Brownian motion with special boundary behaviour, Part II

concept paper for the seminar Stochastische Prozesse und ihre Anwendungen

András Tóbiás

11 February 2016

My task is to present the Sections 3.5.3 and 3.5.4. of Liggett's book. I would like to give a beamer talk, and to draw several graphs to the blackboard. In bullet points, my concept is to include the following four parts:

1. Solving Exercise 3.65: generator of the Euclidean norm of Brownian motion in higher dimensions.
2. (Optional, according to how much time I have. Logically it should come here, but I will shift it to the end of the talk.) Solving Exercise 3.63. b, c: martingales associated to $Y(t) - X(t)$ where $\{X(t)\}$ is a Brownian motion and $\{Y(t)\}$ is its running maximum process.
3. Proving Theorem 3.53.: path continuity of the processes which have generator $\mathcal{L}f(x) = \frac{1}{2}c(x)f''(x)$.
4. Proving Theorem 3.66.: construction of diffusions with generator $\mathcal{L}f(x) = \frac{1}{2}c(x)f''(x)$ under certain conditions, using discrete approximations associated to continuous time Markov chains.

In more detail, I plan to present these four parts as follows.

1 Solution of Exercise 3.65

Exercise (Liggett 3.65)**.** Let $X_1(t), \dots, X_n(t)$ be independent one-dimensional Brownian motions, and consider $Y(t) = \sqrt{\sum_{j=1}^n X_j^2(t)}$. Show that if $f \in C^2[0,\infty)$ has compact support and satisfies $f'(0) = 0$, then f is in the domain of the generator $\mathcal{L}$ of Y, and we have $\mathcal{L}f(y) = \frac{1}{2}f''(y) + \beta\frac{f'(y)}{y}$. Specify $\beta = \beta(n)$.

The problem for $n = 1$, has already been solved by Adrián in the first part the talk, then we have $\beta(1) = 0$ and the process $X_1(t)$ is a reflected Brownian motion. Thus, we may focus on the case $n \geq 2$. We write $\mathbf{X}(t) = (X_1(t), \dots, X_n(t))$ and $|\cdot|$ for Euclidean norm.

Solution. First, by Itō's formula, for $Y^2 = \sum_{j=1}^n X_j^2$ we have

$$\mathrm{d}Y(t)^2 = 2\sum_{i=1}^n X_i(t)\mathrm{d}X_i(t) + \sum_{i=1}^n d[X_i](t) = 2\sum_{i=1}^n X_i(t)\mathrm{d}X_i(t) + ndt.$$

The cross-covariations are zero because of the independence of $X_1, \dots, X_n$.

We claim that

$$\{W_t\} = \left\{ \sum_{i=1}^n \int_0^t \frac{X_i(t)}{|\mathbf{X}(t)|} \mathbb{1}_{\{(\mathbf{X}(t) \neq \mathbf{0})\}} \mathrm{d}X_i(t) \right\}$$

is a continuous local martingale with quadratic variation 1. In fact, it is a continuous local martingale as a sum of stochastic integrals of bounded functions w.r.t. independent Brownian motions. Further, the sum of the quadratic variation of W is given as the sum of the quadratic variations of the summands since

$X_1, \ldots, X_n$ are independent. The quadratic variation of the ith summand equals $\int_0^t \frac{X_i^2(s)}{\sum_{j=1}^n X_j^2(s)} \mathrm{d}s$, thus the q.v. of W_t equals $\int_0^t \frac{\sum_{j=1}^n X_j^2(s)}{\sum_{j=1}^n X_j^2(s)} \mathrm{d}s = t$. By Lévy's characterization, W is a Brownian motion.

Rewriting the sde for Y^2, we get

$$\mathrm{d}Y^2(t) = 2Y(t)\mathrm{d}W(t) + n\mathrm{d}t.$$

Applying Itō's formula to the Itō process Y^2 yields (writing $f : x \mapsto \sqrt{x}$)

$$\mathrm{d}Y(t) = \mathrm{d}\sqrt{Y(t)^2} = \underbrace{\frac{1}{2}\frac{1}{Y(t)}2Y(t)\mathrm{d}W(t) + \frac{1}{2}\frac{1}{Y(t)}n\mathrm{d}t}_{=f'(Y^2(t))\mathrm{d}X(t)} + \underbrace{\frac{1}{2} \times \frac{-1}{4}\frac{1}{Y(t)^3} \times 4Y(t)^2\mathrm{d}t}_{\frac{1}{2}f''(Y^2(t))\mathrm{d}[Y](t)} = \mathrm{d}W_t + \frac{n-1}{2}\frac{\mathrm{d}t}{\mathrm{d}X_t}$$

Thus, Dynkin's formula yields $\mathcal{L}f(y) = \frac{1}{2}f''(y) + \frac{n-1}{2}\frac{f'(y)}{y}$, i.e. $\beta = \frac{n-1}{2}$.

Since $Y(t) = |\mathbf{X}(t)|$, for all $x \in \mathbb{R}^n$ and $f \in C^2([0,\infty))$ with compact support, we have $\mathbb{E}^x[f(Y(t))] = \mathbb{E}^x[f(|X(t)|)]$. In order to ensure that $\mathbb{R}^n \mapsto [0,\infty)$, $x \mapsto f(|x|)$ be twice continuously differentiable, we need to have $f'(0) = 0$. No further conditions needed for a C^2 function f with compact support. This explains why the one given in the exercise is the domain of $\mathcal{L}$.

2 Solution of Exercise 3.63 b, c

Exercise (Liggett 3.63 b)**.** Consider a stochastic process $\{X(t), Y(t)\}$ on $S = \{x, y \in \mathbb{R}^2 : x \le y\}$ (upper diagonal of $\mathbb{R}^2$), such that for $x \le y$, under $\mathbb{P}_{(x,y)}$, $\{X(t), Y(t)\}$ has the following distribution starting at (x, y): $X(t)$ be a standard Brownian motion, and $Y(t) = \max\{y, \max_{0 \le s \le t} X(s)\}$.

Let g be a $2\times$ continuously differentiable function on $\mathbb{R}$ with compact support. We show:

$$g(Y(t)) - g'(Y(t))(Y(t) - X(t))$$

is a martingale.

Solution. Under $\mathbb{P}_{(x,y)}$ this is clearly true under the event $\{Y = y\}$, since X is a martingale. Thus from now on we assume that $y = 0$; if we show that the claim holds in this case, it also holds for $y > 0$.

Using the product formula, we conclude that

$$\begin{aligned}
&g(Y(t)) - g'(Y(t))(Y(t) - X(t)) \\
&\quad = g(Y(t)) - \underbrace{\int_0^t (Y(s) - X(s))\mathrm{d}g''(Y_s)}_{=0} - \int_0^t g''(Y_s)\mathrm{d}(Y(s) - X(s)) - \underbrace{[g''(Y_t), (Y(t) - X(t)]}_{=0} \\
&\quad = g(Y(t)) - \int_0^t g'(Y(s))\mathrm{d}Y(s) + \int_0^t g'(Y(s))\mathrm{d}X(s) \\
&= \int_0^t g'(Y(s))\mathrm{d}Y(s) + \underbrace{\frac{1}{2}\int_0^t g''(Y(s))\mathrm{d}[Y]_s}_{=0} - \int_0^t g'(Y(s))\mathrm{d}Y(s) + \int_0^t g'(Y(s))\,\mathrm{d}X(s) = \int_0^t g'(Y(s))\mathrm{d}X(s)
\end{aligned} \tag{1}$$

is a local martingale. Since g is bounded, also a true martingale[1].

Exercise (Liggett 3.63 c)**.** Let ξ be a random variable with values in an interval $[a, b]$ with $a \le 0 \le b$, furthermore $\mathbb{E}[\xi] = 0$ and ξ has a strictly positive density f on $[a, b]$. Show that $X(\tau) \overset{d}{=} \xi$ for the following stopping time: $\tau = \min\{t \ge 0 : Y(t) \ge \Psi(X(t))\}$, where $\Psi(u) = \begin{cases} \mathbb{E}[\xi|\ \xi \ge u] & \text{if } u < b, \\ u, & \text{if } u \ge b. \end{cases}$ [2].

[1] We note that $Y(t)$ increases only when $X(t) = Y(t)$, and this ensures that all stochastic integrals in (1) make sense.

[2] In the talk I will also explain what is the Skokhorod embedding problem and that here we describe its Azéma–Yor solution.

Solution. For $u \geq 0$, approximating the function $x \mapsto (x-u)^+$ by C^2 functions with compact support, we have by part b) that the stochastic process $\{X(t)\mathbb{1}_{\{Y(t)\geq u\}} - u\mathbb{1}_{\{Y(t)\geq u\}}\}$, which is dominated by the process $\{X(t)+u\}$, is a martingale. Hence, Doob's stopping theorem gives

$$\begin{aligned}0 = \mathbb{E}_{(0,0)}[X(0)\mathbb{1}_{\{Y(0)\geq u\}} - u\mathbb{1}_{\{Y(0)\geq u\}}] &= \mathbb{E}_{(0,0)}(X(\tau)\mathbb{1}_{\{Y(\tau)\geq u\}} - u\mathbb{1}_{\{Y(\tau)\geq u\}} \\ &= \mathbb{E}_{(0,0)}(X(\tau)\mathbb{1}_{\{Y(\tau)\geq u\}} - u\mathbb{P}_{(0,0)}(Y(\tau)\geq u).\end{aligned}$$

Hence, for $u \leq b$ we have $u\mathbb{P}(Y(\tau) \geq \Psi(u)) = \mathbb{E}(X(\tau)\mathbb{1}_{\{Y(\tau)\geq\Psi(u)\}}) = \mathbb{E}(X(\tau)\mathbb{1}_{\{X(\tau)\geq u\}})$.

Differentiating (and noting that the right hand side equals $\int_u^b x\ f_{X(\tau)}(x)\mathrm{d}x$), where $f_{X(\tau)}$ is the density of $(X(\tau))$, gives

$$\Psi'(u)\mathbb{P}_{(0,0)}(Y(\tau) \geq \Psi(u)) - \Psi(u)\mathbb{P}_{(0,0)}(Y(\tau) \in \Psi(\mathrm{d}u)) = -u\mathbb{P}_{(0,0)}(X(\tau) \in \mathrm{d}u).$$

That is, $\mathbb{P}_{(0,0)}(X(\tau) \geq u)\Psi'(u) = [\Psi(u)-u]\mathbb{P}_{(0,0)}(X(\tau) \in \mathrm{d}u)$. On the other hand, elementary computations show that

$$\mathbb{P}(\xi \geq u)\Psi'(u) = [\Psi(u) - u]\mathbb{P}(\xi \in \mathrm{d}u).$$

Consequently, ξ and $X(\tau)$ have the same distribution relative to $\mathbb{P}_{(0,0)}$.

3 Sufficient condition for path continuity of Feller processes

Theorem 1 (Liggett 3.53.)**.** *Suppose $c(\cdot) \in C(\mathbb{R})$ and $0 \leq c(x) \leq K$ for all $x \in \mathbb{R}$, and that $X(t)$ is a Feller process that has generator*

$$\mathcal{L}f(x) = \frac{1}{2}c(x)f''(x) \tag{2}$$

when restricted to C^2 functions with compact support. Then $X(t)$ is a diffusion process.

Here I will follow the proof of Liggett's book, after revising the claim of the Kolmogorov-Centsov theorem.

4 Existence of diffusions

Theorem 2 (Liggett 3.66)**.** *Suppose $c(\cdot)$ is a strictly positive and uniformly bounded C^2 function on $\mathbb{R}$ such that the first three derivatives of $\log c(x)$ are uniformly bounded by $K \geq 0$. Define $\mathcal{L}$ as in (2), for C^2 functions on $\mathbb{R}$ with compact support. Then the closure of $\mathcal{L}$ is the probability generator of a diffusion process.*

Here I will also follow the proof in the book, omitting some uninteresting computations about exact bounds on derivatives, but showing the entire construction using the discrete approximation by continuous time Markov chains. In order to provide a better understanding, I will do the resolvent computation (top of page 131 of Liggett) in more detail than it is in the book. My goal is also to explain about every point of the definition of probability generator why it is satisfied under the conditions of the theorem.

I have described the following part of Liggett's book and used the following references during solving the exercises.

References

[1] T. M. Liggett. *Continuous Time Markov Processes: An Introduction.* AMS Graduate Studies in Mathematics, volume 113, 2010, pp. 124, 128–132.

[2] J. Obłój. *Are the Azéma–Yor processes truly remarkable?* Presentation at the Swiss Probability Seminar, 2007.

[3] A. Göing-Jaeschke and M. Yor. A Survey and Some Generalizations of Bessel Processes. *Bernoulli* Vol. 9, No. 2 (Apr., 2003), pp. 313-349.