Impact of Interferences on Connectivity in Ad Hoc Networks

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***Abstract*—We study the impact of interferences on the connectivity of large-scale ad hoc networks, using percolation theory. We assume that a bi-directional connection can be set up between two nodes if the signal to noise ratio at the receiver is larger than some threshold. The noise is the sum of the contribution of interferences from all other nodes, weighted by a coefficient γ, and of a background noise.**

We find that there is a critical value of γ above which the network is made of disconnected clusters of nodes. We also prove that if γ is nonzero but small enough, there exist node spatial densities for which the network contains a large (theoretically infinite) cluster of nodes, enabling distant nodes to communicate in multiple hops. Since small values of γ cannot be achieved without efficient CDMA codes, we investigate the use of a very simple TDMA scheme, where nodes can emit only every nth time slot. We show that it achieves connectivity similar to the previous system with a parameter γ/n.

***Index Terms*—Ad hoc networks, CDMA, connectivity, interferences, percolation, TDMA.**

I. Introduction

RANDOM graphs associated with the Poisson Boolean model and percolation properties of these graphs have been considered in [1] for analyzing the connectivity of ad hoc networks. Within this context, the Poisson Boolean model assumes that the stations are located according to a planar Poisson point process, and that each station has an independent random power, identically distributed for all stations.

A more physical model based on the signal to interference ratio was used within the context of ad hoc networks in [2]. In this last paper, which departs from a deterministic and finite population setting, all stations are assumed to have the same power, and some attenuation function is given. Station A can receive a signal from station B if the ratio of the power it receives from B to the total power received from all other stations is above a threshold.

The same physical model was analyzed in [3] in the infinite plane case under Poisson assumptions within the context of CDMA networks. The corresponding coverage process has connection with Poisson shot noise processes.

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The aim of the present paper is to bring all these approaches together and to study the connectivity of infinite ad hoc networks under the physical model alluded to above. The parametric setting will be that of an homogeneous Poisson point process. Our main goal within this context is to learn whether the percolation phenomenon that was established in [1] for the case without interference still holds within this more realistic context.

By analogy with CDMA networks, we will introduce some orthogonality factor γ, which can vary from 0 to 1, and which stems from the imperfect orthogonality of the codes used in CDMA. The case with $\gamma = 0$ (perfect orthogonality) boils down to the case considered in [1].

As we will see, there are essential differences between the case $\gamma = 0$ and $\gamma > 0$. In some natural cases, for the same patterns, the first case could have an infinite component of the connectivity graph, whereas the second one could have no infinite component, or even no connectivity at all.

The main result of the paper is that under attenuation functions with finite support, percolation holds under conditions similar to those of the Boolean model of [1] provided the orthogonality factor γ is small enough. In this sense, connectivity of ad hoc networks scales well with the size of the network even in the case of models that take interferences into account. The question whether this also holds true for attenuations of the type considered in practice (e.g., power functions with parameter between 3 and 6), over an infinite support, is still an open problem at this time.

The type of random graphs that are introduced in the paper are of independent interest. In particular, this class of random graphs which are built on the points of a Poisson point process, may simultaneously have infinite components, bounded range (each edge is of bounded length), and bounded degree (each vertex is of bounded degree).

As is it an essential feature, connectivity has received quite a lot of attention in the previous decade already, in the context of packet radio networks, and has gained renewed interest recently in the context of ad hoc and sensor networks. Most results apply to the full connectivity of a network made of a finite number of nodes. A recursive formula giving the average number of hops between two connected nodes is found in [4], whereas the probability that a given number of nodes on a finite interval are all connected is computed in [5]. In the 2-dim. setting, relations between k-connectivity (the property that the graph has a minimal cutset equal to $k \geq 1$) and the node degree are studied in [6], whereas this problem is addressed when the transmission powers of the nodes are different in [7]. When the number of nodes N tend to ∞, and when the distance r below which nodes can connect decreases at a rate slower than $\sqrt{\log N/N}$, Gupta and Kumar have proven that all nodes are almost surely connected [8]. In this paper, we assume that the number N of nodes is not fixed nor on a finite area, but that they are given as points of a Poisson process over the plane $\mathbb{R}^2$. We do not make assumptions on its intensity, so that our results also apply to low density areas. Since the number of nodes is not bounded, some of them will be disconnected. The problem is then related to percolation theory, which is to find the probability that a node belongs an infinite cluster of nodes. Since the pioneering work of Gilbert [9], which started the field of continuum percolation, the exact value of this probability is still an open problem. Some bounds on the critical intensity λ^* below which it is zero have been obtained analytically in [9]–[11] for the Boolean Poisson Model, and numerically by many others [12]. Percolation of a clustered wireless network, in which the users (clients), who are distributed according to a Poisson process, are all covered by base stations that can connect to each other by a wireless link, is studied in [13]. This model reduces to the Poisson Boolean Model if one base station is placed at each client. To our knowledge, the percolation problem has not been addressed so far when interferences from other nodes are taken in account, which is the goal of this paper.

This paper is structured as follows. Section II describes the physical model considered for transmission between two nodes, from which the Poisson Signal To Interference Ratio Graph (STIRG) $\mathcal{G}(\gamma, \lambda)$ is derived. We obtain a bound on the node degree, which shows that is γ is too large, the network is surely disconnected. We also show that for some attenuation functions, no connection is possible for any $\gamma > 0$. It is important to know if a small but nonzero value of γ still enables long range connectivity. In Section III, we prove that it is fortunately the case. We begin this section by some qualitative observations on simulations, and then formally prove our main result. We also drive several bounds on the critical threshold, and describe its asymptotic behavior for large node densities.

Since percolation may hold for very small values of $\gamma > 0$, narrowband communications may not be possible if we let all nodes emit simultaneously. A remedy is to use TDMA, so that each node is allowed to emit every nth time slot. We show in Section IV that a very simple TDMA scheme achieves connectivity similar to the previous one, with γ/n. We prove formally that if the node density is sufficiently high, and whatever the value of γ is, one can find n such that percolation occurs. Finally, we draw some conclusions and future perspective in Section V.

II. Model

We consider a multiple-hop ad hoc network where nodes are distributed according to a Poisson point process of constant spatial intensity λ. Depending on its location, number of neighbors, and battery level, each node i will adjust its emitting power P_i within a given range $[0, P]$, where P is the maximal power of a node, which is finite. The power of the signal emitted by Node i and received by Node j is $P_i L(\boldsymbol{x}_i - \boldsymbol{x}_j)$, where $\boldsymbol{x}_i$ and $\boldsymbol{x}_j$ are the positions of Node i and j in the plane, respectively, and $L(\cdot)$ is the attenuation function in the wireless medium.

We assume that Node i can transmit data to Node j if the signal received by j is strong enough, compared to the thermal noise. Formally, this condition is written as

$$\frac{P_i L(\boldsymbol{x}_i - \boldsymbol{x}_j)}{N_0 + \gamma \sum_{k \neq i,j} P_k L(\boldsymbol{x}_k - \boldsymbol{x}_j)} \geq \beta \tag{1}$$

where N_0 is the power of the thermal background noise and β is the signal to noise ratio required for successful decoding. The coefficient γ is the inverse of the processing gain of the system, it weights the effect of interferences, depending on the orthogonality between codes used during simultaneous transmissions. It is equal to 1 in a narrowband system, and is smaller than 1 in a broadband system that uses CDMA. The physical model of Gupta and Kumar [2] assumes $\gamma = 1$; other models [14] allow γ to be smaller than 1.

Similarly, Node j can transmit data to Node i if and only if

$$\frac{P_j L(\boldsymbol{x}_j - \boldsymbol{x}_i)}{N_0 + \gamma \sum_{k \neq i,j} P_k L(\boldsymbol{x}_k - \boldsymbol{x}_i)} \geq \beta. \tag{2}$$

From conditions (1) and (2), we can build an oriented graph that summarizes the available links between nodes. In order to define *connected components* (or *clusters*), we have to introduce a symmetric relation. In this paper, we choose to neglect unidirectional links, which are difficult to exploit in wireless networks [15]. In other words, we declare that Node i and Node j are *directly connected* if and only if both (1) and (2) are satisfied. This new relation leads to the definition of a nonoriented random graph associated with the Poisson point process. This Poisson signal-to-interference ratio graph (STIRG) is the main object of study in the present paper.

As our model has much more parameters than degrees of freedom, we will focus on the node density λ and the orthogonality factor γ. The other parameter are supposed constant in the sequel. We will thus denote by $\mathcal{G}(\gamma, \lambda)$ the connectivity graph.

A. Bound on the Degree of the Nodes

In the following theorem, we will prove that if $\gamma > 0$, the number of neighbors of each node is bounded from above (note that this is not the case in the Boolean Model with $\gamma = 0$).

Theorem 1: Each node can have at most $1 + 1/\gamma\beta$ neighbors.

Proof: Pick any node (called hereafter Node 0), and let N be the number of its neighbors (i.e., the number of nodes to which Node 0 is connected). If $N \leq 1$, the claim is trivially proven. Suppose next that $N > 1$, and denote by 1 the node whose signal power received by Node 0 is the smallest but is nonzero, namely is such that

$$P_1 L(\boldsymbol{x}_1 - \boldsymbol{x}_0) \leq P_i L(\boldsymbol{x}_i - \boldsymbol{x}_0), \quad i = 2, \ldots, N. \tag{3}$$

Since it is connected to Node 0, (1) imposes that

$$\frac{P_1 L(\boldsymbol{x}_1 - \boldsymbol{x}_0)}{N_0 + \gamma \sum_{i=2}^{\infty} P_i L(\boldsymbol{x}_i - \boldsymbol{x}_0)} \geq \beta. \tag{4}$$

Taking (3) into account, (4) implies that

$$\begin{aligned} P_1 L(\boldsymbol{x}_1 - \boldsymbol{x}_0) &\geq \beta N_0 + \beta\gamma \sum_{i=2}^{\infty} P_i L(\boldsymbol{x}_i - \boldsymbol{x}_0) \\ &\geq \beta N_0 + \beta\gamma(N-1) P_1 L(\boldsymbol{x}_1 - \boldsymbol{x}_0) \\ &\quad + \beta\gamma \sum_{i=N+1}^{\infty} P_i L(\boldsymbol{x}_i - \boldsymbol{x}_0) \\ &\geq \beta\gamma(N-1) P_1 L(\boldsymbol{x}_1 - \boldsymbol{x}_0) \end{aligned}$$

from which we deduce that

$$N \leq 1 + \frac{1}{\beta\gamma}.$$

■

In CDMA cellular networks, this kind of bound is known under the name of pole capacity (see, e.g., [3], [16]).

As a consequence of Theorem 1, we see that if $\gamma > 1/\beta$, each node has at most one neighbor. This is a very general and restrictive condition, that imposes the network to use efficient spread-spectrum encoding in order to keep γ small, or to introduce a scheduling between nodes to avoid having them emitting all the same time. We will investigate such a scheme in Section IV.

B. Shot-Noise

The sum in the denominator of (1) is a random variable that depends on the position of almost all nodes in the network. We can write it as $N_0 + \gamma I(\boldsymbol{x}_j) - \gamma P_i L(\boldsymbol{x}_i - \boldsymbol{x}_j)$ where

$$I(\boldsymbol{x}) = \sum_{i, \boldsymbol{x}_i \neq \boldsymbol{x}} P_i L(\boldsymbol{x}_i - \boldsymbol{x}) \tag{5}$$

is the *interference contribution*. This kind of variable is called a Poisson *shot-noise*. As it is an infinite sum, it may diverge to infinity, making connections impossible.

If we assume that the sequence $\{P_i\}$ is uniformly bounded from below by a strictly positive constant, and that $L(\cdot)$ has the form $L(\boldsymbol{x}) = l(\|\boldsymbol{x}\|)$ where $l(t)$ is a nonincreasing function of t, the necessary and sufficient condition for the sum

$$\sum_i P_i L(\boldsymbol{x}_i - \boldsymbol{x})$$

to be a.s. finite is given in [17]

$$\int_y^{\infty} l(t) t dt < \infty, \qquad \text{for a sufficiently large } y. \tag{6}$$

This condition remains valid if $\inf_i\{P_i\} = 0$ but the sequence $\{\mathbf{1}_{\{P_i < \varepsilon\}}\}$ is i.i.d and independent from the point process, for some $\varepsilon > 0$ (like in Section IV).

We notice that for $l(t) = 1/t^2$, the integral in (6) is divergent and thus no connection is possible in this case whenever $\gamma > 0$.

By letting $y = 0$ in (6), we obtain the condition for *integrability*, which is stronger. This last property holds for all stationary point processes with finite intensity see e.g., [18], and in particular in the homogeneous Poisson case.

C. Attenuation

For the attenuation, the most common function is

$$l(t) = \frac{1}{t^{\alpha}} \tag{7}$$

with α ranging from 3 to 6. It makes sense to assume attenuation to be a bounded function in the vicinity of the antenna. The following two functions:

- $l(t) = A[\max(t, r_0)]^{-\alpha}$
- $l(t) = (1 + At)^{-\alpha}$

with $A > 0$ are bounded modifications of the latter considered in [3].

III. Percolation

As our model is ergodic (it is a deterministic construction on a Poisson point process), the probability that there exists a cluster of infinite size[1] is either 0 or 1, depending on the parameters λ and γ. In the first case, as there are a.s. only finite clusters, the network is said *subcritical*, whereas in the second case, it is said *supercritical*.

In the subcritical phase, long range connections in multiple hops are not possible, contrary to the supercritical phase. It is thus a crucial property to establish in a network.

We begin this central section by the much simpler Boolean model, which is a particular case for our model when $\gamma = 0$. We then make some preliminary observations on simulations to show the difference between the graphs obtained when the interferences are neglected (which amounts to set $\gamma = 0$) or not (when $\gamma > 0$). In a third step, we prove that percolation occurs (i.e., an infinite cluster exists) for small but nonzero values of γ. We finally give some asymptotic results for large node densities.

A. Existence of a Percolation Threshold for $\gamma = 0$

Let us first note that if we let $\gamma = 0$, the model described in Section II becomes equivalent to a generalized Boolean model, where two nodes are connected if and only if they are in a ball of radius r (which can be a deterministic or random value), independently from all the other nodes. Assuming all nodes emit at the maximum power P, this radius r is then constant and found from (1) to be equal to

$$r = \sup\left\{\rho \text{ such that } l(\rho) \geq \frac{\beta N_0}{P}\right\}.$$

For example, for the attenuation function (7), this radius reads $r = (P/(\beta N_0))^{1/\alpha}$. This is the model we have studied in [1], and for which many results from continuous percolation theory apply [10]. The most important one is mentioned above, namely that there is a critical density λ^*, above which the graph contains an infinite connected component.

B. Some Observations on the Graph With $\gamma > 0$

If $\gamma > 0$, it is clear that for the same realization of the spatial point process giving the position of the nodes, the graph obtained with $\gamma > 0$ misses some edges in the graph obtained with $\gamma' = 0$. In other words, $\mathcal{G}(\gamma, \lambda) \subseteq \mathcal{G}(0, \lambda)$. As a result, it

[1]We conjecture moreover that whenever it exists, the infinite cluster is also unique. The proof of this conjecture is out of the scope of this paper.

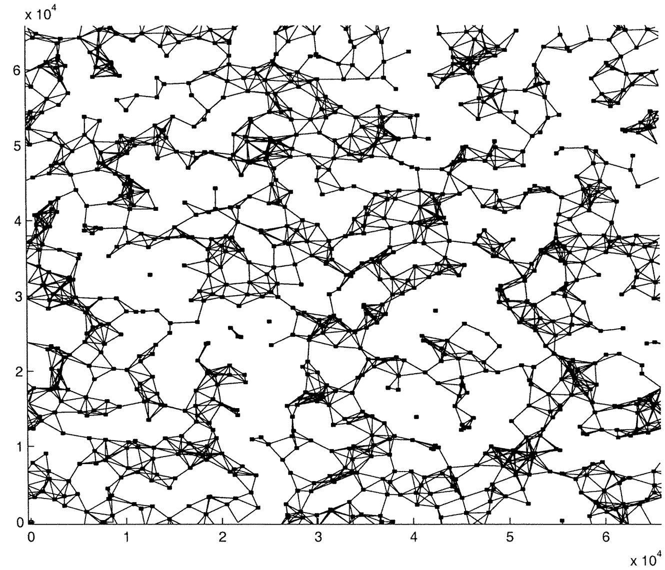

Fig. 1. Example of graph $\mathcal{G}(0, \lambda)$ with no interference (Boolean Model). As the node density is supercritical ($\lambda > \lambda^*$), most of the nodes belong to the same connected component. (This simulation was run in a square of 65 536 × 65 536 pixels with parameters $\lambda = 9.31\dot{1}0^{-4}$, $\beta = 1$, $\gamma = 0$, $N_0 = 1$, $P_i = 100\,000\ \forall\ i$.)

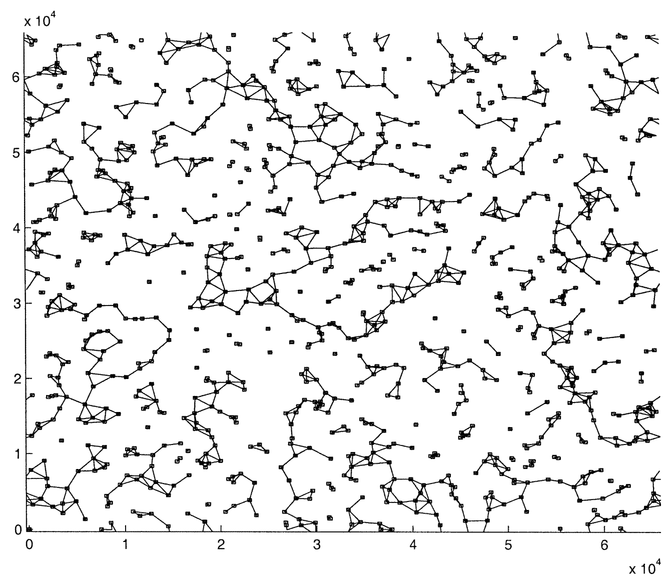

Fig. 2. Example of graph $\mathcal{G}(\gamma, \lambda)$ with interferences ($\gamma = 0.02$). This simulation was run with the same parameters as in Fig. 1, except γ that is now nonzero. Due to the interferences, the graph is split into many small components.

is not sure that percolation still occurs for nonzero values of γ. At least, for $\lambda < \lambda^*$, we are sure that $\mathcal{G}(\gamma, \lambda)$ is always subcritical. However, for $\lambda > \lambda^*$, we know that:

1) For $\gamma = 0$, the network is supercritical;
2) For $\gamma > 1/\beta$, the network is subcritical.

Therefore, there exists a critical value $0 \leq \gamma^*(\lambda) \leq 1/\beta$ at which one observes a phase transition. The Section III-C will prove that $\gamma^*(\lambda)$ is strictly positive for sufficiently large values of λ.

We have computed by simulation the value of the percolation threshold $\gamma^*(\lambda)$, with $L(\boldsymbol{x}) = \max(1, \|\boldsymbol{x}\|)^{-3}$. The simulation results are shown in Figs. 1–4. In the simulations, all nodes

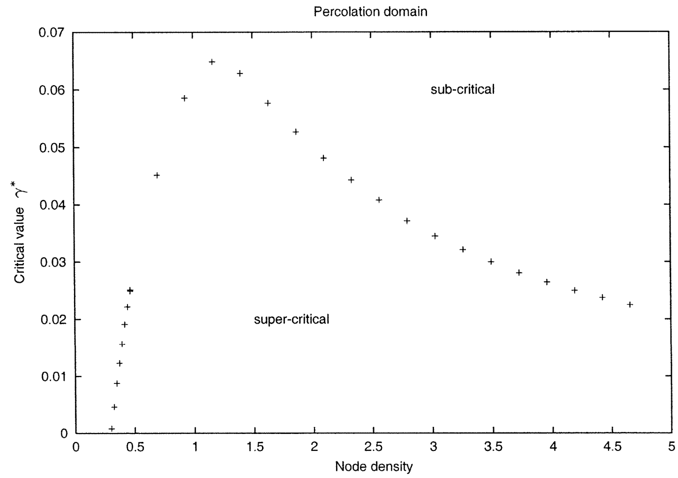

Fig. 3. Critical value of γ as a function of the node density λ. The curve shows the critical value of γ below which the network percolates. [The parameters of this simulation are $\beta = 1$, $N_0 = 10^4$ and $P_i = 10^5\ \forall\ i$].

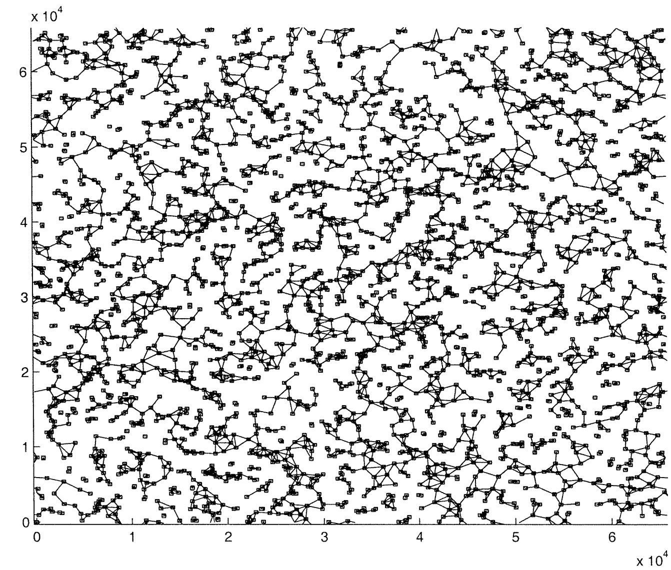

Fig. 4. Barely supercritical graph with interferences. This simulation was run with the same parameters as in Fig. 2, except that the node density is higher ($\lambda = 2.79\dot{1}0^{-3}$). The graph percolates despite the interferences because here $\gamma < \gamma^*(\lambda)$. One can observe that fewer edges are needed to achieve percolation than in Fig. 1.

emit with the same power P. We observe in Figs. 1 and 2 that $\mathcal{G}(0.02, \lambda) \subseteq \mathcal{G}(0, \lambda)$. We observe in Fig. 3 that $\gamma^*(\lambda)$ exhibit a maximum at a certain density $\tilde{\lambda}$. Below $\tilde{\lambda}$, increasing the node density helps for connectivity, whereas after the maximum, the impact of interferences becomes preponderant, and $\gamma^*(\lambda)$ becomes decreasing. Fig. 4 illustrates the percolation phenomenon with γ slightly smaller than $\gamma^*(\lambda)$.

C. Percolation for Nonzero Values of γ

We have shown above that if γ exceeds some finite, positive critical value, percolation does not occur. We want now to show that percolation can occur for nonzero values of γ. We make the simplifying assumption that every node emits at maximal power P: $P_i = P\ \forall\ i$. This corresponds to the worst power assignment for the interfering communications. We need additional assumptions on the attenuation function $L(\cdot)$. In this subsection, we restrict ourselves to nonincreasing, isotropic attenuation functions that have the following additional properties:

$$L(\boldsymbol{x}) = 0 \quad \forall\, \boldsymbol{x} \in \mathbb{R}^2 \text{ s.t. } ||\boldsymbol{x}|| > d \tag{8}$$

$$\frac{\beta N_0}{P} < L(\boldsymbol{x}) < M \quad \forall\, \boldsymbol{x} \in \mathbb{R}^2 \text{ s.t. } ||\boldsymbol{x}|| \leq \delta \tag{9}$$

for some $0 < \delta < d$ and $M > \beta N_0/P$.

We will then prove the following main theorem.

Theorem 2: If the isotropic attenuation function $L(\cdot)$ verifies assumptions (8) and (9), then there exist $\lambda' < \infty$ and a function $\gamma'(\lambda)$ such that

- $\gamma'(\lambda) > 0$ for all $\lambda > \lambda'$;
- if $\lambda > \lambda'$ and $\gamma < \gamma'(\lambda)$, there exists a.s. an infinite connected component in the graph $\mathcal{G}(\lambda, \gamma)$.

This theorem implies therefore that communication between distant nodes is still possible despite interferences. The proof of this central theorem is quite lengthy, and is therefore divided in several intermediate results. The first step is to map the process defined on the continuous plane $\mathbb{R}^2$ onto a discrete grid (lattice) $\mathcal{L}$, whose edges are declared open if certain properties of the Poisson process in their neighborhood are met. The second and more lengthy step is to prove bond percolation, that is, the existence of an infinite path made of open edges, on the dual lattice $\mathcal{L}'$. The third step is then straightforward, as the reverse mapping allows us then to conclude that the network indeed percolates and has an infinite cluster on the continuous plane $\mathbb{R}^2$. The reason for carrying most of the proof on the discrete lattice $\mathcal{L}'$ is that we can then make use of the larger collection of results found in the literature [19] on discrete bond percolation than on continuous percolation.

Step 1: mapping of the graph on a lattice

We begin by constructing a square lattice, denoted by $\mathcal{L}$ over the plane, with edge length d. Let $\mathcal{L}'$ be the dual lattice of $\mathcal{L}$, obtained by putting a vertex in the center of every square of $\mathcal{L}$, and an edge across every edge of $\mathcal{L}$. As $\mathcal{L}$ is square lattice, $\mathcal{L}'$ is simply the same lattice shifted by $d/2$ horizontally and vertically, as depicted in Fig. 5. Note that there exists a one-to-one relation between the edges of $\mathcal{L}'$ and the edges of $\mathcal{L}$. Furthermore, we set the origin O of the plane at a vertex of $\mathcal{L}'$, without any loss of generality.

Let us now consider the original Poisson point process over the plane. Each square of Lattice $\mathcal{L}$ contains in average λd^2 points. We will study bond percolation on Lattice $\mathcal{L}'$. To do this, we will declare some edges *open* and others *closed* depending on the realization of the underlying Poisson point process.

In Lattice $\mathcal{L}$, we divide again each square into K^2 subsquares of size $(d/K) \times (d/K)$, where $K \in \mathbb{N}^*$ is given by

$$K = \left\lceil \frac{\sqrt{5}d}{\delta} \right\rceil. \tag{10}$$

This value has been chosen so that $||\boldsymbol{x}|| \leq \sqrt{5}d/K$ implies that $||\boldsymbol{x}|| \leq \delta$. Next, we introduce a second integer parameter N defined by

$$N = \inf_{\boldsymbol{x} \text{ s.t. } ||\boldsymbol{x}|| \leq \sqrt{5}d/K} \left\lfloor \frac{1}{\gamma M} \left(\frac{L(\boldsymbol{x})}{\beta} - \frac{N_0}{P} \right) \right\rfloor. \tag{11}$$

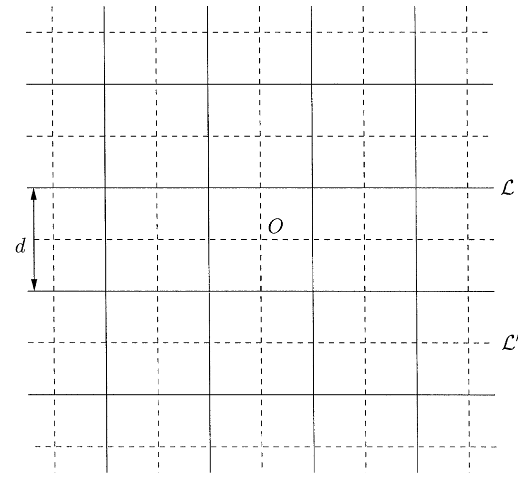

Fig. 5. Lattice $\mathcal{L}$ (plain) and its dual $\mathcal{L}'$ (dashed).

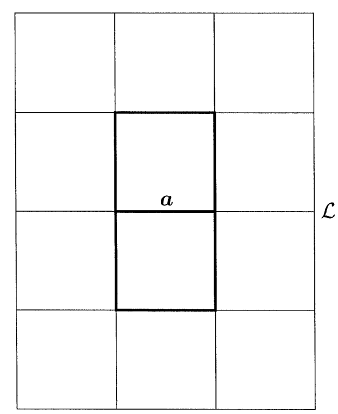

Fig. 6. Conditions for a to be open: both squares in the middle (bold line) must be populated, and the total number of points in the 12 squares must be at most N.

Because of (9) and (10), one can check that $N \in \mathbb{N}$. Combining (9) with (10) and (11), we obtain the following inequality

$$||\boldsymbol{x}|| \leq \frac{\sqrt{5}d}{K} \Rightarrow \frac{L(\boldsymbol{x})P}{N_0 + \gamma NMP} \geq \beta \tag{12}$$

which is more restrictive than the left inequality in (9).

We can now formally define our discrete percolation model from the original, continuous one by introducing a bunch of definitions. We designate by the term "point" the location of a node in the original network, to avoid any confusion with a vertex in the grid $\mathcal{L}$.

Definition 1: A square X of $\mathcal{L}$ is said to be *populated* if all its subsquares contain at least one point.

Definition 2: An edge $\boldsymbol{a}$ of $\mathcal{L}$ is said to be *open* if the following conditions are fulfilled:

- both squares adjacent to $\boldsymbol{a}$ are populated;
- the total number of points located in the two squares adjacent to $\boldsymbol{a}$ and all their direct neighbors (that is, all the squares having at least one vertex in common with these two squares, as depicted in Fig. 6) is less than or equal to $N + 1$.

Definition 3: An edge $\boldsymbol{a}'$ of $\mathcal{L}'$ is said to be *open* if and only if the corresponding edge of $\mathcal{L}$ is open.

Definition 4: A path (in $\mathcal{L}$ or $\mathcal{L}'$) is said to be *open* (resp. *closed*) if all edges forming this path are open (resp. closed).

The above definitions have been chosen such that an open edge guarantees connectivity in the continuous model (see Lemma 4 hereafter). In fact, the first condition ensures a homogeneous population in the squares, whereas the second condition puts a limit to the interference contribution. It is very important to notice that the main difference between this model and the usual discrete percolation models is that here the state of the edges (open or closed) are *not* independent from each other.

Step 2: percolation on lattice $\mathcal{L}'$

We want to know whether percolation occurs in our newly defined discrete model, namely if one can find an infinite open path in $\mathcal{L}'$. Let q be the probability that an arbitrary edge is closed. Actually, q is pretty difficult to compute, but we show in Lemma 2 that q can be made arbitrarily small by choosing suitable values of λ and γ. As a first step, we introduce the following simple but useful lemma:

Lemma 1: Let X be a Poisson random variable of parameter μ, and $0 < \varepsilon < 1$ a positive constant. Then

$$\lim_{\mu\to\infty} \mathbb{P}[X \le (1-\varepsilon)\mu] = 0$$

and

$$\lim_{\mu\to\infty} \mathbb{P}[X < (1+\varepsilon)\mu] = 1.$$

Proof: Using Chebyshev inequality

$$\mathbb{P}[|X-\mu| \ge \varepsilon\mu] \le \frac{Var(X)}{\varepsilon^2\mu^2} = \frac{1}{\varepsilon^2\mu}.$$

Thus

$$\lim_{\mu\to\infty} \mathbb{P}[|X-\mu| \ge \varepsilon\mu] = 0$$

which implies the above results. ■

We can now prove that q can be made as small as necessary.

Lemma 2: For any $q' > 0$, there exists $\lambda' < \infty$ and $\gamma'(\lambda) > 0$ such that

$$\lambda > \lambda' \text{ and } \gamma < \gamma'(\lambda) \Rightarrow q < q'.$$

Proof: Let us find a lower bound to the probability $p = 1 - q$ that an arbitrary edge of $\mathcal{L}'$ is open:

$$\begin{aligned} p &= \mathbb{P}\ (\text{an edge of } \mathcal{L}' \text{is open}) \\ &= \mathbb{P}\Big(\big\{2K^2 \text{subsquares of surface } (d/K)^2 \text{ have} \\ &\qquad \text{at least 1 point each}\big\} \text{and } \big\{\text{a surface of } 12d^2, \\ &\qquad \text{including these subsq,} \\ &\qquad \text{has no more than } N \text{ pts}\big\}\Big) \\ &\ge \mathbb{P}\Big(\big\{2K^2 \text{ subsquares have between } 1 \text{ and} \\ &\qquad \lfloor N/12K^2 \rfloor \text{ pts each}\big\} \text{ and a surface of } 10d^2, \\ &\qquad \text{excluding these subsq, has no more than} \\ &\qquad \lfloor 5N/6 \rfloor \text{ points}\big\}\Big) \\ &= \mathbb{P}^{2K^2}\Big(\text{a subsq of surface } (d/K)^2 \text{ has between} \\ &\qquad 1 \text{ and } \lfloor N/12K^2 \rfloor \text{ pts}\Big) \\ &\quad \times \mathbb{P}\Big(\text{a surface of } 10d^2 \text{ has} \\ &\qquad \text{no more than } \lfloor 5N/6 \rfloor \text{ pts}\Big) \\ &= \mathbb{P}^{2K^2}\left(1 \le X \le N/12K^2\right)\mathbb{P}\left(Y \le 5N/6\right) \end{aligned}$$

where X and Y are two independent Poisson random variables of parameter $\lambda(d/K)^2$ and $10\lambda d^2$, respectively. We take now

$$\gamma'(\lambda) = \frac{1}{12M\lambda d^2(1+\varepsilon)}\left(\frac{Pl\left(\sqrt{5}d/K\right)}{\beta} - N_0\right) \tag{13}$$

for some $\varepsilon > 0$. If $\gamma = \gamma'(\lambda)$, according to (11), we have

$$N = \lfloor 12\lambda d^2(1+\varepsilon) \rfloor.$$

and thus

$$p = \mathbb{P}^{2K^2}\left(1 \le X \le \lambda d^2(1+\varepsilon)/K^2\right)\mathbb{P}(Y \le 10\lambda d^2(1+\varepsilon)).$$

It follows from Lemma 1 that

$$\lim_{\lambda\to\infty} p = 1.$$

For any $q' > 0$, there exists thus λ' such that if $\lambda > \lambda'$, $p > 1 - q'$. As p is a decreasing function of γ, the result holds also when $\gamma < \gamma'(\lambda)$. ■

We have now to cope with the dependence between edges. We observe first that our model is k-dependent, with $k = 3$, which means that if the graphtheoretic distance between two edges is greater than 3, they are independent (see [19, p. 17]). We can then apply results in [20] and prove that our model stochastically dominates an independent model. Super-criticality of the independent model thus implies supercriticality of our model.

However, to keep this paper self-contained, we propose here a simple and constructive way to prove the existence of an infinite open path in our particular case. Moreover, this method provides us an explicit lower bound on the critical value of γ, which we will exploit in Section III-D. We start with the following lemma, which applies to paths in $\mathcal{L}$.

Lemma 3: In $\mathcal{L}$, the probability for a path of length n to be closed is less than or equal to $q^{n/70}$, where q is the probability that an arbitrary edge is closed.

Proof: Let us consider a path of length n in $\mathcal{L}$ and denote by $S = \{\boldsymbol{a}_i\}_{i=1}^n$ the set of the edges forming this path. Let $S' \subseteq S$ be a subset of S. We clearly have that

$$\begin{aligned} \mathbb{P}(\text{the path is closed}) &= \mathbb{P}(\boldsymbol{a} \text{ is closed} \quad \forall\, \boldsymbol{a} \in S) \\ &\le \mathbb{P}(\boldsymbol{a} \text{ is closed} \quad \forall\, \boldsymbol{a} \in S'). \end{aligned}$$

By construction, the event "$\boldsymbol{a}$ is closed" depends on the realization of the Poisson point process in some region of the plane, according to Definition 2. Let us call $R(\boldsymbol{a}) \subset \mathbb{R}^2$ this region. It is the rectangle shown in the middle of Fig. 7. To compute the probability of this event, we will choose S' so that $R(\boldsymbol{a}) \cap R(\boldsymbol{b}) = \emptyset$, $\forall\, \boldsymbol{a}, \boldsymbol{b} \in S'$ and $\boldsymbol{a} \ne \boldsymbol{b}$. In this way, the set of indicator random variables $\mathbf{1}_{\{\boldsymbol{a}_i \text{ is closed}\}}$, taking value 1 if with $\boldsymbol{a}_i$ is closed and 0 otherwise, with $\boldsymbol{a}_i \in S'$, are i.i.d variables with $P(\mathbf{1}_{\{\boldsymbol{a}_i \text{ is closed}\}} = 1) = q$. Therefore

$$\mathbb{P}(\boldsymbol{a} \text{ is closed} \quad \forall\, \boldsymbol{a} \in S') = q^m \quad \text{ with } m = \text{card}(S').$$

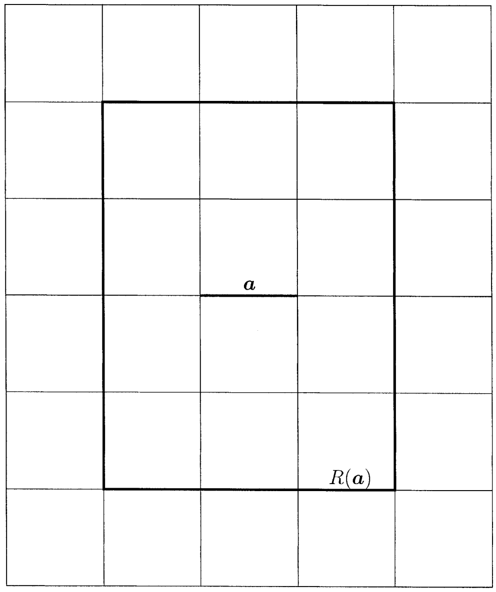

Fig. 7. Edge a and its dependency region $R(a)$. Around a, we drew all edges that have a nondisjoint dependency region with $R(a)$.

We construct S' as follows: we take the first edge of the path $\boldsymbol{a}_1 \in S'$. This edge is the center of a certain region $R(\boldsymbol{a}_1)$ of the plane, as defined above and shown in Fig. 7. Then we follow the path until we find an edge $\boldsymbol{a}_k$ such that $R(\boldsymbol{a}_k) \cap R(\boldsymbol{a}_1) = \emptyset$, and add it to S'. We iterate this last step until we reach the end of the path.

In order to find an upper bound on $\mathbb{P}(\boldsymbol{a} \text{ is closed}, \forall\, \boldsymbol{a} \in S')$, we need a lower bound on $m = \operatorname{card}(S')$. In other words, we need to know how many edges are skipped until we find the next element of S' in our construction scheme. To answer this question, we will simply count the number of edges $\boldsymbol{b}$ in $\mathcal{L}$ that satisfy $R(\boldsymbol{b}) \cap R(\boldsymbol{a}_1) \neq \emptyset$. We see on Fig. 7 that there are 70 of them. We are therefore sure that a path starting with Edge $\boldsymbol{a}_1$ cannot go through more than 70 edges without visiting an edge $\boldsymbol{a}_k$ such that $R(\boldsymbol{a}_k) \cap R(\boldsymbol{a}_1) = \emptyset$. Since the path S has n edges, m is thus bounded by

$$m \geq 1 + \lfloor (n-1)/70 \rfloor .$$

We have finally obtained the upper bound we were looking for, which reads

$$\begin{aligned}\mathbb{P}(\text{the path is closed}) &\leq \mathbb{P}(\boldsymbol{a} \text{ is closed} \quad \forall\, \boldsymbol{a} \in S') \\ &= q^m \\ &\leq q^{1+\lfloor (n-1)/70 \rfloor} \\ &\leq q^{n/70}.\end{aligned}$$

■

We can now prove the theorem.

Theorem 3: If $q < (11 - 2\sqrt{10}/27)^{70}$, the probability that there exists an infinite open path in $\mathcal{L}'$ starting at the origin is strictly greater than zero.

Proof: We will prove this theorem by contradiction: assume that there exists no infinite open path starting at the origin in $\mathcal{L}'$. Then there exists a closed circuit in $\mathcal{L}$ that surrounds the origin. In the sequel, we will find an upper bound to the probability that such a circuit exists. The result is then deduced from the following equation:

$$\begin{aligned}&\mathbb{P}(\exists \text{ an infinite open path starting at the origin in } \mathcal{L}') \\ &\quad = 1 - \mathbb{P}(\exists \text{ a closed circuit in } \mathcal{L} \text{ that surrounds the origin}).\end{aligned} \tag{14}$$

We know from [19, pp. 15–18] that the number $\rho(n)$ of circuits of length n in $\mathcal{L}$ that surround the origin is bounded from above by

$$\rho(n) \leq 4 \cdot n \cdot 3^{n-2}.$$

Among these circuits, some are closed; we denote by $M(n)$ the number of closed circuits of length n in $\mathcal{L}$ that surround the origin. Using Lemma 3, we find a bound to the probability that this number is nonzero

$$\mathbb{P}(M(n) \geq 1) \leq \rho(n) q^{n/70} \leq 4 \cdot 3^{n-2} n q^{n/70}.$$

Consequently

$$\begin{aligned}\mathbb{P}(M(n) \geq 1 \text{ for some } n) &= \sum_{n=1}^{\infty} \mathbb{P}(M(n) \geq 1) \\ &\leq \sum_{n=1}^{\infty} 4 \cdot 3^{n-2} n q^{n/70} \\ &= \frac{4}{3} q^{1/70} \sum_{n=1}^{\infty} n (3q^{1/70})^{n-1} \\ &= \frac{4q^{1/70}}{3(1-3q^{1/70})^2}.\end{aligned}$$

The above expression is strictly smaller than one if

$$q < \left(\frac{11 - 2\sqrt{10}}{27} \right)^{70}$$

in which case we can conclude using (14) that

$$\mathbb{P}(\exists \text{ an infinite open path starting at the origin in } \mathcal{L}') > 0.$$

■

From Theorem 3, we can deduce the following corollary by ergodicity.

Corollary 1: If $q < (11 - 2\sqrt{10}/27)^{70}$, there exists a.s. an infinite open cluster in $\mathcal{L}'$.

Step 3: reverse mapping and percolation on the plane

Now that there is an infinite open cluster in $\mathcal{L}'$ with positive probability, we still need to prove that this yields the existence of an infinite component in the original graph $\mathcal{G}(\gamma, \lambda)$ for suitable values of γ and λ, as defined in Lemma 2. Lemmas 4 and 5 establish this link between the discrete and the continuous models.

Lemma 4: If a square X is populated, and if the total interference level at any point of the square is less than or equal to NM, then all points in the square belong to the same cluster.

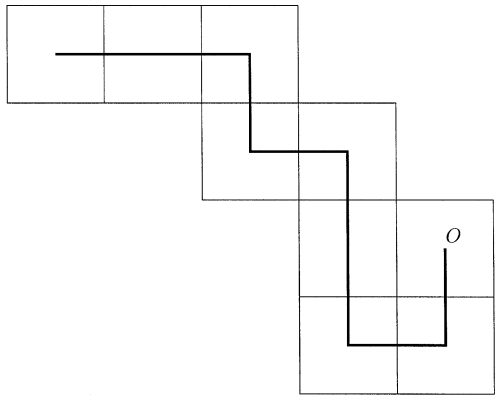

Fig. 8. Open path in $\mathcal{L}'$ (in bold) and its associated sequence of squares (whose sides are edges of $\mathcal{L}$).

Furthermore, if two adjacent squares fulfill the same conditions, all points inside these squares belong to the same cluster.

Proof: We consider two adjacent subsquares, and an arbitrary point in each of them (we know that we can find at least one point in each subsquare, because the square is populated). Because both adjacent subsquares have a side of length d/K, the distance between these two points is at most $\sqrt{5}d/K$. The signal-to-interference ratio is then

$$\frac{P_i L(\boldsymbol{x}_j - \boldsymbol{x}_i)}{N_0 + \gamma \sum_{k \neq i,j} P_k L(\boldsymbol{x}_k - \boldsymbol{x}_i)} \geq \frac{P L(\boldsymbol{x}_j - \boldsymbol{x}_i)}{N_0 + \gamma NMP} \geq \beta$$

where the second inequality follows from (12). Thus each point in a given subsquare is connected to all points in the adjacent subsquares. As a result, since the square is populated, all points in the whole square are connected together.

The second part of the lemma is quite obvious. If two squares are adjacent, we simply consider two adjacent subsquares, one in the first square, one in the other one, and apply the same arguments. ■

Lemma 5: If there exists an infinite open path in $\mathcal{L}'$, then there exists an infinite cluster in the continuous model.

Proof: We consider an infinite open path in $\mathcal{L}'$. Remember that each vertex of $\mathcal{L}'$ is located at the center of a square of $\mathcal{L}$ (see Fig. 8). Along an open path of $\mathcal{L}'$, at each vertex, we find a square that fulfills the conditions given in Definition 2. Let us consider one of these squares, which we will denote by X. As the attenuation function L is zero for distances above d, all interferences in X come from nodes located in X and its direct neighbors (adjacent squares and diagonal neighbors). As the edge is open, according to Definition 2, the total number of points in this neighborhood is less than or equal to $N+1$. The total interference contribution is thus smaller than NMP. We can then apply Lemma 4 and conclude that all points in X are connected together.

Moreover, as two consecutive squares along the infinite open path are adjacent, the sequence of squares form an infinite cluster of connected points. ■

Combining Lemmas 5, 2 and Corollary 1, we have established Theorem 2.

D. Asymptotic Results for Large λ

In Section III-C, we proved that if $\lambda > \lambda'$ and $\gamma < \gamma'(\lambda)$, the network contains an infinite cluster. The function $\gamma'(\lambda)$ is thus a lower bound on the actual threshold $\gamma^*(\lambda)$. We observe furthermore in (13) that $\gamma'(\lambda) = c_1/\lambda$ for some constant c_1.

In this section, we look for an upper bound on $\gamma^*(\lambda)$. Again, we have to assume here that $L(\boldsymbol{x})$ is decreasing with respect to $\|\boldsymbol{x}\|$ and satisfies (9).

We construct a new square lattice $\mathcal{L}''$ over the plane, similar to the previous ones, but with edge length $\delta/2$ instead of d. We assume also that the origin of $\mathbb{R}^2$ is located at the center of a square of $\mathcal{L}''$.

Lemma 6: If there are more than

$$N' = \frac{(1+2\beta\gamma)PM}{\beta^2\gamma N_0} \tag{15}$$

nodes inside a square of $\mathcal{L}''$, all nodes in this square are isolated.

Proof: Pick any node i inside the square, and another node j (inside or outside the square). As $L(\cdot)$ is bounded from above by M, we have

$$PL(\boldsymbol{x}_j - \boldsymbol{x}_i) \leq PM.$$

Because of (9), we also have

$$\begin{aligned}\sum_{k \neq i,j} PL(\boldsymbol{x}_k - \boldsymbol{x}_i) &\geq \sum_{k \text{ in the sq.}, k \neq i,j} PL(\boldsymbol{x}_k - \boldsymbol{x}_i) \\ &\geq \sum_{k \text{ in the sq.}} PL(\boldsymbol{x}_k - \boldsymbol{x}_i) - 2PM \\ &\geq N'P\frac{\beta N_0}{P} - 2PM \\ &= \beta N' N_0 - 2PM.\end{aligned}$$

Therefore, we have

$$\begin{aligned}\frac{PL(\boldsymbol{x}_j - \boldsymbol{x}_i)}{N_0 + \gamma \sum PL(\boldsymbol{x}_k - \boldsymbol{x}_i)} &\leq \frac{PM}{N_0 + \gamma(\beta N' N_0 - 2PM)} \\ &\leq \frac{PM}{\gamma(\beta N' N_0 - 2PM)}.\end{aligned}$$

The above expression is clearly smaller than β when $N' > (1 + 2\beta\gamma)PM/\beta^2\gamma N_0$, which implies that Node i is isolated. ■

We can now define a site percolation model by declaring a square of $\mathcal{L}''$ *open* if it contains at most $2N' = 2(1+2\beta\gamma)PM/\beta^2\gamma N_0$ nodes. It is declared *closed* otherwise. It is clear that each square is open or closed independently from the others. Therefore, the origin is a.s. surrounded by a closed circuit (i.e., a circuit formed by closed squares) in $\mathcal{L}''$ if

$$\mathbb{P}\text{ (a square is closed)} > p_{\text{site}} \tag{16}$$

where p_{site} is the critical site percolation threshold, whose value is around 0.59 (see [19, p. 56]). The number of nodes inside a square is a Poisson random variable of parameter $\lambda\delta^2/4$. Lemma 1 implies that if

$$2N' \leq \frac{(1-\varepsilon)\lambda\delta^2}{4} \tag{17}$$

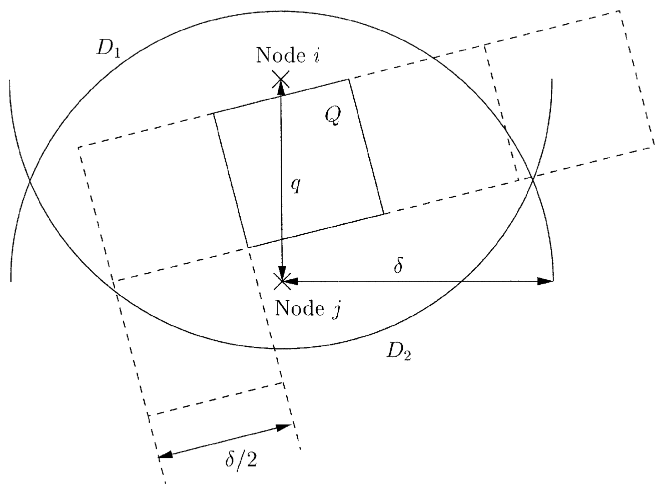

Fig. 9. Chain of closed squares separating the two nodes.

we have

$$\lim_{\lambda\to\infty} \mathbb{P}\text{ (a square is closed)} = 1$$

which means that above a certain value of λ, Inequality (16) holds.

Inequality (17) is verified if

$$\frac{2(1+2\beta\gamma)PM}{\beta^2\gamma N_0} \le \frac{(1-\varepsilon)\lambda\delta^2}{4} \tag{18}$$

which can be recast as

$$\gamma \ge \frac{8PM}{\beta[(1-\varepsilon)\beta\lambda\delta^2 N_0 - 16PM]}.$$

When $\lambda \ge 16PM/\beta\varepsilon\delta^2 N_0$, a sufficient condition is

$$\gamma \ge \frac{8PM}{(1-2\varepsilon)\beta^2\lambda\delta^2 N_0} := \frac{c_2}{\lambda}. \tag{19}$$

We thus proved that for sufficiently high densities, if $\gamma \ge c_1/\lambda$, the origin is a.s. surrounded by a closed circuit in the discrete model. We now have to prove that in this case, the origin belongs to a finite cluster in the continuous model.

Because of Lemma 6, when a site is closed, the square centered on this site contains only isolated nodes. Therefore, in the continuous model, when $\gamma \ge c_1/\lambda$, the origin is surrounded by a chain of closed squares with no link inside. To make sure that the origin belongs to a finite cluster, we have to prove that no link can cross this chain.

Let us consider two nodes i and j, such that Node i is located inside an open square surrounded by the chain, and Node j is also located inside an open square, but on the other side of the chain. As these nodes are separated by the chain of closed squares, the distance between them $q := \|x_i - x_j\|$ is larger than $\delta/2$.

We consider two cases. First, we assume that $\delta/2 < q < \delta$. In this case we construct the disk D_1 of radius δ centered on x_i and the disk D_2 of radius δ centered on x_j, as depicted in Fig. 9. As the chain of closed squares separates x_i and x_j, there exists at least one closed square Q that has a nonempty intersection with the segment $[x_i, x_j]$. Moreover, the shortest distance between $[x_i, x_j]$ and $\mathbb{R}^2\backslash(D_1 \cup D_2)$ is

$$\sqrt{\delta^2 - \frac{q^2}{4}} \ge \frac{\sqrt{3}}{2}\delta.$$

As the diagonal of Q has length $\delta/\sqrt{2}$, Q cannot have a nonempty intersection with $[x_i, x_j]$ and with $\mathbb{R}^2\backslash(D_1 \cup D_2)$ at the same time. Therefore $Q \subset D_1 \cup D_2$.

Furthermore, we count the number of nodes inside three different subsets of Q

$$\begin{aligned} N_1 &= \Phi_\lambda(Q \cap (D_1\backslash D_2)) \\ N_2 &= \Phi_\lambda(Q \cap (D_2\backslash D_1)) \\ N_3 &= \Phi_\lambda(Q \cap D_1 \cap D_2). \end{aligned}$$

As Q is a closed square, we have by assumption $N_1+N_2+N_3 \ge 2N'$. This implies that either

$$N_1 + N_3 \ge N'$$

or

$$N_2 + N_3 \ge N'.$$

Let us assume without loss of generality that the first inequality holds. There are thus at least N' nodes located inside D_1. As D_1 has radius δ, and because of (9), the signal received by Node i from each of these nodes is at least $P\beta N_0/P = \beta N_0$. The SINR at Node i received from Node j is thus upper-bounded by

$$\beta_{ji} \le \frac{PM}{N_0 + \gamma N'\beta N_0}.$$

Plugging the value of N' into this expression, we verify that

$$\beta_{ji} \le \beta$$

which means that no link between Node i and Node j exists. The same is true if $N_2 + N_3 \ge N'$.

Let us now address the case where $q > \delta$ (the case $q = \delta$ appears with probability zero). In this case, we draw the same disks D_1 and D_2, but with radius q. There exists at least one square Q of the chain such that $Q \subset D_1 \cup D_2$. We define N_1, N_2 and N_3 in the same way as above. Thus, either $N_1 + N_3 \ge N'$ or $N_2 + N_3 \ge N'$.

Let us assume without loss of generality that $N_1 + N_3 \ge N'$. This implies that there are at least N' nodes inside D_1. Node j is by construction on the border of D_1. Therefore, all these nodes are closer to Node i than Node j. As we assumed that $l(x)$ is decreasing, the SINR at i from Node j is bounded above by

$$\beta_{ji} \le \frac{Pl(q)}{N_0 + \gamma PN'l(q)} \le \frac{1}{\gamma N'}.$$

From (9) and (15), we verify that

$$N' > \frac{1}{\beta\gamma}$$

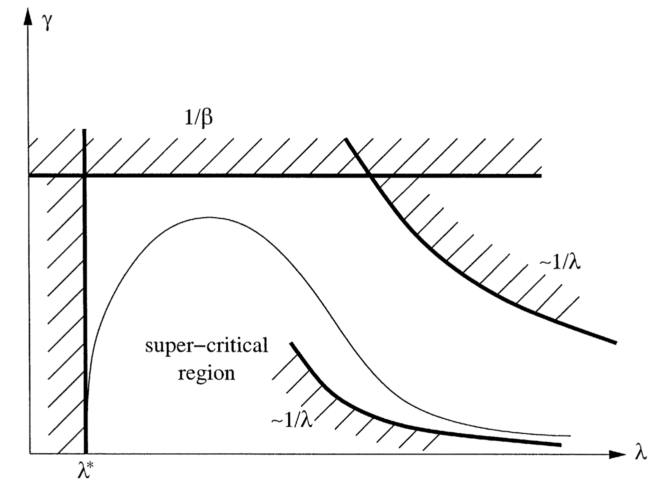

Fig. 10. Illustration of the bounds on the supercritical domain.

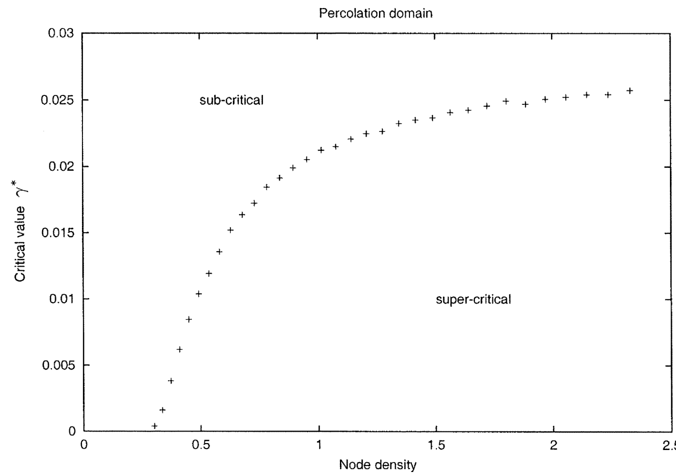

Fig. 11. Critical value of γ for an unbounded attenuation function $L(\boldsymbol{x}) = \|\boldsymbol{x}\|^{-3}$. In this case, the percolation threshold is an increasing function of the node density. [The parameters of this simulation are $\beta = 1$, $N_0 = 10^4$ and $P_i = 10^5 \ \forall\ i$].

and therefore

$$\beta_{ji} \leq \beta$$

meaning that the link cannot exist.

Consequently, we have proved that if the origin is surrounded by a chain of closed squares in the discrete model, then the continuous model is subcritical. We conclude that when (19) holds, the network is subcritical. We have thus obtained an upper bound on the critical value $\gamma^*(\lambda)$ by proving that if $\gamma \geq c_2/\lambda$, the origin belongs a.s. to a finite cluster. As both upper and lower bounds on the critical threshold have this form, we have obtained the asymptotic behavior of the threshold $\gamma^*(\lambda)$ for $\lambda \to \infty$.

Theorem 4: For λ tending to infinity, the critical value of γ has the following asymptotic behavior

$$\gamma^*(\lambda) = \Theta\left(\frac{1}{\lambda}\right).$$

Fig. 10 illustrates the typical shape of the function $\gamma^*(\lambda)$. Note that if $L(\cdot)$ does not fulfill Condition (9), the asymptotic behavior may be dramatically different. For example, take $L(\boldsymbol{x}) = 1/\|x\|^\alpha$, which is not bounded from above for small $\|\boldsymbol{x}\|$. In this case, increasing λ by a factor a is equivalent to dividing N_0 by a factor $a^{\alpha/2}$. It follows that in this case $\gamma^*(\lambda)$ is always an increasing function. Fig. 11 illustrates the case where $L(\boldsymbol{x}) = \|\boldsymbol{x}\|^{-3}$.

IV. TDMA Approach

We can conclude, from the previous sections, that unless γ can be made sufficiently small, long-range communications are impossible if we allow all nodes to emit simultaneously, because the graph $\mathcal{G}(\gamma, \lambda)$ may remain in a subcritical phase for all λ. Having a small γ requires nodes to use CDMA for transmission, which can be complex to implement in an ad hoc network (node synchronization may be difficult in the presence of mobility). An alternative is to avoid having all nodes emitting at the same time, and thus to use a TDMA scheme. We assume that each time interval is divided into n time slots. An optimal TDMA scheme poses also a quite complex challenge to assign the slots to each node, which is clearly beyond the scope of this paper. In this section, we keep the strategy suboptimal but very simple and totally decentralized: each node picks randomly a number i between 1 and n, and only emits during the ith time slot. All nodes are listening at all times. We also assume, for the sake of simplicity, that all nodes emit with the same power P. We denote by $\mathcal{G}^n(\lambda, \gamma)$ the graph obtained by superposing the n graphs derived for each slot.

In the following theorem, we propose an extension of Theorem 2 that proves that with fixed γ, one can reach the supercritical phase by choosing n large enough.

Theorem 5: For a fixed γ, if the isotropic attenuation function $L(\cdot)$ verifies assumptions (8) and (9), then there exist $\lambda' < \infty$ and a function $n'(\lambda, \gamma)$ such that

- $n'(\lambda, \gamma) < \infty$ for all $\lambda > \lambda'$
- if $\lambda > \lambda'$ and $n \geq n'(\lambda, \gamma)$, there exists a.s. an infinite connected component in the graph $\mathcal{G}^n(\lambda, \gamma)$.

We prove Theorem 5 by following the same steps as Theorem 2, except a few modifications. We first need to change a little bit the mapping. The Poisson process is now decomposed in n *subprocesses*, formed by the points emitting in each of the time slots. Because these time slots are picked independently for each node, each subprocess is Poisson with intensity λ/n. Definition 2 is then replaced by the following one:

Definition 5: An edge $\boldsymbol{a}$ of $\mathcal{L}$ is said to be *open* if the following conditions are fulfilled.

- both squares adjacent to $\boldsymbol{a}$ are populated, and
- the total number of points **of each subprocess** located in the two squares adjacent to $\boldsymbol{a}$ and all their direct neighbors is less than or equal to $N + 1$

With this new definition, the probability that an edge is open needs to be recomputed. Lemma 2 is thus replaced by the following.

Lemma 7: For any $q' > 0$, there exists $\lambda' < \infty$ and $n'(\lambda) < \infty$ such that

$$\lambda > \lambda' \text{ and } n \geq n'(\lambda) \Rightarrow q < q'.$$

Proof: We have

$$\begin{aligned} q &= \mathbb{P}\left(\{\text{an edge of } \mathcal{L}' \text{is closed}\}\right) \\ &\leq 2K^2 \mathbb{P}\left(\left\{\text{no point in a subsq. of surf. } d^2/K^2\right\}\right) \\ &\quad + n\mathbb{P}(\{\text{subprocess } i \text{ exceeds } N+1 \text{ pts} \\ &\qquad \text{in a rectangle of surf. } 12d^2\}) \\ &= 2K^2 e^{-\lambda d^2/K^2} + n\mathbb{P}(Z > N+1) \end{aligned} \tag{20}$$

where Z is a Poisson random variable of parameter $12\lambda d^2/n$. The first term can be obviously made arbitrarily small by choosing λ large enough. Let us call λ' the smallest value of λ such that the first term is smaller than $q'/2$.

For the second term, as $N \in \mathbb{N}$, take the case $N = 0$ as an upper bound. Note that in this way, we obtain a bound that is independent from γ

$$\begin{aligned} n\mathbb{P}(Z > N+1) &\leq n\mathbb{P}(Z > 1) \\ &= n\left(1 - e^{-12\lambda d^2/n} - \frac{12\lambda d^2}{n} e^{-12\lambda d^2/n}\right) \\ &\leq n\left(\frac{12\lambda d^2}{n} - \frac{12\lambda d^2}{n} e^{-12\lambda d^2/n}\right) \\ &= 12\lambda d^2\left(1 - e^{-12\lambda d^2/n}\right). \end{aligned}$$

The latter expression tends to zero when n increases. There exists therefore $n'(\lambda) < \infty$ such that $n \geq n'(\lambda)$ implies $n\mathbb{P}(Z > N) < q'/2$.

As both terms of (20) can be made smaller than $q'/2$, we proved that $q < q'$. ■

The remainder of the proof of Theorem 5 is constructed from the same arguments as in Section III-C.

By applying this TDMA strategy, we actually reduce the number of interfering nodes by a factor of n. It is therefore interesting to compare the connectivity of the graph obtained by superposing the n graphs derived for each slot, to that of the original graph $\mathcal{G}(\gamma, \lambda)$ obtained when all nodes emit at the same time. Let us introduce the following notations for the interference contribution, which is another shot noise, at each time slot

$$I_k(\boldsymbol{x}) = \sum_{i \in S_k, \boldsymbol{x}_i \neq \boldsymbol{x}} PL(\boldsymbol{x}_i - \boldsymbol{x})$$

where S_k, $k = 1, \ldots, n$ is the set of the indexes of the nodes that emit during the kth time slot. It follows immediately that

$$\sum_{k=1}^{n} I_k(\boldsymbol{x}) = I(\boldsymbol{x}).$$

The expected values of the interference term (1) in the TDMA scheme is n times lower than in the regular scheme

$$\mathbb{E}[\gamma I_k(\boldsymbol{x})] = \frac{1}{n}\mathbb{E}\left[\gamma I(\boldsymbol{x})\right].$$

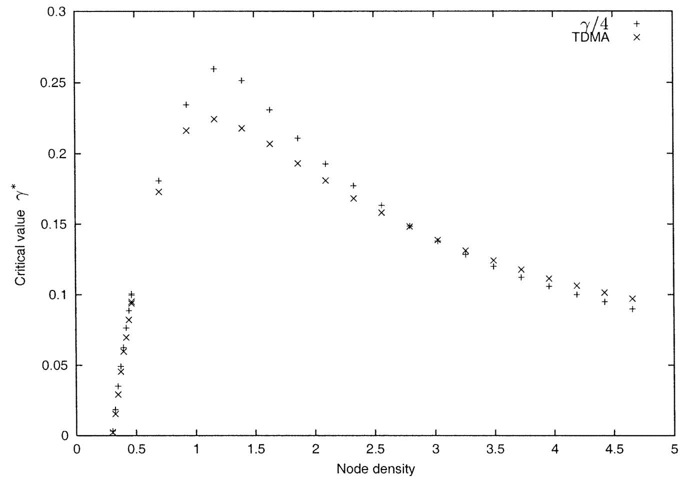

Fig. 12. Comparison between the critical threshold in the TDMA case ($n = 4$ time slots) and in the original model with all nodes allowed to emit simultaneously. To make comparison easier, the critical value in the second case has been multiplied by 4. [Simulation parameters are the same as in Fig. 3].

We computed by simulation the critical threshold $\gamma^*(\lambda)$ in the TDMA scheme. Fig. 12 presents the results, compared to those of the regular scheme. As expected, we observe that the threshold in the TDMA scheme is about n times higher. This means that introducing an n-time slots TDMA system is somehow equivalent to dividing γ by n.

Finally, if the attenuation function has the form $L(\boldsymbol{x}) = \|\boldsymbol{x}\|^{-\alpha}$, which is not bounded, we have observed in [21] that the TDMA scheme performs not only as well as the CDMA scheme with γ divided by n, but even much better, especially for large values of λ.

V. Conclusion

We have studied the connectivity of Poisson signal-to-interference ratio graphs (STIRG) $\mathcal{G}(\gamma, \lambda)$ where γ represents the imperfect orthogonality of the codes used in CDMA, or is set to 1 in a narrowband system.

The STIRG is radically different from the graph obtained in the Boolean Model, where $\gamma = 0$: the node degree is now bounded (Theorem 1), and the existence of an edge between two nodes depends not only on the location of these two nodes, but on the location of all others. We have shown that if γ is too large, all clusters are almost surely finite. Our main result is that percolation, and thus long range communications, are however still possible if γ is small enough, but nonzero (Theorem 2). If this had not been the case, it would have been a serious impediment for multiple hops large scale ad hoc networks.

We have also proven that when the node density λ tends to infinity, the critical value $\gamma^*(\lambda)$ decreases as $1/\lambda$ provided the attenuation function is bounded from above and from below in a small neighborhood of the origin. The main result of this paper is a first picture of the shape of the region in the (λ, γ) plane where percolation occurs.

As a small value of γ requires very efficient and thus complex CDMA codes, an alternative is to use a TDMA system, where each node emits during 1 slot every n time slots. We showed that such a system led to a connectivity similar to the original scheme with an orthogonality factor γ/n. We proved furthermore that if

λ is large enough, one can make the graph reach the supercritical phase by choosing n sufficiently large.

The main restriction in Theorems 2 and 5 is the requirement (8) that the attenuation function $L(\cdot)$ has a finite support. This assumption was used in the proof to find an infinite sequence of open independent edges in the lattice $\mathcal{L}'$, and to prove bond percolation on this lattice. Our simulations show however that this assumption is not necessary.

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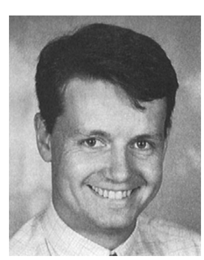

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