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Measuring Graph Robustness via Game Theory

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Abstract: Measuring the reliability of a network is one of the rich and complex areas of combinatorial optimization. Since the precise meaning of reliability highly depends on the application, there is an abundance of reliability metrics that have been proposed. Applying game-theoretical tools for measuring security has become very common. The basic idea is very natural: define a game between two virtual players, the Attacker and the Defender, such that the rules of the game capture the circumstances under which reliability is to be measured. Then analyzing the game might give rise to an appropriate security metric: the better the Attacker can do in the game, the lower the level of security is. This kind of analysis can give rise to new graph reliability metrics and in some cases it can shed a new light on some well-known ones. In this paper we survey a few recent results of this type.

Keywords: Robustness, Game Theory, Nash-equilibrium, Connectivity, Graph Strength

1 Introduction

The problem of measuring the robustness or reliability of a graph arises in many applications. The most widely applied reliability metrics are obviously the connectivity based ones, however, these are unsuitable in many cases. The reason for that is that in many applications the network is almost completely functional if removing some nodes or links results in the loss of only a small number of nodes that are in some sense insignificant or peripheral. Connectivity based metrics (even weighted versions of these) are not capable of capturing this idea as they are only concerned with whether the resulting graph is connected or not.

There is an abundance of recent books and papers on game-theoretical tools for measuring and increasing security. Since all aspects of security are obviously of utmost importance nowadays and game theory as a tool to address related problems presents itself very naturally, the literature on this topic is extremely diverse. Much of the arsenal of game theory has been employed on various applications which very often have little in common besides somehow being related to security. In this paper, however, only the theory of two-player, zero-sum games, the simplest and probably most widely known subfield of game theory will be relied on to address various problems raised by applications concerning the measuring of graph robustness.

All results mentioned in this paper will be based on the following approach. Assume that an input graph G is given (in some cases with a few designated vertices). G will either be directed or undirected depending on the application. Besides that, in most cases certain weight functions will also be part of

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the input: for each edge $e \in E(G)$ or vertex $v \in V(G)$ the "damage" caused by the loss of e or v (or in other words, the "importance" of e or v) will be denoted by $d(e)$ or $d(v)$, respectively; furthermore, the cost of attacking an edge e will be denoted by $c(e)$. In each application, we will define a two-player, zero-sum game on G between two virtual players, the Attacker and the Defender. In all such games, the Attacker will choose (or "attack and destroy") an edge e of G (or more formally: the set of pure strategies of the Attacker will be $E(G)$). Simultaneously (or simply without knowing the Attacker's chosen edge) the Defender will choose a subset of the edges $Z \subseteq E(G)$ that will be thought of as some kind of "communication infrastructure" and the requirements on which will vary in each application (for example, Z can be the edge set of a path or a spanning tree, etc). Regardless of the Defender's choice, the Attacker will have to pay the cost of attack $c(e)$ to the Defender. There will be no further payoff if $e \notin Z$. If, on the other hand, $e \in Z$ then the Defender will pay the Attacker an amount that will be individually defined for each application (and will somehow depend on e, Z and the damage values $d(e)$ and $d(v)$). Since these games will be two-player, zero-sum games by definition, they will have a unique Nash-equilibrium payoff V (which will simply be referred to as the *game value* in this paper) by Neumann's classic Minimax Theorem (see Section 2). Since V is the highest expected gain the Attacker can guarantee himself by an appropriately chosen mixed strategy, it makes sense to say that the reciprocal of V is a valid reliability metric in the sense corresponding to the specific definition of the game.

We remark that it might seem unrealistic in the above described framework that the Defender should receive the cost of attack $c(e)$ from the Attacker (as the Defender is indifferent to the costs and efforts associated with an attack, she is only affected by the damage caused). In other words, it would be more natural to assume that the above given payoffs only describe the Attacker's gain while the Defender's loss depends exclusively on e, Z and the damage values $d(e)$ and $d(v)$ (and is thus always bigger by $c(e)$ than the Attacker's gain). This would also imply that the game is not zero-sum any more. However, it is easily shown that the thus-obtained non-zero-sum game is essentially equivalent to the zero-sum game described above. This equivalency is due to the fact that the sum of the payoffs only depends on the choice of the Attacker and it more precisely means that Nash-equilibria of the two versions of the game are identical and the Attacker's Nash-equilibrium payoff is unique in the non-zero-sum version of the game and it is equal to the (unique) Nash-equilibrium payoff corresponding to the zero-sum version. (An analogous statement would not be true for the Defender.) The proof of this equivalency is a simple exercise (see [10, Lemma 1] for a proof). We will disregard this point in the remainder of the paper and focus on the zero-sum game versions described above.

2 Preliminaries on Game Theory

In this section we very briefly summarize all the necessary background on game theory. A *(finite) two-player, zero-sum game* is given by a matrix M called the *payoff matrix*. Columns of M correspond to one of the players and rows of M to the other, so for the sake of simplicity one can refer to the two players as *Column Player* and *Row Player*. Columns and rows of M are called the *pure strategies* of the respective players. The matrix M defines the game in the following sense: both players choose one of their pure strategies (simultaneously, without knowing each other's choices) and then the corresponding entry of M (that is, the one in the intersection of the chosen row and column) is payed by the Row Player to the Column Player. (Obviously, a negative payment means that in reality it is the Column Player who pays the absolute value of the amount to the Row Player.)

A *mixed strategy* of a player is a probability distribution on their pure strategies. If M is a $k \times n$ matrix then it is natural to store the Column Player's and the Row Player's mixed strategies as n-dimensional column vectors and k-dimensional row vectors, respectively. If we fix a pair of mixed strategies $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{y} \in \mathbb{R}^k$ then the Column Player's expected gain (or, equivalently, the Row Player's expected loss) is obviously $\mathbf{y}M\mathbf{x}$. It is sensible for the Column Player to choose a mixed strategy $\mathbf{x}$ that maximizes his worst case expected gain, therefore he is interested in finding an $\mathbf{x}$ that maximizes the minimum value of $\mathbf{y}M\mathbf{x}$ over all possible mixed strategies $\mathbf{y}$ of the Row Player; in other words, his job is $\max_{\mathbf{x}} \left\{ \min_{\mathbf{y}} \{\mathbf{y}M\mathbf{x}\} \right\}$.

Analogously, the Row Player's task is $\min_{\mathbf{y}}\left\{\max_{\mathbf{x}}\{\mathbf{y}M\mathbf{x}\}\right\}$; that is, she wants to minimize her worst case expected loss. Neumann's classic Minimax Theorem [12] states that these two values are equal for every payoff matrix M: $\max_{\mathbf{x}}\left\{\min_{\mathbf{y}}\{\mathbf{y}M\mathbf{x}\}\right\} = \min_{\mathbf{y}}\left\{\max_{\mathbf{x}}\{\mathbf{y}M\mathbf{x}\}\right\}$. This common value is called the *game value* corresponding to M. Since a pair of mixed strategies $(\mathbf{x}, \mathbf{y})$ that attain the corresponding optima is equivalent to the (more general) notion of a Nash-equilibrium in the special case of two-player, zero-sum games, the game value is also referred to as a (Nash-)equilibrium payoff in the literature (which is known to be unique in this special case). However, in this paper we will keep calling it the game value.

It is useful to mention that the description of the tasks of the two players can be simplified by observing that it is sufficient for a mixed strategy to "guard against" all pure strategies of the other player, that will imply that it also guards against all mixed strategies. For example, if every entry of the column vector $M\mathbf{x}$ is at least μ for a mixed strategy $\mathbf{x}$, that translates to saying that no matter which pure strategy the Row Player picks, the Column Player's expected gain is at least μ. However, this also implies $\mathbf{y}M\mathbf{x} \geq \mu$ for every mixed strategy $\mathbf{y}$ (since $\mathbf{y}M\mathbf{x}$ is a convex combination of the entries of $M\mathbf{x}$). Hence the Column Player's task can also be described as maximizing the minimum entry of $M\mathbf{x}$ over all mixed strategies $\mathbf{x}$ (and the Row Player's case is analogous).

The above also implies (as it is shown in many textbooks, see e.g. [11]) that two-player, zero-sum games are easy to handle algorithmically via linear programming: optimum mixed strategies for the game given by M can be found efficiently by solving the following linear program and its dual:

$$\max\{\mu : M\mathbf{x} \geq \mu \cdot \mathbf{1}, \mathbf{1} \cdot \mathbf{x} = 1, \mathbf{x} \geq \mathbf{0}\}$$

(where $\mathbf{1}$ denotes the all-1 vector). However, since the size of the payoff matrix M will be exponential in the size of the input graph G in all applications mentioned in this paper, this approach will not be viable.

3 Connectivity Based Metrics

The following simple example might illuminate the approach described in the Introduction.

THE *st*-PATH GAME

Input: A connected, undirected graph G and two vertices $s, t \in V(G)$;
The Attacker chooses an edge e of G;
The Defender chooses a path P between s and t;
The Payoff from the Defender to the Attacker is 1 if e is on P and 0 otherwise.

The origin of the following simple claim is unclear, one can regard it as folklore.

Claim 1 *The game value of the st-path game is $\frac{1}{\lambda(s,t)}$, where $\lambda(s,t)$ denotes the edge-connectivity between s and t (that is, the size of the minimum cut separating s and t).*

PROOF: Let C be a cut of size $\lambda(s,t)$ that separates s and t and assume that the Attacker uses the following mixed strategy: he assigns a probability of $\frac{1}{\lambda(s,t)}$ to every edge of C and 0 to the rest of the edges. Since every st-path contains at least one edge from C, this mixed strategy guarantees the Attacker an expected gain of at least $\frac{1}{\lambda(s,t)}$. This proves that game value is at least $\frac{1}{\lambda(s,t)}$.

Now choose $\lambda(s,t)$ pairwise edge-disjoint paths between s and t (which are known to exist by Menger's classic theorem, see [13, Section 9.1]). Assume that the Defender uses the following mixed strategy: she assigns a probability of $\frac{1}{\lambda(s,t)}$ to every chosen path and 0 to the rest of the st-paths. Since the chosen paths are edge-disjoint, this mixed strategy guarantees the Defender an expected loss of at most $\frac{1}{\lambda(s,t)}$. Hence the game value is at most $\frac{1}{\lambda(s,t)}$. $\square$

The relevance of Claim 1 is that it shows that the notion of edge-connectivity between two vertices (viewed as a reliability metric) is well captured by the st-path game. However, the notion of (general) edge-connectivity (the minimum size $\lambda(G)$ of a subset of edges the removal of which disconnects G) is also easy to capture by a similar game:

The path game

Input: A connected, undirected graph G;
First *the Attacker* chooses two distinct nodes $s, t \in V(G)$ and declares them to the Defender;
then *the Defender* chooses a path P between s and t,
and (simultaneously) *the Attacker* chooses an edge e of G;
The Payoff from the Defender to the Attacker is 1 if e is on P and 0 otherwise.

Claim 2 *The game value of the path game is* $\frac{1}{\lambda(G)}$.

Proof: The proof is analogous to that of Claim 1 with the only difference being that the Attacker first chooses the nodes s and t in such a way that they are separated by a minimum cut of G. $\square$

Obviously, the notion of node connectivity of G (either between two specific vertices or in general) can be captured by analogously defined games as the ones in Claims 1 and 2. The following theorem shows that, as one would expect, the weighted versions of these games lead to weighted minimum cuts.

The weighted st-path game

Input: A connected, undirected graph G, two nodes $s, t \in V(G)$, a damage function $d : E(G) \to \mathbb{R}^+$ and a cost function $c : E(G) \to \mathbb{R}$;
The Attacker chooses an edge e of G;
The Defender chooses a path P between s and t;
The Payoff from the Defender to the Attacker is $d(e) - c(e)$ if e is on P and $-c(e)$ otherwise.

Obviously, the above payoffs correspond to the framework described in the Introduction: the cost of attack $c(e)$ must be paid by the Attacker in all cases, but he receives the damage value $d(e)$ if he succeeds in hitting the st-path chosen by the Defender.

The weighted st-path game is considered and solved in the $d(e) \equiv 1$ and the $c(e) \equiv 0$ cases in [6] and [15], respectively. (In [6], a generalization of the $d(e) \equiv 1$ case of the game is also solved: there the Attacker can target a subset of the edges of a given size and the Defender can choose two paths between two source-destination pairs.) The following result, however, seems to be new.

Theorem 3 *For every input of the weighted st-path game the game value is*

$$\max\left\{\left\{\frac{1-q(C)}{p(C)} : C \textit{ is a cut that separates } s \textit{ and } t\right\} \cup \left\{-c(e) : e \in E(G)\right\}\right\},$$

where $p(e) = \frac{1}{d(e)}$ *and* $q(e) = \frac{c(e)}{d(e)}$ *for all* $e \in E(G)$.

Proof: Let the value of the above maximum be μ. Assume first that $\mu = -c(e)$ for some $e \in E(G)$. Then if the Attacker targets e with a probability of 1, his total expected gain is obviously at least μ. Now assume that $\mu = \frac{1-q(C)}{p(C)}$ for a cut C that separates s from t and let the Attacker use the following mixed strategy: assign a probability of $\frac{p(e)}{p(C)}$ to every edge of C and 0 to the rest of the edges. Consider an arbitrary path P between s and t and fix an edge $e \in C$. Then e contributes to the Attacker's expected gain by $\frac{p(e)}{p(C)}(d(e) - c(e)) = \frac{1-q(e)}{p(C)}$ or $\frac{p(e)}{p(C)}(-c(e)) = \frac{-q(e)}{p(C)}$ depending on whether e is on P or not, respectively. Since C obviously contains at least one edge of P, the Attacker's total expected gain is at least $\frac{1-q(C)}{p(C)} = \mu$. Since in all cases the Attacker has a mixed strategy that guarantees him an expected gain of at least μ, the game value is also at least μ.

Replace every edge $e = \{u, v\}$ of G by the directed arcs $e' = \overrightarrow{uv}$ and $e'' = \overrightarrow{vu}$ and let the capacity of both be $\mu \cdot p(e) + q(e)$. Then, by the definition of μ, the capacity of every edge is non-negative and the total capacity of every $s\bar{t}$-cut is at least 1. Therefore there exists a flow f from s to t of overall value 1 by the Ford-Fulkerson theorem. Assume without loss of generality that for every edge e of G either $f(e') = 0$ or $f(e'') = 0$. It is well-known (see [13, Section 10.3]) that f is a non-negative linear combination of characteristic vectors of directed paths from s to t and directed cycles. This implies that there exists a set of (undirected) paths $P_1, P_2, \ldots, P_t$ in G between s and t and corresponding non-negative coefficients $\alpha_1, \alpha_2, \ldots, \alpha_t$ such that $\sum_{i=1}^{t} \alpha_i = 1$ and

$$\sum \left\{ \alpha_i : e \text{ is on } P_i \right\} \leq \mu p(e) + q(e) \tag{1}$$

holds for each edge e.

Now assume that the Defender uses the following mixed strategy: for every $1 \leq i \leq t$ she assigns the probability α_i to P_i (and 0 to the rest of the st-paths). Then if the Attacker targets an edge e then her expected loss is

$$\sum_{i:e\in E(P_i)} \alpha_i(d(e) - c(e)) - \sum_{i:e\notin E(P_i)} \alpha_i c(e) = d(e) \left(\sum_{i:e\in E(P_i)} \alpha_i \right) - c(e) \leq d(e)(\mu p(e) + q(e)) - c(e) = \mu$$

by (1). Therefore the game value is at most μ. $\square$

Corollary 4 ([15]) *If $c(e) = 0$ is assumed for all $e \in E(G)$ then the game value of the weighted st-path game is $\frac{1}{\lambda_p(s,t)}$, where $p(e) = \frac{1}{d(e)}$ for every edge e and $\lambda_p(s,t)$ is the minimum total weight of a cut that separates s and t with respect to the weight function p.*

4 Graph Strength and Related Metrics

The *strength* of a connected graph G was defined by Gusfield [9]. The idea is quite natural: if we remove a subset $U \subseteq E(G)$ of the edges then the efficiency of this "attack" against G can be measured by the ratio of the number of new components created and $|U|$ (that is, the "effort" required for the attack). Then it makes sense to define the reciprocal of the maximum efficiency of an attack to be a security metric: $\sigma(G) = \min\left\{\frac{|U|}{\text{comp}(G-U)-1} : U \subseteq E(G), \text{comp}(G-U) > 1\right\}$, where $\text{comp}(G-U)$ is the number of components of the graph obtained from G by deleting U. This notion was extended to a weighted version by Cunningham [7]:

Definition 5 *Assume that a connected graph G is given with a positive weight function $p : E(G) \to \mathbb{R}^+$ on its edges. Then*

$$\sigma_p(G) = \min\left\{\frac{p(U)}{\text{comp}(G-U) - 1} : U \subseteq E(G), \text{comp}(G-U) > 1\right\}$$

is called the strength *of G with respect to p.*

$\sigma_p(G)$ is computable in strongly polynomial time as it was shown by Cunningham [7]. It was proved in [14] that the following game is capable of capturing the notion of $\sigma_p(G)$. The game resembles the weighted st-path game defined above with the only difference being that the Defender's pure strategies are spanning trees instead of st-paths.

THE SPANNING TREE GAME

Input: A connected, undirected graph G, a damage function $d : E(G) \to \mathbb{R}^+$ and a cost function $c : E(G) \to \mathbb{R}$;

The Attacker chooses an edge e of G;

The Defender chooses a spanning tree T of G;

The Payoff from the Defender to the Attacker is $d(e) - c(e)$ if e is in T and $-c(e)$ otherwise.

Theorem 6 ([14]) *For every input of the spanning tree game the game value is*

$$\max_{\emptyset \neq U \subseteq E(G)} \frac{\mathrm{comp}(G-U) - 1 - q(U)}{p(U)},$$

where $p(e) = \frac{1}{d(e)}$ *and* $q(e) = \frac{c(e)}{d(e)}$ *for all* $e \in E(G)$ *(and* $\mathrm{comp}(G-U)$ *is the number of components of the graph obtained from* G *by deleting* U*). Furthermore, there exists a strongly polynomial algorithm that computes the game value of the spanning tree game and an optimum mixed strategy for both players.*

We remark that the above formula (without a corresponding strongly polynomial algorithm) was shown previously in the special case of $d(e) \equiv 1$ in [8].

Corollary 7 ([14]) *The game value of the spanning tree game is* $\frac{1}{\sigma_p(G)}$ *if* $p(e) = \frac{1}{d(e)}$ *and* $c(e) = 0$ *is assumed for all* $e \in E(G)$.

It is important to mention that Theorem 6 was proved in [14] in a much more general, matroidal setting: the MATROID BASE GAME was defined analogously to the spanning tree game with the only difference being that the Attacker chooses an element of the ground set S of a given matroid $M = (S, \mathcal{I})$ and the Defender chooses a base B of M. Then the following was proved:

Theorem 8 ([14]) *Assume that the matroid* $M = (S, \mathcal{I})$ *and damage and cost functions* $d : S \to \mathbb{R}^+$ *and* $c : S \to \mathbb{R}$, *respectively are given. Then the game value of the matroid base game is equal to*

$$\max_{\emptyset \neq U \subseteq S} \frac{\mathrm{r}(S) - \mathrm{r}(S-U) - q(U)}{p(U)},$$

where $p(s) = \frac{1}{d(s)}$ *and* $q(s) = \frac{c(s)}{d(s)}$ *for all* $s \in S$. *Furthermore, if* M *is given by an independence testing oracle then there exists a strongly polynomial algorithm that computes the game value of the matroid base game and an optimum mixed strategy for both players.*

We remark that the above theorem was essentially known before in the special case of $c \equiv 0$: then one easily shows that solving the matroid base game is equivalent to the *capacitated fractional base packing* problem discussed in [13, Section 42.4], where a strongly polynomial algorithm is given in [13, Theorem 42.7]. However, that algorithm does not seem to generalize to the $c \not\equiv 0$ case. Hence the above theorem can also be regarded as a generalization of [13, Theorem 42.7].

Obviously, Theorem 8 gives rise to a number of natural extensions of the spanning tree game and readily provides the corresponding modifications of the notion of graph strength. For example, one could modify the definition of the spanning tree game by allowing the Defender to choose the edge set of the union of k edge-disjoint spanning trees (where $k \geq 1$ is given); or the Defender could choose a spanning edge set of a given size, etc. (Besides that, the matroid base game turned out to be relevant even in the very special case of the uniform matroid: that came up in an application concerning the security of content-adaptive steganography, see [10].)

In [14] a further generalization of the matroid base game was also considered: the COMMON BASE GAME is almost identical to the matroid base game with the only difference being that the Defender chooses a common base of two matroids given on the common ground set S. Not all results on the matroid base game seem to extend smoothly to the common base game, a strongly polynomial algorithm is only known is certain special cases. We omit the technical details here (see [14]), we only discuss an application of the common base game that yields a new security metric, a directed analogue of graph strength.

Assume that a digraph G is given. Call a subset of the nodes $R \subseteq V(G)$ a *source set* if every node of G is reachable from a node in R via a directed path. A vertex $r \in V(G)$ is a *source node* if $\{r\}$ is a single-element source set. For every arc set $U \subseteq E(G)$, denote by $\mathrm{source}(G-U)$ the minimum cardinality of a source set in the digraph obtained from G by deleting U. (In other words, $\mathrm{source}(G-U)$ is the number of weak components in a maximum size branching of $G-U$.)

Definition 9 *Assume that a directed graph G is given that has a source node; assume further that a positive weight function $p : E(G) \to \mathbb{R}^+$ is given. Then*

$$\overrightarrow{\sigma}_p(G) = \min\left\{\frac{p(U)}{\text{source}(G-U)-1} : U \subseteq S, \text{source}(G-U) > 1\right\}$$

is the directed strength *of G with respect to p.*

It was proved in [14] that $\overrightarrow{\sigma}_p(G)$ is computable is strongly polynomial time.

Recall that an *arborescence* of G is a subset A of the arcs that is a spanning tree of the underlying undirected graph such that the digraph $(V(G), A)$ has a source node. (It is well-known and elementary that the existence of an arborescence is equivalent to the existence of a source node.) Then the following is a straightforward analogue of the spanning tree game:

THE ARBORESCENCE GAME

Input: A directed graph G that has a source node, a damage function $d : E(G) \to \mathbb{R}^+$ and a cost function $c : E(G) \to \mathbb{R}$;

The Attacker chooses an arc e of G;

The Defender chooses an arborescence A of G;

The Payoff from the Defender to the Attacker is $d(e) - c(e)$ if e is in A and $-c(e)$ otherwise.

The following theorem yields an analogous description of the game value as that of Theorem 6.

Theorem 10 ([14]) *For every input of the arborescence game the game value is*

$$\max_{\emptyset \neq U \subseteq E(G)} \frac{\text{source}(G-U) - 1 - q(U)}{p(U)},$$

where $p(e) = \frac{1}{d(e)}$ and $q(e) = \frac{c(e)}{d(e)}$ for all $e \in E(G)$.

As a corollary, we get the analogue of the connection between graph strength and the spanning tree game.

Corollary 11 ([14]) *The game value of the arborescence game is $\frac{1}{\overrightarrow{\sigma}_p(G)}$ if $p(e) = \frac{1}{d(e)}$ and $c(e) = 0$ is assumed for all $e \in E(G)$.*

5 Persistence and Related Metrics

In [7] another reliability metric was defined that is based on a somewhat similar idea to that of graph strength. Assume that a "headquarters" node $r \in V(G)$ is given in a graph G with the special role that every node needs to communicate with r. Assume further that the "importance" of the information stored in a node v is represented by $d(v)$. Then if a subset U of the edges is removed, the efficiency of this "attack" can be measured by the ratio of $|U|$ and the total d-value of the nodes that become unreachable from r. Hence the reciprocal of the maximum efficiency of an attack is again a sensible a reliability metric:

Definition 12 *Assume that a connected graph G, a designated node $r \in V(G)$ and a non-negative weight function $d : (V(G) \setminus \{r\}) \to \mathbb{R}^{\geq 0}$ are given. Then*

$$\pi(G) = \min\left\{\frac{|U|}{\lambda(U)} : U \subseteq E(G), \lambda(U) > 0\right\}$$

is called the persistence *of G, where*

$$\lambda(U) = \sum \{d(v) : v \textit{ is unreachable from } r \textit{ after removing } U\}.$$

It is important to remark the following to avoid confusion. In [7] the above notion was defined on directed graphs (since, although both versions make sense, the directed one generalizes the undirected version by the usual trick of replacing every undirected edge by two directed ones). The notion was not given any specific name, it was regarded as a directed analogue of graph strength $\sigma_p(G)$ (see Definition 5) and was referred to as the "directed model". However, the above notion of $\pi(G)$ is substantially different from and not to be confused with *directed strength* $\overrightarrow{\sigma}_p(G)$ defined in Definition 9. In fact, we believe that $\overrightarrow{\sigma}_p(G)$ is a much closer analogue to $\sigma_p(G)$ than $\pi(G)$. The name *persistence* was coined in [1] for easier reference (and to better distinguish $\pi(G)$ from $\sigma_p(G)$ and $\overrightarrow{\sigma}_p(G)$).

It was proved in [7] that $\pi(G)$ is computable in strongly polynomial time (see [5] for a somewhat simpler description of the algorithm). Despite its apparent naturality and simplicity, the notion of $\pi(G)$ did not seem to receive much attention for more than two decades when it came up in an application: it was argued in [1] and [4] that it is very appropriate for measuring the reliability of wireless sensor networks.

The following game is related to the spanning tree game, but it is capable of capturing the notion of persistence (in the sense seen above). The idea is again natural: if the Attacker targets the edge e and succeeds in hitting the spanning tree T chosen by the Defender, then his gain is assumed to be the total damage he causes: the sum of the d-values of all nodes in the component of $T - e$ not containing r.

The rooted spanning tree game

Input: A connected, undirected graph G, a cost function $c : E(G) \to \mathbb{R}$, a node $r \in V(G)$ and a damage function $d : (V(G) \setminus \{r\}) \to \mathbb{R}^{\geq 0}$;

The Attacker chooses an edge e of G;

The Defender chooses a spanning tree T of G;

The Payoff from the Defender to the Attacker is $\lambda(T,e) - c(e)$, if e is in T and $-c(e)$ otherwise, where

$$\lambda(T,e) = \sum \big\{ d(v) : v \text{ is unreachable from } r \text{ in } T \text{ after removing } e \big\}.$$

Theorem 13 ([2]) *For every input of the rooted spanning tree game, the game value is*

$$\max_{\emptyset \neq U \subseteq E(G)} \frac{\lambda(U) - c(U)}{|U|}.$$

Corollary 14 ([2]) *The game value of the rooted spanning tree game is* $\frac{1}{\pi(G)}$ *if* $c(e) = 0$ *is assumed for all* $e \in E(G)$.

In fact, it is only Corollary 14 that is explicitly stated in [2], however, the proof extends smoothly to show Theorem 13 (we omit the details here). Furthermore, there are two further possible extensions of the rooted spanning tree game and Theorem 13, both follow from trivial modifications of the proof in [2]. The first one is to consider directed graphs instead of undirected ones: the notion of persistence is straightforward in this case (and, as already mentioned above, generalizes the undirected case) and the only change needed in the corresponding game is to consider arborescences rooted in r instead of spanning trees. Secondly, in [7] the notion of $\pi(G)$ was originally defined in a weighted version: $\pi_s(G) = \min\left\{ \frac{s(U)}{\lambda(U)} : U \subseteq E(G), \lambda(U) > 0 \right\}$, where $s : E(G) \to \mathbb{R}^+$ is a given positive weight function. In order to capture this weighted version of persistence, the definition of the rooted spanning tree game would be needed to be modified in the following way: the Attacker's gain would be $\frac{\lambda(T,e)}{s(e)} - c(e)$ if he hits the Defender's spanning tree T (and $-c(e)$ otherwise). Admittedly, there does not seem to be any natural intuition behind this definition, however, the game value in this case turns out to be $\max_{\emptyset \neq U \subseteq E(G)} \frac{\lambda(U) - q(U)}{s(U)}$, where $q(e) = s(e) \cdot c(e)$ for all e, which does indeed yield the reciprocal of $\pi_s(G)$ in the $c(e) \equiv 0$ case.

To conclude this paper, we mention an interesting variant of the rooted spanning tree game (only in its simplest case, with no weight functions) that lacks the role of a special headquarters node. The intuitive idea behind the definition is quite natural: if the Attacker succeeds in hitting the spanning tree chosen by the Defender then the bigger component of $T-e$ can still be regarded as a functioning network, so the Defender's loss (and thus the Attacker's gain) is the size of the smaller component.

The "you decide, I choose" spanning tree game

Input: A connected, undirected graph G;
The Attacker chooses an edge e of G;
The Defender chooses a spanning tree T of G;
The Payoff from the Defender to the Attacker is the number of nodes in the *smaller* component of $T-e$ if e is in T and 0 otherwise.

The following theorem expresses the game value of the above game in terms of a special multicommodity flow problem.

Theorem 15 ([3]) *Consider the following multicommodity flow problem for an arbitrary undirected, connected graph G. A flow f_v corresponds to every vertex $v \in V(G)$; the target node for f_v is v and every other vertex is assumed to be a source node for f_v which all produce a common flow amount of α_v. All edges of G have a capacity of 1 and they are all undirected. (In other words: every edge can carry commodities in both directions, but the total value of all carried amounts in both directions add up to at most 1.) The objective is to maximize $\sum_{v\in V(G)} \alpha_v$. Then if we denote the maximum value of this problem by $\mu(G)$ then the game value of the "you decide, I choose" spanning tree game is $\frac{1}{\mu(G)}$.*

The above theorem suggests that $\frac{1}{\mu(G)}$ could be regarded as a special, new graph reliability metric (that corresponds to the intuition behind the "you decide, I choose" spanning tree game). Obviously, $\mu(G)$ can be computed in polynomial time via linear programming. However, not much more than that is known about $\mu(G)$ (see [3] for some further details).

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