$$E(G_4) = \{45, 51, 41, 13\}, E(G_5) = \{51, 12, 52, 24\}.$$

Now we consider the linear space the characteristic vectors of the edge sets of these five graphs $G_i, i \in \{1, 2, 3, 4, 5\}$ generate. These graphs can be defined as the elements of the family $\mathcal{G} = \{G_I : I \subseteq [5]\}$, where

$$G_I = \oplus_{i \in I} G_i,$$

meaning that $V(G_I) = [5]$ and $E(G_I)$ contains exactly those edges that appear in an odd number of the graphs G_i with $i \in I$.

Note that every edge of the underlying K_5 on $[5]$ appears in exactly two of the graphs $G_1, \ldots, G_5$, therefore for $I = [5]$ we have that G_I is the empty graph just as $G_\emptyset$ is. This implies that for every $I \subseteq [5]$ and $\overline{I} := [5] \setminus I$ we have $G_I = G_{\overline{I}}$, thus every graph in our graph family has exactly two representations as G_I for some $I \subseteq [5]$. (The two representations are given by I and $\overline{I}$ as we have seen. It also follows that if $J \neq I, \overline{I}$ then $G_J \neq G_I$, otherwise we would have $G_{J \oplus I}$ be the empty graph for $J \oplus I \notin \{\emptyset, [5]\}$ contradicting that every edge appears exactly twice in the sets $E(G_i),\ i = 1, \ldots, 5.$) Thus we have indeed $\frac{1}{2}2^5 = 16$ graphs in our family matching our upper bound for $n = 4$.

We have to show that the symmetric difference of any two of our graphs contains a triangle. Since our construction is closed for the symmetric difference operation this is equivalent to say that all graphs in our family except the empty graph contains a triangle. Since $G_I = G_{\overline{I}}$ it is enough to prove that G_I contains a triangle for all $1 \leq |I| \leq 2, I \subseteq [5]$. This is easy to see when $|I| = 1$. For subsets with $|I| = 2$ it is enough to check this for $I = \{1, 2\}$ and $I = \{1, 3\}$ by the rotational symmetry of our construction. But these two cases are easy to check: $G_{\{1,2\}}$ contains the triangles on the triples of vertices $1, 2, 4$ and $1, 3, 4$, while $G_{\{1,3\}}$ contains the triangle on vertices $1, 2, 3$. □

Proposition 4.16.

$$M_{K_3}(6) = 64.$$

Proof. The value of the upper bound given by Proposition 4.9 is 2^6 for $L = K_3$ and $n = 6$, so we need to prove only the lower bound.

To this end we give a construction of 64 graphs forming a graph family code on $[6]$ for K_3. The construction will have several similarities to that in Proposition 4.15 though with somewhat less symmetry. But again our graphs will form a vector space in the sense of Remark 6 to be specified through a set of seven generators that altogether cover each one of the edges of the underlying K_6 exactly twice, so every member of our graph family will have exactly two representations by the generators just as in the proof of Proposition 4.15. Here are the details.

Think about the 6 vertices $1, \dots, 6$ as being on a circle in the vertices of a regular hexagon in their natural order as we go around the circle. Our first four generator graphs are the following four edge-disjoint triangles (plus three isolated points) given by their edge sets as follows.

$$E(G_1) = \{12, 23, 13\}, E(G_2) = \{34, 45, 35\},$$

$$E(G_3) = \{56, 16, 15\}, E(G_4) = \{24, 46, 26\}.$$

The other three graphs are three K_4's (plus two isolated vertices) that are rotations of each other, in particular,

$$E(G_5) = \{12, 24, 45, 15, 14, 25\}, E(G_6) = \{23, 35, 56, 26, 25, 36\},$$

$$E(G_7) = \{34, 46, 16, 13, 36, 14\}.$$

It is easy to check that the above seven graphs cover each edge of the underlying K_6 exactly twice. Just as in the proof of Proposition 4.15 this implies that the generated family of graphs of the form

$$G_I = \oplus_{i \in I} G_i$$

where I runs through all subsets of [7] contains exactly two representations of this form for each of its members, namely

$$G_I = G_J \text{ if and only if } J = [7] \setminus I.$$

Thus our family has $2^6 = 64$ members that matches our upper bound. Now we have to show that the symmetric difference of every pair of our graphs contains a triangle. Since the family is closed under symmetric difference this is equivalent to every G_I except $G_\emptyset = G_{[7]}$ containing a triangle. To show this we consider the representation of each of our graphs as G_I where I contains at most one of the three K_4 generators, that is $|I \cap \{5, 6, 7\}| \leq 1$. When $I \cap \{5, 6, 7\} = \emptyset$ but I itself is nonempty then this is trivial as in such a case G_I is the union of some of the edge-disjoint graphs $G_1, \dots, G_4$ each of which is a triangle itself. In case $|I \cap \{5, 6, 7\}| = 1$, then by symmetry we may assume w.l.o.g. that $I \cap \{5, 6, 7\} = \{5\}$. Then if we also have $\{1, 2\} \subseteq I$ then the triangles on vertices $1, 3, 4$ and $2, 3, 5$ (and two more) will be contained in G_I. So we may assume that at least one of G_1 and G_2 is not part of our representation of G_I and by symmetry, we may assume $2 \notin I$. But then to avoid the triangles on vertices $1, 4, 5$ and $2, 4, 5$ being in G_I we need both $3 \in I$ and $4 \in I$. In this case, however, we will have the triangle on vertices $4, 5, 6$ present in G_I. This completes the proof. ☐

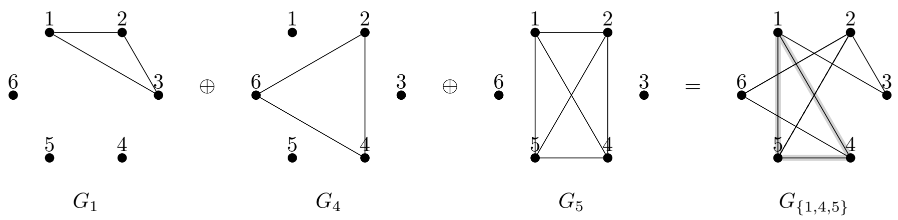

Figure 4.1: Graphs G_1, G_4, G_5 and their generated graph $G_{\{1,4,5\}}$ in the proof of Proposition 4.16.

Recall $\mathcal{C}_{\text{odd}}$ be the class of all graphs containing an odd cycle. Since $ex(n, \mathcal{C}_{\text{odd}}) = ex(n, K_3)$ the upper bound of Proposition 4.9 is also $2^{\binom{n}{2} - \lceil \frac{n}{2} \rceil \lfloor \frac{n}{2} \rfloor}$ for $M_{\mathcal{C}_{\text{odd}}}(n)$. Since $K_3 \cong C_3$ is an odd cycle, we obviously have $M_{K_3}(n) \leq M_{\mathcal{C}_{\text{odd}}}(n)$ and so by Propositions 4.14, 4.15 and 4.16 the previous upper bound is also sharp for $M_{\mathcal{C}_{\text{odd}}}(n)$ when $n \in \{3, 4, 5, 6\}$. Although we could not prove that $M_{K_3}(7)$ is also equal to this upper bound, we can show this at least for $M_{\mathcal{C}_{\text{odd}}}(7)$.

Proposition 4.17.

$$M_{\mathcal{C}_{odd}}(7) = 2^9.$$

Proof. The upper bound $2^{\binom{n}{2}-\lceil\frac{n}{2}\rceil\lfloor\frac{n}{2}\rfloor}$ is equal to 2^9, so it is enough to prove that this is also a lower bound. This we do similarly as in the proofs of Propositions 4.15 and 4.16.

Again, we think about the seven vertices forming the set $[7]$ as the vertices of a regular 7-gon around a cycle in their natural order. We define $7+3=10$ simple graphs $G_1,\dots,G_7$ and $G_8,\dots,G_{10}$ that will generate our family. Let G_1 be the triangle with edges $12, 24, 14$ and $G_2,\dots,G_7$ be its six possible rotated versions, that is the triangles with edge sets $\{23, 35, 25\}, \{34, 46, 36\}, \dots, \{17, 13, 37\}$, respectively. Note that these seven triangles cover all pairs of vertices exactly once, that is, they form a Steiner triple system. The three other graphs G_8, G_9, G_{10} are three edge-disjoint seven-cycles, namely those with edge sets

$$\{12, 23, 34, 45, 56, 67, 17\}, \{13, 35, 57, 27, 24, 46, 16\}, \{14, 47, 37, 36, 26, 25, 15\},$$

respectively. Note that these three graphs also cover all pairs of vertices exactly once and that the edge sets of a G_i for $i \in [7]$ and G_j with $j \in \{8, 9, 10\}$ intersect in exactly one element. Since our ten graphs cover the edges of the underlying K_7 exactly twice, just as in the proofs of Propositions 4.15 and 4.16 the generated family

$$\oplus_{i\in I} G_i$$

as I runs over all subsets of $\{1,\dots,10\}$ will have exactly 2^9 distinct members each of which is represented by two subsets of $\{1,\dots,10\}$, some I and its complement. All we are left to show for proving $M_{\mathcal{C}_{odd}}(7) \geq 2^9$ is that each such G_I except $G_\emptyset = G_{[10]}$ contains an odd cycle. If $I \subseteq [7]$, this is obvious and so is also if $I \subseteq \{8, 9, 10\}$. When both $I \cap [7]$ and $I \cap \{8, 9, 10\}$ are nonempty, then we consider that representation G_I which has $|I \cap [7]| \leq 3$. If we have $|I \cap \{8, 9, 10\}| = 1$ then whichever 7-cycle we have (that is, whichever of G_8, G_9, G_{10}) it will have two consecutive edges that do not appear in either of the at most three triangles. If we take the first pair of such edges (as we go along our 7-cycle in an appropriate direction) for which the previous one is an edge of one of our triangles (since we take at least one triangle and each triangle intersects each 7-cycle, such an edge must exist), then the construction ensures that these two consecutive edges close up to a K_3 in our G_I. In case we have two 7-cycles in our G_I representation, then those create 7 distinct K_3's in their union. Each of our triangles intersects exactly three of those seven K_3's created, so if we have $|I \cap [7]| \leq 2$ then at least one of these seven K_3's remain untouched. Thus we are left with the case of two 7-cycles and exactly three triangles. For this case let us switch to the complementary representation with four triangles and one 7-cycle. By symmetry, we may assume that our 7-cycle is G_8. If the four triangles are such that two consecutive edges of G_8 do not appear in any of them then we can finish the argument as before. If this is not the case, then the four triangles must leave three such edges of G_8 uncovered which form a matching. Because of symmetry we may assume that these are the edges $12, 34, 56$. This also tells us exactly which are the four triangles we have in the representation of G_I, namely those that contain the remaining four edges, that is, G_2, G_4, G_6 and G_7. In this case G_I contains the K_3, for example, on the vertices $2, 5, 6$. Finally, if we have all the three 7-cycles in our representation then the complementary representation has no 7-cycle at all and this case we have already covered. This completes the proof. □

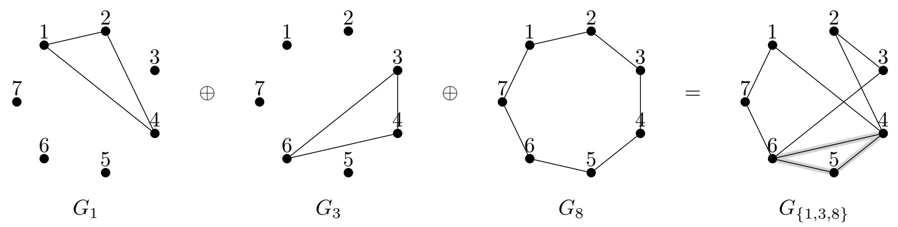

Figure 4.2: Graphs G_1, G_3, G_8 and their generated graph $G_{\{1,3,8\}}$ in the proof of Proposition 4.17.

4.4 Open problems

In the final section of our paper [j2] we have listed some related open problems. Some of these have been investigated or even answered since then. Here we will list all the problems and give reference to their solutions if there is one.

Linear codes

Problem 1. *For what graph families $\mathcal{F}$ is it true that $M_{\mathcal{F}}(n)$ is achieved by a linear graph family code, that is one that is closed under the symmetric difference operation?*

Our results here include examples where this is the case as well as ones in which it is not. Indeed in Theorem 4.7 the precise answer is n or $n+1$, and if this is not a power of 2 there is no optimal linear solution. Another family of examples in which the optimal family cannot be achieved by a linear example is that in which the family $\mathcal{F}$ is the family of all graphs with at most $2r$ edges, where r is chosen so that the sum

$$\sum_{i=0}^{r} \binom{\binom{n}{2}}{i}$$

is not a power of 2. Indeed, by a theorem of Kleitman [Kle66] (for usual codes) the size of the optimum family here is the size of the family of all graphs with at most r edges.

Although it is not exactly an answer to this question, in [Alo24] Alon investigated the dual problem, $D_{\mathcal{F}}(n)$. In particular, he was interested in the cases, where $\mathcal{F}$ contains all copies of a single graph H on n vertices. So his question was about the maximum size of a graph family – which he called H-code – the symmetric difference of no two members of which is a graph isomorphic to H. He looked at graph families $\mathcal{F}$ like cliques, stars and matchings and also investigated the asymptotics of $d_{\mathcal{F}}(n) = \frac{D_{\mathcal{F}}(n)}{2^{\binom{n}{2}}}$. He also studied the linear variant of these problems, that is, the version when the H-code is closed under symmetric difference. For this linear version, in [Ver25] Versteegen provided a general upper bound on $D_{\mathcal{F}}(n)$ when $\mathcal{F}$ contains all copies of an arbitrary single graph H.

Asymmetric differences

The construction in the proof of Theorem 4.2 has the property that for any two of its graphs G and G' with an equal number of edges their two *asymmetric* differences

$$G \setminus G' = ([n], E(G) \setminus E(G')) \text{ and } G' \setminus G = ([n], E(G') \setminus E(G))$$

are isomorphic. This suggests the following question.

Problem 2. *What is the maximum possible size of a graph family $\mathcal{A}$ of graphs on n vertices satisfying that if $A, A' \in \mathcal{A}$ then $A \setminus A'$ and $A' \setminus A$ are isomorphic?*

In [GJS23] Gishboliner, Jin and Sudakov completely resolved our question by showing that the maximum possible size of such a family is exactly $2^{\frac{1}{2}(\binom{n}{2}-\lfloor\frac{n}{2}\rfloor)}$ and even characterized all the extremal constructions.

Phase transitions

Theorems 4.5 and 4.7 show a huge difference between requiring a spanning path or a spanning star in the symmetric differences. One may wonder what happens "in between". Note that if we formulate this "in betweenness" so that we want to have a spanning tree with diameter at most k, then while with $k = 2$ we are at Theorem 4.7 and with $k = n - 1$ at Theorem 4.5, already for $k = 3$ we get the same result as for $k = n - 1$ by the construction in the proof of Theorem 4.2. (This is simply because complete bipartite graphs contain spanning trees of diameter at most 3.) So it seems plausible to formulate questions in terms of more specific "natural" sequences of spanning trees $T_1, T_2, \ldots$. (In the problem below the notation $M_{T_n}(n)$ is meant to denote the largest possible cardinality of a family of graphs on vertex set $[n]$ such that the symmetric difference of any two of them contains T_n as a subgraph.)

Problem 3. *For what "natural" sequences $T_1, T_2, \ldots, T_i, \ldots$ of trees (with T_i having exactly i vertices for every i) will the value of $M_{T_n}(n)$ grow only linearly in n? A similar question is valid if T_i is replaced by $\mathcal{T}_i$, some "natural" family of i-vertex trees.*

In [Bai+24] Bai, Gao, Ma and Wu partially solved this problem by proving the following result.
Theorem ([Bai+24]) *For infinitely many n and all integers $3 \leq \ell \leq \frac{n-1}{12logn} + 2$, we have*

$$M_{F_\ell}(n) \geq 2^{(n-2)},$$

where F_ℓ denotes the family of graphs containing a spanning tree that has exactly ℓ leaves. In particular, this holds whenever $n \geq 64$ and $n = p$ or $n = 2p - 1$ for odd primes p.

Exact cardinality

Propositions 4.14, 4.15, 4.16, 4.17 showed that the upper bound of Proposition 4.9 can be sharp for small values of n for the requirement that a triangle or at least an odd cycle is contained in the symmetric differences. It would be interesting to know whether this can also happen for large values of n.

Problem 4. *Is*

$$M_{K_3}(n) = 2^{\binom{n}{2}-\lceil\frac{n}{2}\rceil\lfloor\frac{n}{2}\rfloor}$$

true always or at least for infinitely many values of n? Even if this is not so, does the analogous equality hold for $M_{\mathcal{C}_{odd}}(n)$?

Note that there are much better known estimates for the number of triangle-free graphs on n labeled vertices than the one we have used here, in fact, it is known that almost all of these graphs are bipartite [EKR76]. While this improves the gap between the upper and lower bounds that follow from our proofs for $M_{K_3}(n)$, it is still far from determining its precise value.

Publications

Number of publications:	5
Number of peer-reviewed journal papers (written in English):	3
Number of articles in journals indexed by WoS or Scopus:	3
Number of publications (in English) with at least 50% contribution of the author:	2
Number of peer-reviewed publications:	3
Number of citations:	13
Number of independent citations:	10

Publications Linked to the Theses

Journal Papers

[*j*1] Anna Gujgiczer and Gábor Simonyi. On multichromatic numbers of widely colorable graphs. *Journal of Graph Theory* 100(2), 2022, pp. 346–361. DOI: 10.1002/jgt.22785.

[*j*2] Noga Alon, Anna Gujgiczer, János Körner, Aleksa Milojević, and Gábor Simonyi. Structured codes of graphs. *SIAM Journal on Discrete Mathematics* 37(1), 2023, pp. 379–403. DOI: 10.1137/22M1487989.

[*j*3] Anna Gujgiczer and Gábor Simonyi. Critical subgraphs of Schrijver graphs for the fractional chromatic number. *Graphs and Combinatorics* 40, 2024. DOI: 10.1007/s00373-024-02782-9.

Additional Publications (Not Linked to Theses)

International Conference and Workshop Papers

[*c*4] Anna Gujgiczer, Gábor Simonyi, and Gábor Tardos. On the generalized Mycielskian of complements of odd cycles. In: *Proceedings of the 12th Japanese-Hungarian Symposium on Discrete Mathematics and Its Applications*, pp. 485–488. 2023.

[*c*5] Anna Gujgiczer, Márton Elekes, Oszkár Semeráth, and András Vörös. Towards model-based support for regression testing. In: *24th PhD Mini-Symposium (Minisy@ DMIS 2017)*, pp. 26–29. 2017.

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