Practice session 2

Exercise 2.1: Consider the following TBox:

$$\begin{aligned}\mathsf{Father} &\equiv \mathsf{Human} \sqcap \neg\mathsf{Female} \sqcap \exists\mathsf{hasChild}.\top\\ \mathsf{FatherOfGirls} &\equiv \mathsf{Father} \sqcap \forall\mathsf{hasChild}.\mathsf{Female}.\end{aligned}$$

With respect to the above TBox consider the reasoning task of checking if FatherOfGirls is subsumed by ¬Female. First, reduce this task into a satisfiability check. Next, using the technique of expansion of named concepts, reduce this task to the satisfiability check of a concept C w.r.t. an empty TBox. What is this concept C?

Exercise 2.2: Consider the TBox $\mathcal{T}$ consisting of the following two axioms:

$$\begin{aligned}\mathsf{Person} &\equiv \exists\mathsf{hasParent}.\mathsf{Person}\\ \mathsf{Person} &\sqsubseteq \mathsf{Man} \sqcup \mathsf{Woman}.\end{aligned}$$

Transform this TBox to a form which contains subsumption axioms only. Build the $C_{\mathcal{T}}$ internalisation concept of the TBox:

$$C_{\mathcal{T}} = (\neg C_1 \sqcup D_1) \sqcap (\neg C_2 \sqcup D_2) \sqcap \cdots \sqcap (\neg C_n \sqcup D_n),$$

assuming that the TBox $\mathcal{T}$ contains the concept subsumption axioms $C_i \sqsubseteq D_i$, $i = 1, \ldots, n$.

Exercise 2.3: Consider the task of checking the satisfiability of the concept C w.r.t. the TBox $\mathcal{T} = \{T\}$, where

$$T = \exists \mathsf{hasChild}^{-}.\mathsf{Optimist} \sqsubseteq \forall \mathsf{hasChild}.\mathsf{Optimist}$$
$$C = \mathsf{Optimist}.$$

(a) Construct the $C_{\mathcal{T}}$ internalisation concept of $\mathcal{T}$.

(b) Use the technique of internalisation to reduce the above satisfiability problem to a satisfiability problem w.r.t. a TBox containing no concept axioms, i.e. construct the concept $C_{C,\mathcal{T}}$ and the TBox $\mathcal{T}_{C,\mathcal{T}}$:

$$C_{C,\mathcal{T}} = C \sqcap C_{\mathcal{T}} \sqcap \forall U.C_{\mathcal{T}}. \tag{2.1}$$

Recall that $\mathcal{T}_{C,\mathcal{T}}$ contains all the role axioms in $\mathcal{T}$, the axiom $\mathsf{Trans}(U)$, and the axiom $R \sqsubseteq U$ for every role R which appears in either C or $\mathcal{T}$. If inverse roles are also permitted by the DL language in question, then the axiom $\mathsf{Inv}(R) \sqsubseteq U$ is also to be included in $\mathcal{T}_{C,\mathcal{T}}$, for every role R appearing in either C or $\mathcal{T}$.

(c) Build an interpretation $\mathcal{I}_0$ consisting of a child, a parent, and a grandparent. Select a non-empty interpretation for concept C so that $\mathcal{I}_0$ satisfies $\mathcal{T}$. Extend $\mathcal{I}_0$ to an interpretation $\mathcal{I}_1$ that demonstrates that $C_{C,\mathcal{T}}$ is satisfiable w.r.t. $\mathcal{T}_{C,\mathcal{T}}$.

(d) Construct an interpretation $\mathcal{I}_2$ that demonstrates that $C_{C,\mathcal{T}}$ is satisfiable w.r.t. $\mathcal{T}_{C,\mathcal{T}}$ (i.e. $\mathcal{I}_2 \models \mathcal{T}_{C,\mathcal{T}}$ and $(C_{C,\mathcal{T}})^{\mathcal{I}_2}$ is non-empty), but which, at the same time, is *not* a model of $\mathcal{T}$. Try to make the domain of $\mathcal{I}_2$ as small as possible.

(e) Consider the following interpretation $\mathcal{I}_3$:

$$\Delta^{\mathcal{I}_3} = \{\mathtt{a}_1, \mathtt{a}_2, \mathtt{a}_3, \mathtt{b}_1, \mathtt{b}_2, \mathtt{b}_3, \mathtt{c}_1, \mathtt{c}_2, \mathtt{c}_3,\}$$
$$\mathsf{hasChild}^{\mathcal{I}_3} = \{\langle \mathtt{a}_1, \mathtt{a}_2\rangle, \langle \mathtt{a}_2, \mathtt{a}_3\rangle, \langle \mathtt{b}_1, \mathtt{b}_2\rangle, \langle \mathtt{b}_2, \mathtt{b}_3\rangle, \langle \mathtt{c}_1, \mathtt{c}_2\rangle, \langle \mathtt{c}_2, \mathtt{c}_3\rangle\}$$
$$\mathsf{Optimist}^{\mathcal{I}_3} = \{\mathtt{a}_1, \mathtt{a}_2, \mathtt{b}_1, \mathtt{c}_2\}$$
$$U^{\mathcal{I}_3} = \{\langle \mathtt{a}_i, \mathtt{a}_j\rangle | i \neq j\} \cup \{\langle \mathtt{b}_i, \mathtt{b}_j\rangle | i \neq j\} \cup \{\langle \mathtt{c}_i, \mathtt{c}_j\rangle | i \neq j\}.$$

Show that $\mathcal{I}_3$ is a model of the TBox $\mathcal{T}_{C,\mathcal{T}}$. Using the interpretation $\mathcal{I}_3$ construct the meanings of the three subterms of $C_{C,\mathcal{T}}$, as shown in (2.1). Build the interpretation of $C_{C,\mathcal{T}}$ in $\mathcal{I}_3$.

Does $\mathcal{I}_3$ demonstrate that $C_{C,\mathcal{T}}$ is satisfiable w.r.t. $\mathcal{T}_{C,\mathcal{T}}$? Is $\mathcal{I}_3$ a model of $\mathcal{T}$?

Try to transform the interpretation $\mathcal{I}_3$ in such a way that another interpretation $\mathcal{I}'_3$ is obtained that is a model of $\mathcal{T}$ and in which C is non-empty.