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Graph Coloring Parameters and Graph Codes

Ph.D. Thesis Booklet

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1 Introduction

In graph theory, a much-studied graph parameter is the chromatic number, which is used in practice for problems such as frequency or time allocation. In many cases, the behavior of the chromatic number is difficult to understand. An example of this is how this parameter behaves in graph products. In 1966 Stephen Hedetniemi formulated the conjecture that the chromatic number of the so-called tensor product of two graphs is equal to the minimum of the chromatic number of the factors. It is clear, however, that the chromatic number of the product is at most the chromatic number of the factors. Therefore, the conjecture essentially asked whether the reverse inequality holds as well. This question remained unanswered for a long time, but in 2019, it was disproved [Shi19]. The first counterexample found was very large both in terms of the vertex number of the factors and their chromatic number. Later, smaller counterexamples were found [Zhu21; Tar22; Wro20; Tar23] and now the conjecture is fully settled, meaning that for any number c if both factors have chromatic numbers greater than c, we know whether their product can be c-colorable or not.

Other interesting, well-studied and closely related graph parameters are the so-called fractional chromatic number and multichromatic numbers. In the earlier counterexamples to Hedetniemi's conjecture the fractional chromatic number turned out to be an important parameter and in the later counterexamples the multichromatic numbers of some special graph classes came into play. My first thesis addresses some questions within this topic. It is also worth mentioning that Hedetniemi-type problems in which we consider other parameters of the graphs involved in place of the chromatic number, were formulated as well. In the case of the fractional chromatic number it is known that the Hedetniemi-type conjecture is true [Zhu11].

Multichromatic numbers are closely related to Kneser graphs - as those parameters can be expressed with homomorphisms to corresponding Kneser graphs - a famous graph class whose chromatic number was determined by Lovász in his celebrated paper [Lov78], where he proved that the already known upper bound that was conjectured to be tight is tight indeed. However, in general, those graphs are not vertex critical for this parameter, meaning that after a vertex removal the chromatic number does not necessarily decrease. Schrijver observed that special induced subgraphs, now called Schrijver graphs, have the same chromatic number as the Kneser graph (with the same parameters), and they are vertex critical for that. Moreover, Kneser and Schrijver graphs (with the same parameters) share the same fractional chromatic number as well [Tal03; ST06], but even the Schrijver graph is not critical for that (except for some special cases). My second thesis focuses on finding induced subgraphs of Schrijver graphs with the same fractional chromatic number, which are also vertex-critical for that parameter.

A research direction different from the ones mentioned above is to investigate the maximum size of graph families where some relation of any two members of the family (considered as the codewords) satisfies some prescribed condition. An example of this is the famous conjecture of Simonovits and Sós [SS76] proven by Ellis, Filmus and Friedgut [EFF12], that determines the maximum possible cardinality of a family of graphs on n labeled vertices in which the intersection of any two members contains a triangle. The role of the intersection can be replaced, to get new interesting questions, e.g. by the symmetric difference of the edge sets of the two graphs. It is what we can arrive to if the basic code distance problem (how many binary sequences of a given length can be given at most if any two differ in at least a given number of coordinates) is modified so that we do not prescribe the minimum distance of any two codewords but require that they differ in some specific structure. Apart from the containment of a triangle it is also interesting to examine global conditions like connectedness or Hamiltonicity.

2 Summary of the theses

The first two theses are related to special graphs classes. These graphs serve as universal graphs for some coloring parameters, meaning, that if a graph G has the required coloring parameter then it has a homomorphism to the corresponding special graph. We say that a graph G admits a homomorphism to a graph H if there exists an edge preserving map from the vertex set of G to the vertex set of H and we denote the existence of such a homomorphism by $G \to H$. One can easily see that, for example, the chromatic number can be expressed in such a way. A graph G has chromatic number at most c if and only if it has a homomorphism to K_c, the complete graph on c vertices. In the first and the second theses the universal graphs (or their subgraphs) for the so-called s-wide coloring and multicoloring are explored.

The third thesis is more directly related to information theory, codewords which can be defined on graphs are investigated there.

The theorem numbers in the following summary chapters of these theses generally align with the numbering in the theses themselves, but not in every case. The reason for this is that not all theorems are included in the summary, a different order sometimes seemed more advantageous for the concise description, and some theorems have been merged.

2.2 Multichromatic Numbers of Widely Colorable Graphs

As mentioned in the Introduction, related to the Hedetniemi conjecture, a certain multichromatic number of a special graph class became interesting. This graph class plays an important role in the theory of wide colorings. A vertex-coloring of a graph is called s-wide if the two endvertices of every walk of length $2s-1$

receive different colors in it. It is easy to see that this is one possible generalization of the term coloring in graph theory, as 1-wide coloring is equivalent to the proper graph coloring. It can be shown that a graph is s-widely colorable with t colors if and only if it admits a homomorphism into the following universal graph [ST06] denoted by $W(s,t)$ some special cases of which appeared in the related question.

$$V(W(s,t)) = \{(x_1 \dots x_t) : \ \forall i \ x_i \in \{0,1,\dots,s\}, \exists! i \ x_i = 0, \ \exists j \ x_j = 1\},$$

$$E(W(s,t)) = \{\{(x_1 \dots x_t),(y_1 \dots y_t)\} : \forall i \ |x_i - y_i| = 1 \textit{ or } x_i = y_i = s\}.$$

If we set $s = 1$, then we get $W(1,t) = K_t$ by the definition, which is in line with our earlier observation that the complete graphs are universal graphs for proper colorings.

Multicoloring is when we color the vertices of a graph G with n colors in such a way that every vertex receives k distinct colors and if two vertices u and v are adjacent then the set of colors received by u is disjoint from the set of colors received by v. Formally, it is a function $f : v \mapsto \{c_1,\dots,c_k\}$ where for $\forall i \in [k]$ $c_i \in [n]$, such that if $\{u,v\} \in E(G)$ then $f(u) \cap f(v) = \emptyset$ (where $[k] = \{1,2,\dots,k\}$ and similarly $[n] = \{1,2,\dots,n\}$). Such colorings were first considered by Geller and Stahl, see [GS75; Sta76]. Stahl [Sta76] introduced the corresponding multichromatic number $\chi_k(G)$ as the minimum number of colors needed for such a coloring, called a k-fold coloring. (This graph parameter can also be expressed by the existence of a homomorphism into some universal graph as discussed in the next section.)

The fractional chromatic number $\chi_f(G)$ can be defined as

$$\chi_f(G) = \inf_k \left\{ \frac{\chi_k(G)}{k} \right\}.$$

With my advisor in [j1] we have determined the exact values for the k-th multichromatic numbers for the above mentioned $W(s,t)$ universal graphs in cases when $k \le s$.

This work was motivated by a question of Tardif in [Tar22], where he constructed a counterexample graph pair G, H to the Hedetniemi conjecture, where G and H had large chromatic numbers, more than 14, but their product was 14-colorable. In that counterexample G was $W(3,9)[K_4]$, the graph which is obtained by blowing up each vertex of $W(3,9)$ into a clique of size 4, fully connecting the cliques corresponding to originally adjacent vertices in $W(3,9)$. It is easy to see that the chromatic number of this graph is exactly the 4-th multichromatic number of $W(3,9)$. In hope for constructing smaller counterexamples in a similar way he asked whether $\chi(W(3,t)[K_3]) = \chi_3(W(3,t))$ is large, in particular, for $t = 8$ more than 12 and for $t = 7$ more than 11. He also observed that in general

$$\chi_k(W(s,t)) \ge t + 2(k-1)$$

holds. In other words, he asked if strict inequality is true in the special case when $s = k = 3$ and $t = 7$ or $t = 8$. We have answered his question in the negative and generalized the result to all t and $k \leq s$:

Theorem 2.2.1. *If $k \leq s$, then*

$$\chi_k(W(s,t)) = t + 2(k-1).$$

We also showed that this result cannot be generalized for arbitrarily large k (with respect to s).

Theorem 2.2.2. *For all pairs of positive integers $t \geq 3$ and $s \geq 1$ there exists some threshold $k_0 = k_0(s,t) > s$ for which*

$$\chi_k(W(s,t)) > t + 2(k-1)$$

whenever $k \geq k_0$.

We also managed to prove the following theorems about the fractional chromatic number of a $W(s,t)$ graph. For that we have used some previous results concerning Mycielki graphs s-wide colorability [BS05; SST24; GJS04; ST06]. The Mycielskian $M(G)$ of a graph G is a result of a graph operation, introduced by Mycielski [Myc55], which does not increase the clique number of the graph G, but it increases its chromatic number. The construction can be generalised (see Chapter 1 of the dissertation) to get h-level Mycielskians $M_h(G)$, where the original construction $M(G) = M_2(G)$. The effect of the original Mycielski construction, $M_2(G)$, on the fractional chromatic number were investigated in [LPU95], where a simple function was given:

$$\chi_f(M(G)) = \chi_f(G) + \frac{1}{\chi_f(G)}.$$

For a general h, the fractional chromatic number $\chi_f(M_h(G))$ was studied by Tardif in [Tar01]. He proved that the value of $\chi_f(G)$ also determines $\chi_f(M_h(G))$.

$$\chi_f(M_h(G)) = \chi_f(G) + \frac{1}{\sum_{i=0}^{h-1}(\chi_f(G)-1)^i}.$$

Using this result we managed to prove the following two theorems by showing the existence of homomorphisms from $M_{3s-2}(W(s,t))$ to $W(s,t+1)$ and from $W(s,t+1)$ to $M_s(W(s,t))$.

Theorem 2.2.3.

$$\chi_f(W(s,t)) + \frac{\chi_f(W(s,t))-2}{(\chi_f(W(s,t))-1)^{3s-2}-1} \leq \chi_f(W(s,t+1)) \leq \chi_f(W(s,t)) + \frac{\chi_f(W(s,t))-2}{(\chi_f(W(s,t))-1)^{s}-1}$$

Theorem 2.2.4. *For any fixed positive integer s we have*

$$\lim_{t\to\infty} \chi_f(W(s,t)) = \infty.$$

2.3 Critical Subgraphs of Schrijver Graphs for the Fractional Chromatic Number

As the $W(s,t)$ graphs were universal graphs for wide-colorings, Kneser graphs are the universal graphs for multicolorings, meaning that the k-th multichromatic number of a graph is at most n if and only if it admints a homomorphism to the Kneser graph $KG(n,k)$. For positive integers $n \geq 2k$ the Kneser graph $KG(n,k)$ is defined on the vertex set that consists of the $\binom{n}{k}$ k-element subsets of $[n]$ with two such subsets forming an edge if and only if they are disjoint:

$$V(KG(n,k)) = \binom{[n]}{k}$$

$$E(KG(n,k)) = \{\{A,B\} : A \cap B = \emptyset\}.$$

Kneser [Kne55] observed that the chromatic number of $KG(n,k)$ is at most $n-2k+2$ and conjectured that this upper bound is tight. This was proved many years later by Lovász in his celebrated paper [Lov78] using the Borsuk-Ulam theorem. Soon afterwards Schrijver [Sch78] found that a certain induced subgraph $SG(n,k)$ of $KG(n,k)$, now called Schrijver graph, still has chromatic number $n-2k+2$ and moreover, it is also vertex-critical for this property, that is, deleting any of its vertices the chromatic number becomes smaller.

The fractional chromatic number of $KG(n,k)$ is $\frac{n}{k}$ (which is a simple consequence of the Erdős-Ko-Rado theorem [EKR61]). Schrijver graphs $SG(n,k)$ share this fractional chromatic value [Tal03; ST06], but most Schrijver graphs are not vertex-critical for this parameter (the only exceptions are the trivial cases) and this suggested the problem of finding critical subgraphs of Schrijver graphs for the fractional chromatic number.

In a joint paper [j3] with my advisor we worked on this problem. We defined a natural property for the sets representing the vertices and named the subgraph formed by the vertices satisfying this property $Q(n,k)$ (the formal definition of $Q(n,k)$ can be found in Chapter 3 of the dissertation). A basic property of these graphs is the following:

Theorem 2.3.1. *Let $n \geq 2k$ and $\ell \geq 2$ be any positive integer. Then the graphs $Q(n,k)$ and $Q(\ell n, \ell k)$ are isomorphic.*

Based on the above theorem, when studying the properties of $Q(n,k)$ graphs, we can always assume that $\gcd(n,k) = 1$.

Theorem 2.3.2. *Assume $n \geq 2k$, $\gcd(n,k) = 1$ and let a and b be the smallest positive integers for which $ak = bn - 1$. The graph $Q(n,k) \subseteq SG(n,k)$ satisfies the following properties.*

- $\chi_f(Q(n,k)) = \frac{n}{k} = \chi_f(SG(n,k))$.
- $\forall U \in V(Q(n,k))\ \ \chi_f(Q(n,k) \setminus \{U\}) = \frac{a}{b} < \frac{n}{k}$, *that is $Q(n,k)$ is vertex-critical for the fractional chromatic number.*
- *$Q(n,k)$ contains an induced subgraph isomorphic to $Q(a,b)$.*

While proving this result we realised that the above theorem is true because the found special subgraph is isomorphic to another known graph, the circular complete graph, $K_{n/k}$, which is the universal graph for yet another coloring parameter, the circular chromatic number. The definitions of the circular complete graph $K_{n/k}$ for $n \geq 2k$ and the related circular chromatic number χ_c are the following:

$$V(K_{n/k}) = \{0, 1, \dots, n-1\}$$

$$E(K_{n/k}) = \{\{i,j\} : k \leq |i-j| \leq n-k\},$$

$$\chi_c(G) = \min\left\{\frac{p}{q} : p \leq |V(G)|, G \to K_{p/q}\right\}.$$

Theorem 2.3.3. *$Q(n,k)$ is isomorphic with the circular complete graph $K_{n/k}$ whenever $\gcd(n,k) = 1$.*

It was known for circular complete graphs that they are vertex-critical for the fractional chromatic number, but edge-criticality was not studied before (neither for the fractional nor for the circular chromatic number). We also investigated this question. For that we called an edge $\{i,j\} \in E(K_{n/k})$ a *shortest edge* if $|i-j| = k$ or $|i-j| = n-k$. (The name comes from the fact that these are the shortest edges when the vertices are arranged in order along a circle.)

Theorem 2.3.4. *If $\gcd(n,k) = 1$, $e \in E(K_{n/k})$ and a, b are defined as the smallest positive integers for which $ak = bn - 1$ then*

$$\chi_f(K_{n/k} \setminus \{e\}) = \chi_c(K_{n/k} \setminus \{e\}) = \begin{cases} \frac{a}{b} & \text{if } e \text{ is a shortest edge} \\ \frac{n}{k} & \text{otherwise.} \end{cases}$$

Finally, we proved that $SG(n,k)$ itself is vertex critical for the fractional chromatic number only in some trivial cases.

Theorem 2.3.5. $\forall U \in V(SG(n,k))\ \ \chi_f(SG(n,k) \setminus \{U\}) < \chi_f(SG(n,k))$ *if and only if one of the following holds: $k = 1, n = 2k$, or $n = 2k+1$.*

2.4 Graph Codes

In a joint work [j2] with Noga Alon, János Körner, Aleksa Milojević and Gábor Simonyi we investigated the maximum size of graph families on a common vertex set of cardinality n such that the symmetric difference of the edge sets of any two members of the family satisfies some prescribed condition. Note, that if the prescribed condition is just to contain at least d edges, then we get back the basic code distance problem: How many codewords of length $\binom{n}{2}$ can be given such that any two of them differ in at least d coordinates?

In this subsection I will list some of the results that we had (see Chapter 4 of the dissertation for more). We considered global properties like connectedness, Hamiltonicity as well as local properties like containment of a triangle and some more. Formally all these can be described by saying that the graph defined by the symmetric difference of the edge sets of any two of our graphs belong to a prescribed family of graphs (namely those that are connected, contain a Hamiltonian cycle, or contain a triangle, etc.)

Let $\mathcal{F}$ be a fixed class of graphs. A graph family $\mathcal{G}$ on n labeled vertices is called $\mathcal{F}$-good if for any pair $G, G' \in \mathcal{G}$ the graph $G \oplus G'$ defined by

$$V(G \oplus G') = V(G) = V(G') = [n],$$

$$E(G \oplus G') = \{e : e \in (E(G) \setminus E(G')) \cup (E(G') \setminus E(G))\}$$

belongs to $\mathcal{F}$.

Let $M_{\mathcal{F}}(n)$ denote the maximum possible size of an $\mathcal{F}$-good family on n vertices. We were interested in the value of $M_{\mathcal{F}}(n)$ for various classes $\mathcal{F}$. The followings theorems give this value in some cases we considered.

Theorem 2.4.1. *Let $\mathcal{F}_c$ denote the class of connected graphs and $\mathcal{F}_{2c}$ the class of 2-connected graphs. Then*

$$M_{\mathcal{F}_c}(n) = 2^{n-1},\ M_{\mathcal{F}_{2c}}(n) = 2^{n-2}.$$

Theorem 2.4.2. *Let $\mathcal{F}_{Hp}$ denote the class of graphs containing a Hamiltonian path and $\mathcal{F}_{Hc}$ denote the class of graphs containing a Hamiltonian cycle. Then for infinitely many values of n we have*

$$M_{\mathcal{F}_{Hp}}(n) = 2^{n-1},\ M_{\mathcal{F}_{Hc}}(n) = 2^{n-2}.$$

In the above listed theorems for proving the maximality of $M_{\mathcal{F}}(n)$ for the family $\mathcal{F}$ in question we used the following lemma.

Lemma 2.4.1. *For any graph class $\mathcal{F}$ we have*

$$M_{\mathcal{F}}(n) \cdot D_{\mathcal{F}}(n) \leq 2^{\binom{n}{2}},$$

where $D_{\mathcal{F}}(n)$ denotes the "dual" of $M_{\mathcal{F}}(n)$, i.e. the maximum possible size of a graph family on n labeled vertices, the symmetric difference of no two members of which belongs to $\mathcal{F}$. Note that denoting by $\overline{\mathcal{F}}$ the class containing exactly those graphs that do not belong to $\mathcal{F}$ we actually have $D_{\mathcal{F}}(n) = M_{\overline{\mathcal{F}}}(n)$. In all of the proofs of the above mentioned theorems we cunstructed $\mathcal{F}$-good and $\overline{\mathcal{F}}$-good families, A and B respectively, of "matching sizes", meaning that $|A| \cdot |B| = 2^{\binom{n}{2}}$, proving that they are both maximal. However, this technique does not work for every class of graphs.

Theorem 2.4.3. *Let $\mathcal{F}_S$ denote the class of graphs containing a spanning star, that is a vertex connected to all other vertices in the graph. Then we have*

$$M_{\mathcal{F}_S}(n) = \begin{cases} n+1 & \text{if } n \text{ is odd} \\ n & \text{if } n \text{ is even.} \end{cases}$$

The dual family does not have a mathcing size, as

$$2^{\binom{n}{2} - \lceil \frac{n}{2} \rceil} \leq D_{\mathcal{F}_S}(n) \leq 2^{\binom{n}{2} - \frac{n}{2}}.$$

For local conditions we could also use Lemma 2.4.1.

Theorem 2.4.4. *Let $\mathcal{F}_{K_3}$ denote the class of graphs containing a triangle. Then we have*

$$M_{\mathcal{F}_{K_3}}(n) \leq 2^{\binom{n}{2} - \lceil \frac{n}{2} \rceil \lfloor \frac{n}{2} \rfloor}.$$

This upper bound is sharp when $n \leq 6$.

The above theorem is just a special case of a more general one, which brings extremal graph theory in the picture. Let $ex(n, G)$ denote the maximum number of edges an n-vertex graph can have without containing a subgraph isomorphc to G and let $\mathcal{F}_G$ denote the class of graphs containing the graph G as a subgraph.

Theorem 2.4.5.

$$M_{\mathcal{F}_G}(n) \leq 2^{\binom{n}{2} - ex(n,G)}.$$

It turns out that asymptotically this upper bound is tight. To state that formally, we also defined a capacity-type asymptotic invariant and we showed that this invariant is upper bounded by a simple function of the chromatic number. Let

$$R_{\mathcal{F}_G}(n) := \frac{2}{n(n-1)} \log_2 M_{\mathcal{F}_G}(n)$$

and call the following always-existing limit the distance capacity:

$$DC(\mathcal{F}_G) := \lim_{n \to \infty} R_{\mathcal{F}_G}(n).$$

Using the Erdős-Stone-Simonovits theorem [ES46; ES66], stating that

$$\lim_{n\to\infty} \frac{ex(n,G)}{\binom{n}{2}} = 1 - \frac{1}{\chi(G)-1},$$

we get $DC(\mathcal{F}_G) \leq \frac{1}{\chi(G)-1}$. Moreover, equality can also be proven.

Theorem 2.4.6. *If $\chi(G) \geq 2$ then we have $DC(\mathcal{F}_G) = \frac{1}{\chi(G)-1}$.*

3 Application of the New Results

This thesis mainly concerns theoretical results that are interesting on their own right and connected to various parts of graph theory. Nevertheless, in the next subsection the connection of the fractional chromatic number and the multichromatic numbers to information theory will be explained, providing a more application based point of view of the results of the first two theses. For the last thesis, as it was already mentioned, defining codewords as graphs is a generalization of the classical code distance problem, therefore no further explanation is needed for its relevance to information theory.

3.1 Shannon capacity

Several problems in information theory lead to the definition of special graph parameters and the most famous example of this is the Shannon capacity of graphs [Sha56], which is the tight upper bound on the rate at which information can be transmitted over a discrete, memoryless communication channel with zero error probability.

One can model the communication channel as a graph: the transmittable letters are the vertices and a pair of them are connected if and only if they are distinguishable by the receiver. We consider two t-length codewords distinguishable if they are distinguishable in at least one index. Generally, we are interested in the maximum number of pairwise distinguishable t-length codewords.

Definition 3.1.1. *For two graphs G and H their OR-product $G \cdot H$ is defined as follows*

$$V(G \cdot H) = V(G) \times V(H),$$
$$E(G \cdot H) = \{\{(g_1,h_1)(g_2,h_2)\} : g_1, g_2 \in V(G), h_1, h_2 \in V(H),$$
$$\{g_1, g_2\} \in E(G) \text{ or } \{h_1, h_2\} \in E(H)\}.$$

Let G^t denote the t-fold OR-product of G by itself. By definition, the pairwise distinguishable t-length messages form a clique in G^t for a channel modeled by a graph G, so the question is to determine the clique number $\omega(G^t)$.

One can easily see that this value is always at most $|V(G)|^t$. Furthermore, the clique number is super-multiplicative with respect to the OR-product, meaning that for every pair of graphs G and H, the inequality $\omega(G \cdot H) \geq \omega(G) \cdot \omega(H)$ holds. So it makes sense to normalize this value by taking the t^{th} root. In fact, we are interested in the asymptotics of this value. The formal definition of the Shannon capacity is given below. (In the literature it is sometimes defined differently, by the complementary graph.)

Definition 3.1.2. *The Shannon capacity of a graph G is defined as*

$$C(G) := \limsup_{t\to\infty} \sqrt[t]{\omega(G^t)}.$$

The value of the Shannon capacity is unknown even for graphs with a very simple structure, for example the exact value is not known for any odd cycle longer than 5 (the case of the 5-cycle is a famous result of László Lovász [Lov79]). From the work of Bohman and Holzman [BH03] we know that the Shannon capacity of odd cycles (or their complements in the different interpretation of the problem) is strictly greater than its trivial lower bound 2. This lower bound, given by Bohman and Holzman, was recently improved in [Zhu25]. Due to the considerable difficulty to determine this parameter, even in smaller cases, it is already an interesting result if only some bound is given. It follows from the definition of Shannon capacity, that $\omega(G)$, the clique number of the graph G, is always a lower bound. And certain graph coloring parameters can serve as upper bounds.

Lemma 3.1.1. *Let $\varphi(G)$ be a graph parameter. If the following two conditions hold then $C(G) \leq \varphi(G)$.*

1. $\omega(G) \leq \varphi(G)$,
2. $\varphi(G \cdot H) \leq \varphi(G) \cdot \varphi(H)$ *holds for every pair of graphs G and H.*

The fractional chromatic number satisfies these two conditions, therefore, that as well as $\chi_k(G)/k$ for every k are all upper bounds for this difficult to determine parameter.

Remark. An interesting fact is that the chromatic number (as a special case of $\chi_k(G)/k$ where $k = 1$) also satisfies these conditions. Hence, for those graphs where $\omega(G) = \chi(G) = c$ the Shannon capacity is known, $C(G) = c$ as well. This was the original motivation of Claude Berge to investigate perfect graphs (cf. [Ber97]).

It is also worth noting that the fractional chromatic number of graphs can be interpreted as an information theoretic parameter. In the case where feedback is allowed on the channel, a single graph alone cannot fully model it. However, among the memoryless channels that can be modeled by a given graph it will be true for the worst one that the fractional chromatic number gives the theoretical upper bound on the rate at which information can be transmitted over that channel with zero error probability [Sha56]. Furthermore, this parameter is similar to the Shannon capacity in another way as well, as it can be expressed as the normalized value of the chromatic number of a corresponding power graph [BS74; MP71].

Remark. As it was mentioned in the Introduction, Hedetniemi-type conjectures can be formulated for other graph parameters as well. The question is interesting whenever the value of that parameter for the product is at most as large as the values of the parameter for the factors. The Shannon capacity satisfies this condition. However, we do not know if the analogous conjecture holds for the Shannon capacity or not. In [Sim21] a lower bound on the Shannon capacity of a product graph were given and some graphs are shown that may provide counterexamples.

4 Publication List

Number of publications:	5
Number of peer-reviewed journal papers (written in English):	3
Number of articles in journals indexed by WoS or Scopus:	3
Number of publications (in English) with at least 50% contribution of the author:	2
Number of peer-reviewed publications:	3
Number of citations:	12
Number of independent citations:	9

4.1 Publications Linked to the Theses

Journal Papers

[*j*1] Anna Gujgiczer and Gábor Simonyi. On multichromatic numbers of widely colorable graphs. *Journal of Graph Theory* 100(2), 2022, pp. 346–361. DOI: 10.1002/jgt.22785.

[*j*2] Noga Alon, Anna Gujgiczer, János Körner, Aleksa Milojević, and Gábor Simonyi. Structured codes of graphs. *SIAM Journal on Discrete Mathematics* 37(1), 2023, pp. 379–403. DOI: 10.1137/22M1487989.

[*j*3] Anna Gujgiczer and Gábor Simonyi. Critical subgraphs of Schrijver graphs for the fractional chromatic number. *Graphs and Combinatorics* 40, 2024. DOI: 10.1007/s00373-024-02782-9.

4.2 Additional Publications (Not Linked to Theses)

International Conference and Workshop Papers

[*c*4] Anna Gujgiczer, Gábor Simonyi, and Gábor Tardos. On the generalized Mycielskian of complements of odd cycles. In: *Proceedings of the 12th Japanese-Hungarian Symposium on Discrete Mathematics and Its Applications*, pp. 485–488. 2023.

[*c*5] Anna Gujgiczer, Márton Elekes, Oszkár Semeráth, and András Vörös. Towards model-based support for regression testing. In: *24th PhD Mini-Symposium (Minisy@ DMIS 2017)*, pp. 26–29. 2017.

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