Properties of proof systems

- Important properties of a proof system:
  - Soundness: if $U \vdash V$ then $U \models V$ (what we prove is true)
  - Completeness: if $U \models V$ then $U \vdash V$ (what is true can be proven)
- Gödel's completeness theorem (1929) states that a proof system for FOL using modus ponens is complete (and sound, of course)
- This can be reformulated as:
  the two kinds of consequence – semantic and syntactic – are the same

$$\boxed{\models \equiv \vdash}$$

see the logo of the Association for Logic Programming (ALP):
`https://www.cs.nmsu.edu/ALP`

Association for Logic Programming

Issues with FOL: it is not powerful enough

- FOL is not powerful enough, as it is not possible to uniquely describe arithmetic on natural numbers using FOL
- The set of natural numbers has the following property:

Every integer can be obtained from 0 by adding 1
a finite number number of times (*)

- Property (*) cannot be transformed to a FOL formula, and therefore FOL axiomatisations of arithmetic (e.g. by Peano) have so called non-standard models: in these models there are integers that cannot be reached from 0 by a finite number of incrementation steps
- Gödel's incompleteness theorem states that there is an arithmetic formula φ that is true in the (**single**) model of natural numbers but cannot be proven (equivalent to stating "I am not provable")
- This is not contradicting Gödel's completeness theorem, as there is a non-standard model in which formula φ does not hold

Issues with FOL: it is too powerful

- FOL is too powerful, as it is not (fully) decidable
- A logic is (fully) decidable if there is an algorithm which, given the question if $S \vdash \alpha$ is guaranteed to terminate with a yes or no answer
- FOL is semi-decidable: there is an algorithm (e.g. FOL resolution) that is guaranteed to terminate if $S \vdash \alpha$ holds, but may not terminate if $S \vdash \alpha$ does not hold
- In the past $\sim$ 30 years some subsets of FOL, called Description Logics, have been identified and shown to be (fully) decidable: for these sublanguages there are algorithms that return a yes/no answer to the question: $S \vdash \alpha$?
- We will learn about Description Logics, used mostly in the Semantic Web, in the final part of the course

Part II

Declarative Programming with Prolog

Contents

Prolog in the family of programming languages

Programming paradigms – programming languages

- **Imperative**
  - *Fortran*
  - *Algol*
  - *C*
  - *Java*
  - *Python*
  - ...
- **Declarative**
  - **Functional**
    - *LISP*
    - *ML*
    - *Haskell*
    - ...
  - **Logic**
    - *SQL*
    - *Prolog*
    - *Constraint Prog.*
    - ...

Prolog

- Birth date: 1972, designed by Alain Colmerauer, Robert Kowalski
- First public implementation (Marseille Prolog):
  1973, interpreter in Fortran, A. Colmerauer, Ph. Roussel
- Second implementation (Hungarian Prolog):
  1975, interpreter in CDL, Péter Szeredi

`http://dtai.cs.kuleuven.be/projects/ALP/newsletter/nov04/nav/articles/szeredi/szeredi.html`

- First compiler (Edinburgh Prolog, DEC-10 Prolog):
  1977, David H. D. Warren (current syntax introduced)
- Wiki: `https://en.wikipedia.org/wiki/Prolog`

Prolog – PROgramming in LOGic: standard (Edinburgh) syntax

Standard syntax	English	Marseille syntax
`has_p(b, c).`	`% b has a parent c.`	`+has_p(b, c).`
`has_p(b, d).`	`% b has a parent d.`	`+has_p(b, d).`
`has_p(d, e).`	`% d has a parent e.`	`+has_p(d, e).`
`has_p(d, f).`	`% d has a parent f.`	`+has_p(d, f).`
	`% for all GC, GP, P holds`	
`has_gp(GC, GP) :-`	`% GC has grandparent GP if`	`+has_gp(*GC, *GP)`
`has_p(GC, P),`	`% GC has parent P and`	`-has_p(*GC,*P)`
`has_p(P, GP).`	`% P has parent GP.`	`-has_p(*P,*GP).`

FOL: $\forall GC, GP.\ (\mathtt{has\_gp}(GC, GP) \leftarrow \exists P.(\mathtt{has\_p}(GC, P) \wedge \mathtt{has\_p}(P, GP)))$

- Program execution is SLD resolution, which can also be viewed as pattern-based procedure invocation with backtracking
- Dual semantics: declarative and procedural
  - Slogan: WHAT rather than HOW
    (focus on the logic first, but then think over Prolog execution, too).

Prolog clauses and predicates - some terminology

- A Prolog program is a sequence of *clauses*
- A clause represents a statement, it can be
  - a *fact*, of the form '*head*.', e.g. `has_parent(a,b).`
  - a *rule*, of the form '*head* `:-` *body*.',
    e.g. `has_gp(GC, GP) :- has_p(GC, P), has_p(P, GP).` (*)
- Read '`:-`' as 'if', '`,`' as 'and'
- A *fact* can be viewed as having an empty body, or the body `true`
- A *body* is comma-separated list of *goals*, also named *calls*
- A *head* as well as a *goal* has the form *name*`(`*argument*`,...)`, or just *name*
- A functor of a *head* or a *goal* (or a term, in general) is *F*`/`*N*, where *F* is the name of the term and *N* is the number of args (also called *arity*). Example: the functor of the head of (*) is `has_gp/2`
- The functor of a clause is the functor of its head.
- The collection of clauses with the same functor is called a *predicate* or *procedure*
- Clauses of a predicate should be contiguous (you get a warning, if not)

And what happened to the *function* symbols of FOL?

- Recall: In FOL, atomic predicates have arguments that are terms, built from variables using function symbols, e.g. $lseq(plus(X, 2), times(Y, Z))$
- In maths this is normally written in *infix operator* notation as $X + 2 \leq Y \cdot Z$
- In Prolog, graphic characters (and sequences of such) can be used for both relation and function names: `=<( +(X,2), *(Y,Z) )` (1)
- As a "syntactic sweetener", Prolog supports operator notation in user interaction, i.e. (1) is normally input and displayed as `X+2 =< Y*Z`. However, (1) is the internal, *canonical* format
- The built-in predicate (BIP) `write/1` displays its arg. using operators, while `write_canonical/1` shows the canonical form

```
| ?- write(1 - 2 =< 3*4).             ⟹ 1-2=<3*4
| ?- write_canonical(1 - 2 =< 3*4).   ⟹ =<(-(1,2),*(3,4))
```

- Notice that the predicate arguments are not evaluated, function names act as *data constructors* (e.g. the op. `-` is used not only for subtraction)
- Prolog is a symbolic language, e.g. symbolic derivation is easy
- However, doing arithmetic requires special built-in predicates

Prolog built-in predicates (BIPs) for unification and arithmetic

- Unification. `X = Y`: unifies `X` and `Y`. Examples:

```
| ?- X = 1-2, Z = X*X.            ⟹   X = 1-2, Z = (1-2)*(1-2)
| ?- U = X/Y, c(X,b)=c(a,Y).      ⟹   U = a/b, X = a, Y = b
| ?- 1-2*3 = X*Y.                 ⟹   no (unification unsuccessful)
```

- Arithmetic evaluation. `X is A`: `A` is evaluated, the result is unified with `X`. `A` must be a ground arithmetic expression (ground: no free vars inside)

```
| ?- X = 2, Y is X*X+2.           ⟹   X = 2, Y = 6 ?
| ?- X = 2, 7 is X*X+2.           ⟹   no
| ?- X = 6, 7-1 is X.             ⟹   no
| ?- X is f(1,2).                 ⟹   'Type Error'
```

- Arithmetic comparison. `A =:= B`: `A` and `B` are evaluated to numbers. Succeeds iff the two numbers are equal.
  (Both `A` and `B` have to be ground arithmetic expressions.)

```
| ?- X = 6, 7-1 =:= X.            ⟹   X = 6
| ?- X = 6, X*X =:= (X+3)*(X-2).  ⟹   X = 6
| ?- X = 6, X+3 =:= 2*(X-2).      ⟹   no
| ?- X = 6, X+3 =:= 2*(Y-2).      ⟹   'Instantiation Error'
```

Further BIPs: `A < B`, `A > B`, `A =< B` ($\leq$), `A >= B` ($\geq$), `A =\= B` ($\neq$),

An example: cryptarithmetic puzzle

- Consider this cryptarithmetic puzzle: `AD*AD = DAY`.
  Here each letter stands for a *different* digit, initial digits cannot be zeros.
  Find values for the digits `A`, `D`, `Y`, so that the equation holds.
- We'll use a library predicate `between/3` from library `between`.

```
% between(+N, +M, ?X): X is an integer such that N =< X =< M,
%                      Enumerates all such X values.
```

- I/O mode notation for pred. arguments (used only in comments):
  `+`: input (bound), `-`: output (unbound var.), `?`: arbitrary.
- To load a library: (in SICStus) include the line below in your program:

```
:- use_module(library(between)).
```

  In SWI Prolog the predicate is loaded automatically.
- The Prolog predicate for solving the `AD*AD = DAY` puzzle:

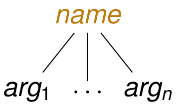
```
ad_day(AD, DAY) :-
    between(1, 9, A), between(1, 9, D), between(0, 9, Y),
    A =\= D, A =\= Y, D =\= Y,
    DAY is D*100+A*10+Y, AD is A*10+D,
    AD * AD =:= DAY.
```

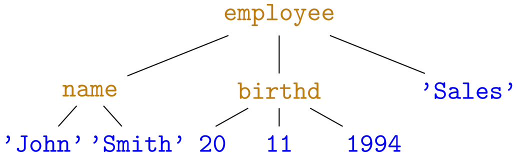
- Solve this puzzle yourself: `GO+TO=OUT`

Data structures in Prolog

Prolog is a dynamically typed language, i.e. vars can take arbitrary values.
Prolog data structures correspond to FOL terms. A Prolog term can be:

- `var` (variable), e.g. `X`, `Sum`, `_a`, `_`; the last two are *void* (don't care) vars
  (If a var occurs once in a clause, prefix it with `_`, or get a WARNING!!!
  Multiple occurrences of a single `_` symbol denote different vars.)
- `constant` (0 argument function symbol):
  - `number (integer or float)`, e.g. `3`, `-5`, `3.1415`
  - `atom` (symbolic constant, cf. enum type), e.g. `a`, `susan`, `=<`, `'John'`
- `compound`, also called record, structure (*n*-arg. function symbol, $n > 0$)
  A compound takes the form: *name*`(`arg_1`,...,`arg_n`)`, where
  - *name* is an atom, arg_i are arbitrary Prolog terms
  - e.g. `employee(name('John','Smith'),birthd(20,11,1994),'Sales')`
  - Compounds can be viewed as trees





Variables in Prolog: the logic variable

- A variable cannot be assigned (unified with) two distinct ground values:

```
| ?- X = 1, X = 2.                  ⟹  no
```

- Two variables may be unified and then assigned a (common) value:

```
| ?- X = Y, X = 2.                  ⟹  X = 2, Y = 2 ?
```

- The above apply to a single branch of execution. If we backtrack over a branch on which the variable was assigned, the assignment is undone, and on a new branch another assignment can be made:

```
   has_p(b, c).        has_p(b, d).        has_p(d, e).
| ?- has_p(b, Y).                   ⟹  Y = c ? ; Y = d ? ; no
```

- A logic variable is a "first class citizen" data structure, it can appear inside compound terms:

```
| ?- Emp = employee(Name,Birth,Dept), Dept = 'Sales',
           Name = name(First,Last), First = 'John'.
     ⟹     Emp = employee(name('John',Last),Birth,'Sales') ?
```

- The `Emp` data structure represents an arbitrary employee with given name `John` who works in the `Sales` department

The logic variable (cont'd)

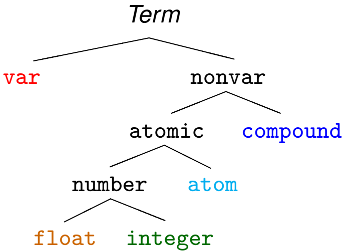
- A variable may also appear several times in a compound, e.g. `name(X,X)` is a Prolog term, which will match the first argument of the employee/3 record, iff the person's first and last names are the same:

```
employee(1, employee(name('John','John'),birthd(2000,12,21),'Sales')).
employee(2, employee(name('Ann','Kovach'),birthd(1988,8,18 ),'HR')).
employee(3, employee(name('Peter','Peter'),birthd(1970,2,12 ),'HR')).

| ?- employee(Num, Emp), Emp = employee(name(_X,_X),_,_).
Num = 1, Emp = employee(name('John','John'),birthd(2000,12,21),'Sales') ? ;
Num = 3, Emp = employee(name('Peter','Peter'),birthd(1970,2,12),'HR') ? ; no
```

- If a variable name starts with an underline, e.g. `_X`, its value is not displayed by the interactive Prolog shell (often called the *top level*)

Classification of Prolog terms

- The taxonomy of Prolog terms – corresponding built-in predicates (BIPs)



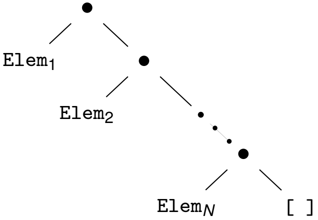
`var(X)`	`X` is a variable
`nonvar(X)`	`X` is not a variable
`atomic(X)`	`X` is a constant (atom or number)
`compound(X)`	`X` is a compound
`number(X)`	`X` is a number
`atom(X)`	`X` is an atom
`float(X)`	`X` is a floating point number
`integer(X)`	`X` is an integer

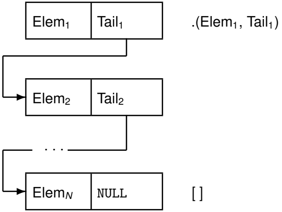
- The five coloured BIPs correspond to the five basic term types.
- Two further type-checking BIPs:
  - `simple(X)`: `X` is not compound, i.e. it is a variable or a constant.
  - `ground(X)`: `X` is a constant or a compound with no (uninstantiated) variables in it.

Another syntactic “sweetener” – list notation

- A Prolog list `[a,b,...]` represents a sequence of terms (cf. linked list)

```
| ?- L = [a,b,c], write_canonical(L).
'.'(a,'.'(b,'.'(c,[])))
```





(Since version 7, SWI Prolog uses `'[|]'`, instead of `'.'` :-((((.)

- The *head* of a list is its first element, e.g. `L`'s head: `a`
  the *tail* is the list of all but the first element, e.g. `L`'s tail: `[b,c]`
- One often needs to split a list to its head and tail: `List = .(Head, Tail).`
  The “square bracketed” counterpart: `List = [Head|Tail]`
- Further sweeteners: $[E_1,E_2,\ldots,E_n|\mathtt{Tail}] \equiv [E_1|[E_2|\ldots,[E_n|\mathtt{Tail}]\ldots]]$
  $[E_1,E_2,\ldots,E_n] \equiv [E_1,E_2,\ldots,E_n|[]]$

Open ended and proper lists

- Example:

```
% head0(L): L's first element is 0.
head0(L) :- L = [0|_].  % '_' is a void, don't care variable

% singleton(L): L has a single element.
singleton([_]).

| ?- singleton(L1). ⇒       L1 = [_A]      % L1 = [_A|[]] is a proper list
| ?- head0(L2).     ⇒       L2 = [0|_A]    % L2 is an open ended list
```

- A Prolog term is called an *open ended* (or *partial*) list iff
  - either it is an unbound variable,
  - or it is a nonempty list structure (i.e. of the form `[_|_]`) and its tail is *open ended*,

  i.e. if sooner or later an unbound variable appears as the tail.
- A list is *closed* or *proper* iff sooner or later an `[]` appears as the tail
- Further examples: `[X,1,Y]` is a proper list, `[X,1|Z]` is open ended.

Working with lists – some practice

(Each occurrence of a void variable (_) denotes a different variable.)

`| ?- [1,2] = [X|Y].`	$\Longrightarrow$	`X = 1, Y = [2] ?`
`| ?- [1,2] = [X,Y].`	$\Longrightarrow$	`X = 1, Y = 2 ?`
`| ?- [1,2,3] = [X|Y].`	$\Longrightarrow$	`X = 1, Y = [2,3] ?`
`| ?- [1,2,3] = [X,Y].`	$\Longrightarrow$	`no`
`| ?- [1,2,3,4] = [X,Y|Z].`	$\Longrightarrow$	`X = 1, Y = 2, Z = [3,4] ?`
`| ?- L = [a,b], L = [_,X|_].`	$\Longrightarrow$	`..., X = b ?  % X = 2nd elem`
`| ?- L = [a,b], L = [_,X,_|_].`	$\Longrightarrow$	`no ?  % length >= 3, X = 2nd elem`
`| ?- L = [1|_], L = [_,2|_].`	$\Longrightarrow$	`L = [1,2|_A] ? % open ended list`

Programming with lists – simple example

- Recall: I/O mode notation for pred. arguments (only in comments): `+`: input (bound), `-`: output (unbound var.), `?`: arbitrary.
- Write a predicate that checks if all elements in a list are the same. Let's call such a list `A`-boring, where `A` is the element appearing repeatedly.

```
% boring(+L, ?A): List L is A-boring.
```

- Transform the following English statements to Prolog clauses
  - `[]` is `A`-boring for every `A`
  - List `L` is `A`-boring, if
    `L`'s head equals `A` and
    `L`'s tail is `A`-boring.
- Remember, you can read '`:-`' as 'if', '`,`' as 'and'

Programming with lists – further examples

- Given a list of numbers, calculate the sum of the list elements.

  ```
  % sum(+L, ?Sum): L sums to Sum. (L is a list of numbers.)
  ```

  - Transform the following English statements to Prolog clauses
    - `[]` sums to 0.
    - A list with head `H` and tail `T` sums to `Sum` if
      `T` sums to `Sum0` and
      `Sum` is the value of `Sum0+H`.
  - Remember, you can do arithmetic calculations with '`is`'
- Given two arbitrary lists, check that they are of equal length.

  ```
  % same_length(?L1, ?L2): Lists L1 and L2 are of equal length.
  ```

  - Transform the following English statements to Prolog clauses
    - `[]` has the same length as `[]`
    - `L1` and `L2` are of equal length if
      the tail of `L1` and the tail of `L2` are of equal length.

Another recursive data structure – binary tree

- A binary tree data structure can be defined as being
  - either a leaf (`leaf`) which contains an integer (`value`)
  - or a node (`node`) which contains two subtrees (`left`,`right`)
- Defining binary tree structures in C and Prolog:

```
% Declaration of a C structure
enum treetype Leaf, Node;
struct tree {
  enum treetype type;
  union {
    struct { int value;
           } leaf;

    struct { struct tree *left;
             struct tree *right;
           } node;
  } u;
};
```

```
% No need to define types in Prolog
% A type-checking predicate can be
% written, if this check is needed:

% is_tree(T): T is a binary tree
is_tree(leaf(Value)) :-
    integer(Value).
is_tree(node(Left,Right)) :-
    is_tree(Left),
    is_tree(Right).
```

Recall: `integer(Value)` is a BIP which succeeds if and only if `V` is an integer.

Calculating the sum of numbers in the leaves of a binary tree

- Calculating the sum of the leaves of a binary tree:
  - if the tree is a leaf, return the integer in the leaf
  - if the tree is a node, add the sums of the two subtrees

```c
% C function (declarative)
int tree_sum(struct tree *tree) {
  switch(tree->type) {
  case Leaf:
   return tree->u.leaf.value;
  case Node:
   return
    tree_sum(tree->u.node.left) +
    tree_sum(tree->u.node.right);
    }
}
```

```prolog
% Prolog procedure
% tree_sum(+T, ?S):
% The sum of the leaves
% of tree T is S.
tree_sum(leaf(Value), S) :-
        S = Value.
tree_sum(node(Left,Right), S) :-

        tree_sum(Left, S1),
        tree_sum(Right, S2),
        S is S1+S2.
```

Sum of Binary Trees – a sample run

```
% sicstus
SICStus 4.3.5 (...)
| ?- consult(tree).     % alternatively: compile(tree). or [tree].
% consulting /home/szeredi/examples/tree.pl...
% consulted /home/szeredi/examples/tree.pl in module user, (...)
| ?- tree_sum(node(leaf(5),
                   node(leaf(3), leaf(2))), Sum).
Sum = 10 ? ; no
| ?- tree_sum(leaf(10), 10).
yes
| ?- tree_sum(leaf(10), Sum).
Sum = 10 ? ; no
| ?- tree_sum(Tree, 10).
Tree = leaf(10) ? ;
! Instantiation error in argument 2 of is/2
! goal:  10 is _73+_74
| ?- halt.
```

The cause of the error: the built-in arithmetic is one-way: the goal `10 is S1+S2` causes an error!

Contents

Two Prolog execution models

- The Goal Reduction model
  - a reformulation of the resolution proof technique
  - good for visualizing the search tree
- The Procedure Box model
  - reflects actual implementation better
  - used by the Prolog trace mechanism

Goal reduction vs. resolution – a propositional example

```
get_fined :-    driving_fast, raining.              (1)
driving_fast :- in_a_hurry.                         (2)
...
in_a_hurry.                                         (3)
raining.                                            (4)
```

- To show that the goal `get_fined` holds, goal reduction repeatedly *reduces* it to other goals using clauses `(1)-(4)`
- When an empty goal (true) is obtained the goal gets proved.

```
(g1)  get_fined                 % (g1) is reduced, using (1), to     (g2)
(g2)  driving_fast, raining     % (g2) is reduced, using (2), to     (g3)
(g3)  in_a_hurry,   raining     % (g3) is reduced, using (3), to     (g4)
(g4)  raining                   % (g4) is reduced, using (4), to     (g5)
(g5)  ■ (empty goal) ≡ true
```

Goal reduction vs. resolution (cnt'd)

```
+get_fined      -driving_fast -raining.        (1)
+driving_fast   -in_a_hurry                    (2)
...
+in_a_hurry.                                   (3)
+raining.                                      (4)
```

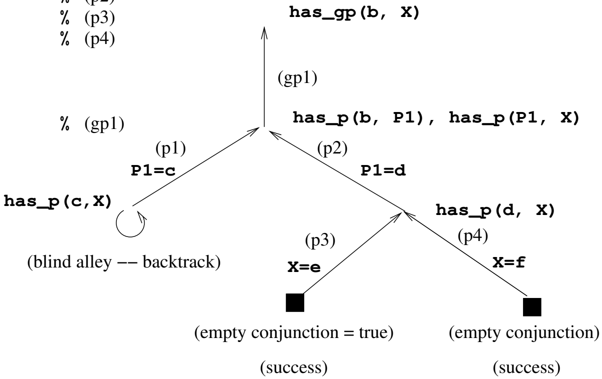
- To show that `get_fined` holds, resolution does an indirect proof
- Assume `get_fined` does not hold, deduce false (contradiction) using clauses (1)–(4)

```
(g1)  -get_fined              % (g1) and          (1) implies (g2)
(g2)  -driving_fast -raining  % (g2) and          (2) implies (g3)
(g3)  -in_a_hurry   -raining  % (g3) and          (3) implies (g4)
(g4)  -raining                % (g4) and          (4) implies (g5)
(g5)   □ (empty clause) ≡ false
```

The Goal Reduction model – the grandparent example

- Goal reduction takes a goal, i.e. a *conjunction* of subgoals G and using a clause C reduces it to goal G', so that $G' \rightarrow G$
- E.g. reducing G = `has_gp(b, X)` using (gp1) gives G' = `has_p(b, P1), has_p(P1, X)`

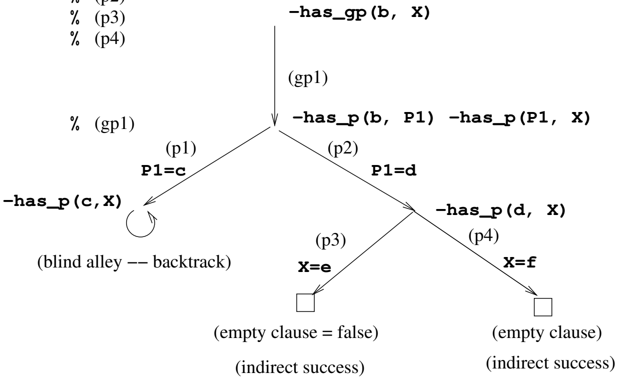
```
has_p(b, c).              % (p1)
has_p(b, d).              % (p2)
has_p(d, e).              % (p3)
has_p(d, f).              % (p4)

has_gp(GC, GP) :-
        has_p(GC, P),
        has_p(P, GP).     % (gp1)

| ?- has_gp(b, X).
```



Resolution – same example

- Resolution takes a negated goal *NG* (which is a *disjunction* of neg. literals) and using a clause *C* deduces new negated goal *NG′*, so that $NG \rightarrow NG'$
- E.g. resolving NG = `-has_gp(b, X)` using (gp1) gives
  NG' = `-has_p(b, P1) -has_p(P1, X)`

```
+has_p(b, c).              % (p1)
+has_p(b, d).              % (p2)
+has_p(d, e).              % (p3)
+has_p(d, f).              % (p4)

+has_gp(GC, GP)
        -has_p(GC, P),
        -has_p(P, GP).     % (gp1)

-has_gp(b, X).
```



The Goal Reduction model (ADVANCED)

Goal reduction: a goal is viewed as a conjunction of subgoals

- Given a goal $G = A, B, \ldots$ and a clause (A `:-` $D, \ldots$)
  $G' = B, \ldots, D, \ldots$ is obtained as the new goal

Goal reduction is the same as resolution, but viewed as backwards reasoning

- Resolution:
  - to prove `A`$\wedge$ `B` $\wedge$ `...`, we negate it obtaining $\neg G_0 =$ `-A -B ...`
  - resolution step : clause $Cl =$ (`+A -D ...`) resolved with $\neg G_0$
    produces $\neg G_1 =$ `-D ...-B ...`

    $\neg G_n \wedge Cl \rightarrow \neg G_{n+1}$ (resolution)
  - success of indirect proof: reaching an empty clause $\square \equiv$ false
- Goal reduction:
  - to prove `A`$\wedge$ `B` $\wedge$ `...`, we start with $G_0 =$ `A, B, ...`
  - reduction step : using $Cl =$ (`A :- D, ...`) one can reduce G_0 to
    $G_1 =$ `D, ..., B, ...`

    $G_{n+1} \wedge Cl \rightarrow G_n$ (reduction)
  - success of the reduction proof: reaching an empty goal $\blacksquare \equiv$ true
- the (resolution) and (reduction) reasoning rules are equivalent!

The definition of a goal reduction step

Reduce a goal G to a new goal G' using a program clause Cl_i:

- Split goal G into the first subgoal G_F and the residual goal G_R
- **Copy** clause Cl_i, i.e. rename all variables to new ones, and split the copy to a head H and body B
- **Unify** the goal G_F and the head H
  - If the unification fails, exit the reduction step with failure
  - If the unification succeeds with a substitution σ, return the new goal $G' = (B, G_R)\sigma$ (i.e. apply σ to both the body and the residual goal)

E.g., slide 105: $G =$ `has_gp(b, X)` using (gp1) $\Rightarrow G' =$ `has_p(b, P1),has_p(P1, X)`

Reduce a goal G to a new goal G' by executing a built-in predicate (BIP)

- Split goal G into the first, BIP subgoal G_F and the residual goal G_R
- **Execute** the BIP G_F
  - If the BIP fails then exit the reduction step with failure
  - If the BIP succeeds with a substitution σ then return the new goal $G' = G_R\sigma$

The goal reduction model of Prolog execution – outline

- This model describes how Prolog builds and traverses a search tree
- A web app for practicing the model: `https://ait.plwin.dev/P1-1`
- The inputs:
  - a Prolog program (a sequence of clauses), e.g. the `has_gp` program
  - a goal, e.g. `:- has_gp(b, GP).`
    extended with a special goal, carrying the solution: `answer(Sol)`:

```
:- has_gp(b, GP),answer(GP).      % Who are the grandparents of a?
:- has_gp(Ch,GP),answer(Ch-GP).   % Which are the child-gparent pairs?
```

- When only an `answer` goal remains, a solution is obtained
- Possible outcomes of executing a Prolog goal:
  - Exception (error), e.g. `:- Y = apple, X is Y+1.`
    (This is not discussed further here)
  - Failure (no solutions), e.g. `:- has_p(c, P), answer(P).`
  - Success (1 or more solutions), e.g. `:- has_p(d, P), answer(P).`

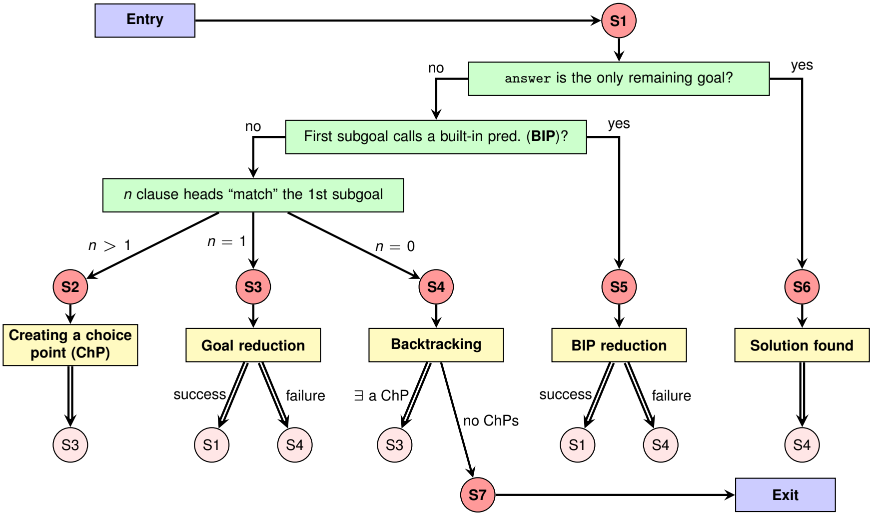
The main data structures used in the model

- There are only two (imperative, mutable) variables in this model: `Goal`: the current goal sequence, `ChPSt` the stack of choice points (ChPs)
- If, in a reduction step, two or more clause heads unify (match) the first subgoal, a new `ChPSt` entry is made, storing:
  - the list of clauses with possibly matching heads
  - the current goal sequence (i.e. `Goal`)

ChPoint name	Clause list	Goal	
CHP2	[p3,p4]	(4)	`hasP(d,Y),answer(b-Y).`
CHP1	[p2,p3,p4]	(2)	`hasP(X,P),hasP(P,Y),answer(X-Y).`

- At a failure, the top entry of the `ChPSt` is examined:
  - the goal stored there becomes the current `Goal`,
  - the first element of the list of clauses is removed, the second is remembered the as the “current clause”,
  - if the list of clauses is now a singleton, the top entry is removed,
  - finally the `Goal` is reduced, using the current clause.
- If, at a failure, `ChPSt` is empty, execution ends.

The flowchart of the Prolog goal reduction model



(Double arrows indicate a jump to the step in the pink circle, i.e. execution continues at the given red circle.)

Remarks on the flowchart

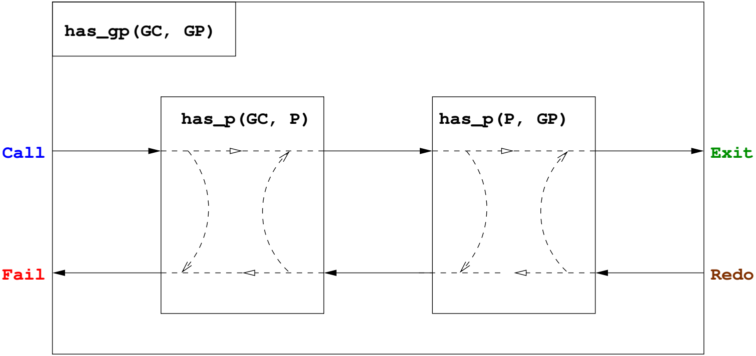
- There are seven different execution steps: **S1–S7**, where **S1** is the initial (but also an intermediate) step, and **S7** represents the final state.
- The main task of **S1** is to branch to one of **S2–S6**:
  - when `Goal` contains an `answer` goal only ⇒ **S6**;
  - when the first subgoal of `Goal` calls a BIP ⇒ **S5**;
  - otherwise the first subgoal calls a user predicate. Here a set of clauses is selected which *contains* all clauses whose heads match the first subgoal (this may be a *superset* of the matching ones). Based on the number of clauses ⇒ **S2**, **S3** or **S4**.
- **S2** creates a new `ChPSt` entry, and ⇒ **S3** (to reduce with the first clause).
- **S3** performs the reduction. If that fails ⇒ **S4**, otherwise ⇒ **S1**.
- **S4** retrieves the next clause from the top `ChPSt` entry, if any (⇒ **S3**), otherwise execution ends (⇒ **S7**).
- In **S5**, similarly to **S3**, if the BIP succeeds ⇒ **S1**, otherwise ⇒ **S4**.
- In **S6**, the solution is displayed and further solutions are sought (⇒ **S4**).

The Procedure Box execution model – example

- The procedure box execution model of `has_gp`

```
has_gp(GC, GP) :- has_p(GC, P), has_p(P, GP).
```

```
has_p(b, c).
has_p(b, d).
has_p(d, e).
has_p(d, f).
```



Prolog tracing, based on the four port box model

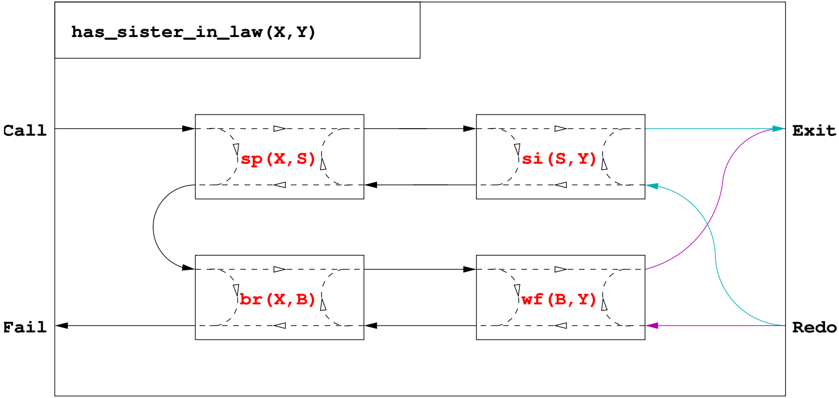
```
| ?- consult(gp3).
% consulting gp3.pl...
% consulted gp3.pl ...
yes
| ?- listing.
has_gp(Ch, G) :-
        has_p(Ch, P),
        has_p(P, G).

has_p(b, c).
has_p(b, d).
has_p(d, e).
has_p(d, f).

yes
| ?- trace.
% The debugger will ...
yes
```

```
| ?- has_gp(Ch, f).
Det? BoxId  Depth Port  Goal
          1       1 Call: has_gp(Ch,f) ?
          2       2 Call: has_p(Ch,P) ?
?         2       2 Exit: has_p(b,c) ?
          3       2 Call: has_p(c,f) ?
          3       2 Fail: has_p(c,f) ?
          2       2 Redo: has_p(b,c) ?
?         2       2 Exit: has_p(b,d) ?
          4       2 Call: has_p(d,f) ?
          4       2 Exit: has_p(d,f) ?
                    No choice left in box 4, box removed (no ?)
?         1       1 Exit: has_gp(b,f) ?
Ch = b ? ;
          1       1 Redo: has_gp(b,f) ?
          2       2 Redo: has_p(b,d) ?
?         2       2 Exit: has_p(d,e) ?
          5       2 Call: has_p(e,f) ?
          5       2 Fail: has_p(e,f) ?
          2       2 Redo: has_p(d,e) ?
          2       2 Exit: has_p(d,f) ?
                    No choice left in box 2, box removed  (no ?)
          6       2 Call: has_p(f,f) ?
          6       2 Fail: has_p(f,f) ?
          1       1 Fail: has_gp(Ch,f) ?
no
| ?-
```

The procedure-box of multi-clause predicates

'Sister in law' can be one's spouse's sister; or one's brother's wife:

```
has_sister_in_law(X, Y) :-
    has_spouse(X, S), has_sister(S, Y).
has_sister_in_law(X, Y) :-
    has_brother(X, B), has_wife(B, Y).
```



The procedure-box of a “database” predicate of facts

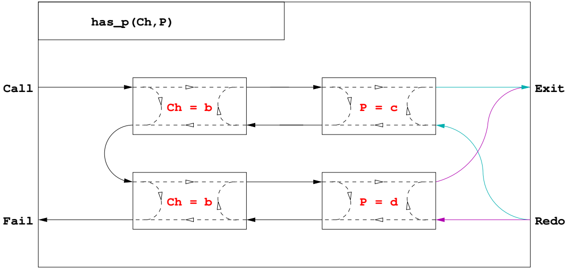
- In general in a multi-clause predicate the clauses have different heads
- A database of facts is a typical example:

```
has_p(b, c).
has_p(b, d).
```

- These clauses can be massaged to have the same head:

```
has_p(Ch, P) :-  Ch = b, P = c.
has_p(Ch, P) :-  Ch = b, P = d.
```

- Consequently, the procedure-box of this predicate is this:



Contents

Summary – syntax of Prolog predicates, clauses

Example

```
% A  predicate with two clauses, the functor is: tree_sum/2
tree_sum(leaf(Val), Val).         %                 clause 1, fact
tree_sum(node(Left,Right), S) :-  %         head    \
    tree_sum(Left, S1),           % goal    \       |
    tree_sum(Right, S2),          % goal    | body  | clause 2, rule
    S is S1+S2.                   % goal    /       /
```

Syntax

⟨ program ⟩	::=	⟨ predicate ⟩ . . .	{i.e. a sequence of predicates}
⟨ predicate ⟩	::=	⟨ clause ⟩ . . .	{with the same functor}
⟨ clause ⟩	::=	⟨ fact ⟩.␣ |	
		⟨ rule ⟩.␣	
⟨ fact ⟩	::=	⟨ head ⟩	
⟨ rule ⟩	::=	⟨ head ⟩:-⟨ body ⟩	{clause functor = head functor}
⟨ body ⟩	::=	⟨ goal ⟩, . . .	{i.e. a seq. of goals sep. by commas}
⟨ head ⟩	::=	⟨ callable term ⟩	{atom or compound}
⟨ goal ⟩	::=	⟨ callable term ⟩	{or a variable, if instantiated to a callable}

Prolog terms (canonical form)

Example – a clause head as a term

```
% tree_sum(node(Left,Right), S)    % compound term, has the
% ________ ________________  _     % functor tree_sum/2
%     |          |           |
% compound name  \         argument, variable
%                 \ - argument, compound term
```

Syntax

⟨ term ⟩	::=	⟨ variable ⟩ |	{has no functor}
		⟨ constant ⟩ |	{⟨ constant ⟩/0}
		⟨ compound term ⟩ |	{⟨ comp. name ⟩/⟨ # of args ⟩}
		. . . extensions . . .	{lists, operators}
⟨ constant ⟩	::=	⟨ atom ⟩ |	{symbolic constant}
		⟨ number ⟩	
⟨ number ⟩	::=	⟨ integer ⟩ | ⟨ float ⟩	
⟨ compound term ⟩	::=	⟨ comp. name ⟩ (⟨ argument ⟩, . . .)	
⟨ comp. name ⟩	::=	⟨ atom ⟩	
⟨ argument ⟩	::=	⟨ term ⟩	
⟨ callable term ⟩	::=	⟨ atom ⟩ | ⟨ compound term ⟩	

Lexical elements

Examples

```
% variable:       Fact FACT _fact X2 _2 _
% atom:           fact ≡ 'fact' 'István' [] ; ',' += ** \= ≡ '\\='
% number:         0 -123 10.0 -12.1e8
% not an atom:    !=, István
% not a number:   1e8 1.e2
```

Syntax

⟨ variable ⟩	::=	⟨ capital letter ⟩⟨ alphanum ⟩... | _ ⟨ alphanum ⟩...
⟨ atom ⟩	::=	'⟨ quoted char ⟩...' | ⟨ lower case letter ⟩⟨ alphanum ⟩... | ⟨ sticky char ⟩... | ! | ; | [] | {}
⟨ integer ⟩	::=	{signed or unsigned sequence of digits }
⟨ float ⟩	::=	{ a sequence of digits with a compulsory decimal point in between, with an optional exponent}
⟨ quoted char ⟩	::=	{any non ' and non \ character} | \ ⟨ escaped char ⟩
⟨ alphanum ⟩	::=	⟨ lower case letter ⟩ | ⟨ upper case letter ⟩ | ⟨ digit ⟩ | _
⟨ sticky char ⟩	::=	+ | - | * | / | \ | $ | ^ | < | > | = | ` | ~ | : | . | ? | @ | # | &

Comments and layout in Prolog

- Comments
  - From a `%` character till the end of line
  - From `/*` till the next `*/`
- Layout (spaces, newlines, tabs, comments) can be used freely, except:
  - No layout allowed between the name of a compound and the "`(`"
  - If a prefix operator (see later) is followed by "`(`", these have to be separated by layout
  - Clause terminator (`.␣`): a stand-alone full stop (i.e., one not preceded by a sticky char), followed by layout
- The recommended formatting of Prolog programs:
  - Write clauses of a predicate continuously, no empty lines between
  - Precede each pred. by an empty line and a spec (head comment)

    ```
    % predicate_name(A1, ..., An): A declarative sentence (statement)
    % describing the relationship between terms A1, ..., An
    ```
  - Write the head of the clause at the beginning of a line, and prefix each goal in the body with an indentation of a few (8 recommended) spaces.

Contents

Disjunctions

- Disjunctions (i.e. subgoals separated by “or”) can appear as goals
- A disjunction is denoted by semicolon (“;”)
- Enclose the **whole** disjunction in parentheses, align chars `(`, `;` and `)`

```
has_sister_in_law(X, Y) :-
    (   has_spouse(X, S), has_sister(S, Y)
    ;   has_brother(X, B), has_wife(B, Y)
    ).
```

- The above predicate is equivalent to:

```
has_sister_in_law(X, Y) :- has_spouse(X, S), has_sister(S, Y).
has_sister_in_law(X, Y) :- has_brother(X, B), has_wife(B, Y).
```

- A disjunction is itself a valid goal, it can appear in a conjunction:

```
has_ancestor(X, A) :-
    has_parent(X, P), (   A = P
                      ;   has_ancestor(P, A)
                      ).
```

Can you make an equivalent variant which does not use “;”?

Disjunctions, continued

- An example with multiple disjunctions:

```
% first_1(L): the first nonzero element of L is 1.
first_1([A,B,C]) :-
        (   A = 1
        ;   A = 0,
            (   B = 1
            ;   B = 0, C = 1
            )
        ).
```

- Note: the `V=Term` goals can no longer be got rid of in disjunctions
- Comma binds more tightly than semicolon, e.g.
  `p :- ( q, r ; s )` ≡ `p :- ((q, r) ; s).`
  Please, never enclose disjuncts (goals on the sides of `;`) in parentheses!
- You can have more than two-way "or"s:
  `p :- ( a ; b ; c ; ...)` which is the same as
  `p :- ( a ; (b ; (c ; ...)))`
- Please, do not use the unnecessary parentheses (colored red)!

Expanding disjunctions to helper predicates

- Example: `p :- q, (r ; s).`

  Distributive expansion inefficient, as it calls `q` twice:

  ```
  p :- q, r.
  p :- q, s.
  ```

- For an efficient solution introduce a helper predicate. Example:

  ```
  t(X, Z) :-
        p(X,Y),
        (   q(Y,U), r(U,Z)
        ;   s(Y, Z)
        ;   t(Y), w(Z)
        ),
        v(X, Z).
  ```

- Collect variables that occur both inside and outside the disj. – `Y,Z`.
- Define a helper predicate – `aux(Y,Z)` – with these vars as args, transform each disjunct to a separate clause of the helper predicate:

  ```
  aux(Y, Z) :- q(Y,U), r(U,Z).
  aux(Y, Z) :- s(Y, Z).
  aux(Y, Z) :- t(Y), w(Z).
  ```

- Replace the disjunction with a call of the helper predicate:

  ```
  t(X, Z) :- p(X, Y), aux(Y, Z), v(X, Z).
  ```

The if-then-else construct

- When the two branches of a disjunction exclude each other, use the if-then-else construct `( condition -> then ; else )`. Example:

```
% pow(A, E, P): P is A to the power E.
pow(A, E, P) :-                              pow1(A, E, P) :-
    (   E > 0, E1 is E-1,     ⟹                 (   E > 0 -> E1 is E-1,
        pow(A, E1, P1),                              pow(A, E1, P1),
        P is A*P1                                    P is A*P1
    ;   E = 0, P = 1                             ;   E = 0, P = 1
    ).                                           ).
```

- `pow1` is about 25% faster than `pow` and requires much less memory
- The atom `->` is a standard operator
- The construct `( Cond -> Then ; Else )` is executed by first executing `Cond`. If this succeeds, `Then` is executed, otherwise `Else` is executed.
- **Important**: Only the first solution of `Cond` is used for executing `Then`. The remaining solutions are discarded!
- Note that `( Cond -> Then ; Else )` looks like a disjunction, but it is not
- The else-branch can be omitted, it defaults to `false`.

Defining “childless” using if-then-else

- Given the `has_parent/2` predicate, define the notion of a `childless` person
- If we can find a child of a GIVEN person, then `childless` should fail, otherwise it should succeed.

```
% childless(+Person): A given Person has no children
childless(Person) :-      (   has_parent(_, Person) -> fail
                          ;   true
                          ).
```

- What happens if you call `childless(P)`, where `P` is an unbound var? Will it enumerate childless people in `P`? No, it will simply fail.
- The above if-then-else can be simplified to:

```
childless(Person) :- \+ has_parent(_, Person).
```

- “`\+`” is called Negation by Failure, “`\+ G`” runs by executing `G`:
  - if `G` fails “`\+ G`” succeeds.
  - if `G` succeeds “`\+ G`” fails (ignoring further solutions of `G`, if any)
- Since a failed goal produces no bindings, “`\+ G`” will never bind a variable.
- Read “`\+`” as “not provable”, cf. $\nvdash$ tilted slightly to the left.

Open and closed world assumption

```
has_parent(a, b).  has_parent(a, c).  has_parent(c, d).        (1)-(3)
```

- Does `(1)-(3)` imply that `a` is childless: $\varphi = \forall x.\neg \mathtt{has\_parent}(x, \mathtt{a})$?
- No. Although `has_parent(Ch, a)` cannot be proven, φ does not hold!
- But in the world of databases we do conclude that `a` is childless...
- Databases use the Closed World Assumption (CWA): anything that cannot be proven is considered false.
- Mathematical logic uses the Open World Assumption (OWA)
  - A statement *S* follows from a set of statements *P* (premises), if *S* holds in any world (interpretation) that satisfies *P*.
  - thus φ is not a logical consequence of `(1)-(3)`
- Classical logic (OWA) is monotonic:
  the more you know, the more you can deduce
- Negation by failure (CWA) is non-monotonic:
  add the fact "`has_parent(e, a).`" to `(1)-(3)` and `\+ has_parent(_, a)` will fail.

Checking inequality – siblings and cousins

```
has_p('Charles', 'Elizabeth').  has_p('Andrew',  'Elizabeth').
has_p('William', 'Charles').    has_p('Beatrice', 'Andrew').
has_p('Harry',   'Charles').    has_p('Eugenie', 'Andrew').
```

- Recall homework L4, define predicate `has_sibling/2`, first attempt:

```
has_sibling0(A, B) :- \+ A = B, has_p(A, P), has_p(B, P).
```

- `has_sibling0` does not work properly, e.g. this goal fails:

```
| ?- has_sibling0('Charles', X).
```

  because `\+ 'Charles' = X` fails (as `'Charles' = X` succeeds)

- Negated goals should be instantiated as much as possible, therefore always place them at the end of the body:

```
has_sibling(A, B) :- has_p(A, P), has_p(B, P), \+ A = B.
```

- Define `has_cousin/2` (using `has_gp/2`, the “has grandparent” predicate)

```
has_cousin(A, B) :-
        has_gp(A, GP), has_gp(B, GP), \+ has_sibling(A, B), A \= B.
```

- Note that the BIP `A \= B` is equivalent to `\+ A = B`

The relationship of if-then-else and negation

- Negation can be fully defined using if-then-else

```
\+ p         ≡         (   p -> false
                       ;   true
                       )
```

- If-then-else can be transformed to a disjunction with a negation:

```
(   cond -> then                (   cond, then
;   else              ⟹        ;   \+ cond, else
)                               )
```

These are equivalent only if `cond` succeeds at most once.
The if-then-else is more efficient (no choice point left).

- As semicolon is associative, there is no need to use nested parentheses `(...)` if multiple if-then-else branches are present (and please don't):

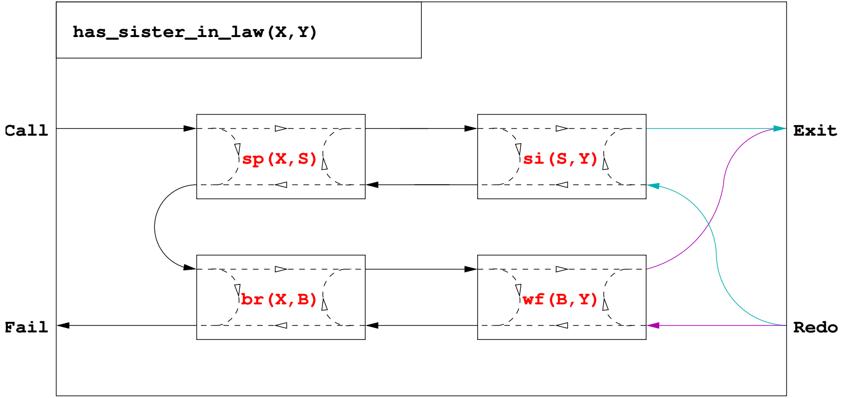
```
(   cond1 -> then1                    (   cond1 -> then1
;   (   cond2 -> then2                ;   cond2 -> then2
    ;   (   (...)   )       ⟹        ;   (...)
    )
;   else                              ;   else
)                                     )
```

The procedure-box of disjunctions

A disjunction can be transformed into a multi-clause predicate

```
has_sister_in_law(X, Y) :-
    (   has_spouse(X, S), has_sister(S, Y)
    ;
        has_brother(X, B), has_wife(B, Y)
    ).
```

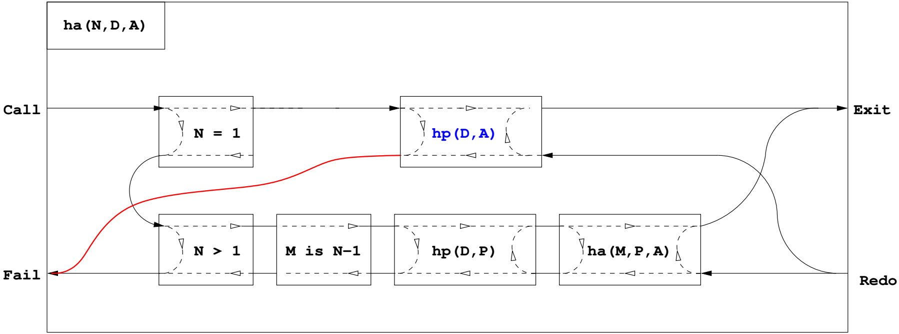
```
has_sister_in_law(X, Y) :-
    has_spouse(X, S), has_sister(S, Y).
has_sister_in_law(X, Y) :-
    has_brother(X, B), has_wife(B, Y).
```



The procedure box for if-then-else

```
% ha(+N, ?D, ?A): D has A as their Nth generation ancestor (N>0 int)
% The 1st,     2nd,          3rd generation ancestors are
%      parents, grandparents, great-grandparents etc.
ha(N, D, A) :-
        (   N = 1 -> hp(D, A)                        % hp(D, A): D has a parent A
        ;   N > 1, M is N-1, hp(D, P), ha(M, P, A)
        ).
```

.

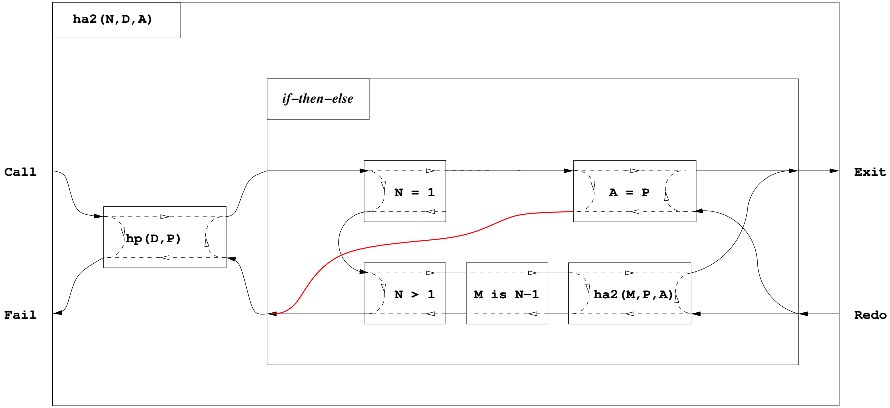


- Failure of the "then" part leads to failure of the whole if-then-else construct

The if-then-else box, continued

- When an if-then-else occurs in a conjunction, or there are multiple clauses, then it requires a separate box

```
ha2(N, D, A) :- hp(D, P), (   N = 1 -> A = P
                          ;   N > 1, M is N-1, ha2(M, P, A)
                          ).
```



Contents

Introducing operators

- Example: `S is -S1+S2` is equivalent to: `is(S, +(-(S1),S2))`
- Syntax of terms using operators

  ⟨ comp. term ⟩ ::=

	⟨ comp. name ⟩ (⟨ argument ⟩, . . .)	{so far we had this}
|	⟨ argument ⟩ ⟨ operator name ⟩ ⟨ argument ⟩	{infix term}
|	⟨ operator name ⟩ ⟨ argument ⟩	{prefix term}
|	⟨ argument ⟩ ⟨ operator name ⟩	{postfix term}
|	(⟨ term ⟩)	{parenthesized term}

  ⟨ operator name ⟩ ::= ⟨ comp. name ⟩ {if declared as an operator}

- The built-in predicate for defining operators:

  `op(Priority, Type, Op)` or `op(Priority, Type, [Op₁,Op₂,...])`:
  - `Priority`: an int. between 1 and 1200 – smaller priorities bind tighter
  - `Type` determines the placement of the operator and the associativity: infix: `yfx`, `xfy`, `xfx`; prefix: `fy`, `fx`; postfix: `yf`, `xf` (`f` – op, `x`, `y` – args)
  - `Op` or $\mathtt{Op}_i$: an arbitrary atom
- The call of the BIP `op/3` is normally placed in a directive, executed immediately when the program file is loaded, e.g.:

```
:- op(800, xfx, [has_tree_sum]).        leaf(V) has_tree_sum V.
```

Characteristics of operators

Operator properties implied by the operator type

Type			Class	Interpretation
left-assoc.	right-assoc.	non-assoc.		
`yfx`	`xfy`	`xfx`	infix	`X f Y` $\equiv$ `f(X, Y)`
	`fy`	`fx`	prefix	`f X` $\equiv$ `f(X)`
`yf`		`xf`	postfix	`X f` $\equiv$ `f(X)`

Parentheses implied by operator priorities and associativities

- `a/b+c*d` $\equiv$ `(a/b)+(c*d)` as the priority of `/` and `*` (400) is less than the priority of `+` (500) smaller priority = **stronger** binding
- `a-b-c` $\equiv$ `(a-b)-c` as operator `-` has type `yfx`, thus it is left-associative, i.e. it binds to the left, the leftmost operator is parenthesized first (the position of `y` wrt. `f` shows the direction of associativity)
- `a^b^c` $\equiv$ `a^(b^c)` as `^` has type `xfy`, therefore it is right-associative
- `a=b=c` $\Longrightarrow$ syntax error, as `=` has type `xfx`, it is non-associative
- the above also applies to different operators of same type and priority: `a+b-c+d` $\equiv$ `((a+b)-c)+d`

Standard built-in operators

Standard operators

1200	xfx	`:- -->`
1200	fx	`:- ?-`
1100	xfy	`;`
1050	xfy	`->`
1000	xfy	`','`
900	fy	`\+`

700	xfx	`= \= =..`
		`< =< =:= =\=`
		`> >= is`
		`== \==`
		`@< @=< @> @>=`

500	yfx	`+ - /\ \/`
400	yfx	`* / // rem` `mod << >>`
200	xfx	`**`
200	xfy	`^`
200	fy	`- \`

Further built-in operators of SICStus Prolog

1150	fx	`mode public dynamic`
		`volatile discontiguous`
		`initialization multifile`
		`meta_predicate block`

1100	xfy	`do`
900	fy	`spy nospy`
550	xfy	`:`
500	yfx	`\`
200	fy	`+`

Operators – additional comments

- The “comma” is heavily overloaded:
  - it separates the arguments of a compound term
  - it separates list elements
  - it is an `xfy` op. of priority 1000, e.g.: `(p:-a,b,c)`≡`:-(p,',' (a,',' (b,c)))`
- Ambiguities arise, e.g. is `p(a,b,c)` $\stackrel{?}{\equiv}$ `p((a,b,c))`?
- Disambiguation: if the outermost operator of a compound argument has priority $\geq$ 1000, then it should be enclosed in parentheses

```
| ?- write_canonical((a,b,c)).  ⇒  ','(a,','(b,c))

| ?- write_canonical(a,b,c).    ⇒  Error: ! write_canonical/3 does not exist

| ?- write_canonical((hgp(A,B):-hp(A,C),hp(C,B))).

                                ⇒  :-(hgp(A,B),','(hp(A,C),hp(C,B)))
```

- Note: an unquoted comma (`,`) is an operator, but not a valid atom

Functions and operators allowed in arithmetic expressions

- The Prolog standard prescribes that the following functions can be used in arithmetic expressions:

plain arithmetic:

`+X, -X, X+Y, X-Y, X*Y, X/Y,`
`X//Y` (int. division, truncates towards 0),
`X div Y` (int. division, truncates towards $-\infty$),
`X rem Y` (remainder wrt. `//`),
`X mod Y` (remainder wrt. `div`),
`X**Y, X^Y` (both denote exponentiation)

conversions:

`float_integer_part(X), float_fractional_part(X), float(X),`
`round(X), truncate(X), floor(X), ceiling(X)`

bit-wise ops:

`X/\Y, X\/Y, xor(X,Y), \ X` (negation), `X<<Y, X>>Y` (shifts)

other:

`abs(X), sign(X), min(X,Y), max(X,Y),`
`sin(X), cos(X), tan(X), asin(X), acos(X), atan(X),`
`atan2(X,Y), sqrt(X), log(X), exp(X), pi`

Uses of operators

- What are operators good for?
  - to allow usual arithmetic expressions, such as in `X is (Y+3) mod 4`
  - processing of symbolic expressions (such as symbolic derivation)
  - for writing the clauses themselves
    (`:-`, `','`, `;` ... are all standard operators)
    - clauses can be passed as arguments to meta-predicates:
      `asserta( (p(X):-q(X),r(X)) )`
  - to make Prolog data structures look like natural language sentences (controlled English), e.g. Smullyan's island of knights and knaves (knights always tell the truth, knaves always lie):
    We meet natives A and B, A says: one of us is a knave.
    `| ?- solve_puzzle(A says A is a knave or B is a knave).`
  - to make data structures more readable:
    `acid(sulphur, h*2-s-o*4).`

Classical symbolic computation: symbolic derivation

- Write a Prolog predicate which calculates the derivative of a formula built from numbers and the atom `x` using some arithmetic operators.

```
% deriv(Formula, D):  D is the derivative of Formula with respect to x.
deriv(x, 1).
deriv(C, 0) :-                  number(C).
deriv(U+V, DU+DV) :-            deriv(U, DU), deriv(V, DV).
deriv(U-V, DU-DV) :-            deriv(U, DU), deriv(V, DV).
deriv(U*V, DU*V + U*DV) :-      deriv(U, DU), deriv(V, DV).
```

```
| ?- deriv(x*x+x, D).       ⟹    D = 1*x+x*1+1 ? ; no

| ?- deriv((x+1)*(x+1), D).
                            ⟹    D = (1+0)*(x+1)+(x+1)*(1+0) ? ; no

| ?- deriv(I, 1*x+x*1+1).   ⟹    I = x*x+x ? ; no

| ?- deriv(I, 2*x+1).       ⟹    no

| ?- deriv(I, 0).           ⟹    no
```

Contents

Concatenating lists

- Let `L1` ⊕ `L2` denote the concatenation of `L1` and `L2`, i.e. a list consisting of the elements of `L1` followed by those of `L2`.
- Building `L1` ⊕ `L2` in an imperative language (A list is either a `NULL` pointer or a pointer to a head-tail structure):
  - Scan `L1` until you reach a tail which is `NULL`
  - Overwrite the `NULL` pointer with `L2`
- If you still need the original `L1`, you have to copy it, replacing its final `NULL` with `L2`. A recursive definition of the ⊕ (concatenation) function:

```
L1 ⊕ L2 =  if L1 == NULL return L2
            else L3 = tail(L1) ⊕ L2
            return a new list structure whose head is head(L1)
                                        and whose tail is L3
```

- Transform the above recursive definition to Prolog:

```
% app0(A, B, C):  the conc(atenation) of A and B is C
app0([], L2, L2).         % The conc. of [] and L2 is L2.
app0([X|L1], L2, L) :-    % The conc. of [X|L1] and L2 is L if
    app0(L1, L2, L3),     % the conc. of L1 and L2 is L3 and
    L = [X|L3].           % L's head is X and L's tail is L3.
```

Efficient and multi-purpose concatenation

- Drawbacks of the `app0/3` predicate:
  - Uses “real” recursion (needs stack space proportional to length of `L1`)
  - Cannot split lists, e.g. `app0(L1,[3],[1,3])` $\rightsquigarrow$ infinite loop
- Apply a generic optimization: eliminate variable assignments
  - Remove goal `Var = T`, and replace occurrences of variable `Var` by `T`

  Not applicable in the presence of disjunctions or if-then-else
- Apply this optimization to the second clause of `app0/3`:

```
app0([X|L1], L2, L) :- app0(L1, L2, L3), L = [X|L3].
```

- The resulting code (renamed to `app`, also available as the BIP `append/3`)

```
% app(A, B, C): The conc. of A and B is C, i.e.C = A⊕B
app([], L2, L2).             % The conc. of [] and L2 is L2.
app([X|L1], L2, [X|L3]) :-   % The conc. of [X|L1] and L2 is [X|L3] if
    app(L1, L2, L3).         % the conc. of L1 and L2 is L3.
```

  - This uses constant stack space and can be used for multiple purposes, thanks to Prolog allowing open ended lists

Tail recursion optimization

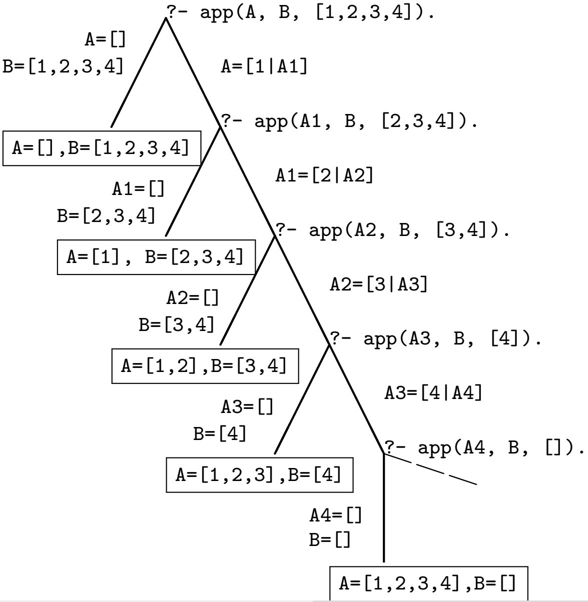
- Tail recursion optimization (TRO), or more generally last call optimization (LCO) is applicable if
  - the goal in question is the last to be executed in a clause body, and
  - no choice points exist in the given predicate.
- LCO is applicable to the recursive call of `app/3`:

```
app([], L, L).
app([X|L1], L2, [X|L3]) :- app(L1, L2, L3).
```

- This feature relies on open ended lists:
  - It is possible to build a list node *before* building its tail
  - This corresponds to passing to append a pointer to the location where the resulting list should be stored.
- Open ended lists are possible because unbound variables are *first class* objects, i.e. unbound variables are allowed inside data structures. (This type of variable is often called the logic variable).

Splitting lists using `append`

```
% app(L1, L2, L3):
% L1 ⊕ L2 = L3.
app([], L, L).
app([X|L1], L2, [X|L3]) :-
    app(L1, L2, L3).

| ?- app(A, B, [1,2,3,4]).
A = [], B = [1,2,3,4] ? ;
A = [1], B = [2,3,4] ? ;
A = [1,2], B = [3,4] ? ;
A = [1,2,3], B = [4] ? ;
A = [1,2,3,4], B = [] ? ;
no
```



How does the "openness" of arguments affect `append(L1,L2,L3)`?

- `L2` is never decomposed ("looked inside") by append, whether it is open ended, does not affect execution
- If `L1` is closed, `append` produces at most one answer

```
| ?- append([a,b], Tail, L).          ⟹  L = [a,b|Tail] ? ; no
| ?- append([a,b], [c|T], L).         ⟹  L = [a,b,c|T] ?  ; no
| ?- append([a,b], [c|T], [_,_,d,_]). ⟹  no
```

- If `L3` is closed (of length *n*), `append` produces at most $n+1$ solutions, where `L1` and `L2` are closed lists (also see previous slide):

```
| ?- append(L1,L2,[1,2]).  ⟹  L1=[], L2=[1,2] ? ; L1=[1], L2=[2] ? ;
                               L1=[1,2], L2=[] ? ; no
| ?- append([1,2], L, [1,2,3,4,5]).  ⟹  L = [3,4,5] ? ; no
| ?- append(L1,[4|L2],[1,2,3,4,5]).  ⟹  L1 = [1,2,3],L2 = [5] ? ; no
| ?- append(L1,[4,2],[1,2,3,4,5]).  ⟹  no
```

- The search may be infinite: if **both** the 1st **and** the 3rd arg. is open ended

```
| ?- append([1|L1], [a,b], L3).          ⟹
                  L1 = [], L3 = [1,a,b] ? ;
                  L1 = [_A], L3 = [1,_A,a,b] ? ;
                  L1 = [_A,_B], L3 = [1,_A,_B,a,b] ? ;     ad infinitum :-((((
| ?- append([1|L1], L2 , [2|L3]).          ⟹  no
```

Eight ways of using `append(L1,L2,L3)` (safe or unsafe)

```
  :- mode append(+, +, +).   % checking if L1 ⊕ L2 = L3 holds
| ?- append([1,2], [3,4], [1,2,3,4]).     ⟹  yes

  :- mode append(+, +, -).   % appending L1 and L2 to obtain L3
| ?- append([1,2], [3,4], L3).            ⟹  L3 = [1,2,3,4] ? ; no

  :- mode append(+, -, +).   % checking if L1 is a prefix of L3, obtaining L2
| ?- append([1,2], L2, [1,2,3,4]).         ⟹  L2 = [3,4] ? ; no

  :- mode append(+, -, -).   % prepending L1 to an open ended L2 to obtain L3
| ?- append([1,2], [3|L2], L3).           ⟹  L3 = [1,2,3|L2] ? ; no

  :- mode append(-, +, +).   % checking if L2 is a suffix of L3 to obtain L1
| ?- append(L1, [3,4], [1,2,3,4]).         ⟹  L1 = [1,2] ? ; no

  :- mode append(-, -, +).   % splitting L3 to L1 and L2 in all possible ways
| ?- append(L1, L2, [1]).  ⟹  L1=[],L2=[1] ? ; L1=[1],L2=[] ? ; no

  :- mode append(-, +, -). (see prev. slide) and :- mode append(-, -, -).
| ?- append(L1, L2, L3). ⟹  L1=[], L3=L2 ? ; L1=[A], L3=[A|L2] ? ;
                            L1=[A,B], L3=[A,B|L2] ? ...
```

Variation on `append` — appending three lists

- Recall: `append/3` has **finite** search space, if its 1st **or** 3rd arg. is closed.
  `append(L,_,_)` completes in $\leq n+1$ reduction steps when `L` has length n
- Let us define `append(L1,L2,L3,L123)`: `L1` $\oplus$ `L2` $\oplus$ `L3` = `L123`. First attempt:

```
append(L1, L2, L3, L123) :-
        append(L1, L2, L12), append(L12, L3, L123).
```

  - Inefficient: `append([1,...,100],[1,2,3],[1], L)` – 203 and not 103 steps...
  - Not suitable for splitting lists – may create an infinite choice point
- An efficient version, suitable for splitting a given list to three parts:

```
% L1 ⊕ L2 ⊕ L3 = L123,
% where either both L1 and L2 are closed, or L123 is closed.
append(L1, L2, L3, L123) :-
        append(L1, L23, L123), append(L2, L3, L23).
```

  - `L3` can be open ended or closed, it does not matter
  - Note that in the first `append/3` call either `L1` or `L123` is closed.
    If `L1` is closed, the first `append/3` produces an open ended list:
    `| ?- append([1,2], L23, L123).` $\Longrightarrow$ `L123 = [1,2|L23]`

The BIP `length/2` – length of a list

- `length(?List, ?N)`: list `List` is of length `N`

```
| ?- length([4,3,1], Len).              Len = 3 ? ;
                                        no
| ?- length(List, 3).                   List = [_A,_B,_C] ? ;
                                        no
| ?- length([[4,1,3],[2,8,7]], Len).    Len = 2 ? ;
                                        no

| ?- length(L, N).                      L = [], N = 0 ? ;
                                        L = [_A], N = 1 ? ;
                                        L = [_A,_B], N = 2 ? ;
                                        L = [_A,_B,_C], N = 3 ?  ...
```

- `length/2` has an infinite search space if the first argument is an open ended list and the second is a variable.

Appending a list of lists

- Library `lists` contains a predicate `append/2`
  see e.g. `https://www.swi-prolog.org/search?for=append%2F2`

```
% append(LL, L): LL is a closed list of lists.
%                L is the concatenation of the elements of LL.
```

- Conditions for safe use (finite search space):
  - Each element of `LL` is a closed list

    ```
    | ?- append([[1,2],[3],[4,5]], L).  ⟹  L = [1,2,3,4,5] ? ; no
    ```
  - `L` is a closed list

    ```
    | ?- append([L1,L2,L3], [1,2]), L1 \= [],
       ⟹                 L1 = [1],   L2 = [],    L3 = [2] ? ;
                         L1 = [1],   L2 = [2],   L3 = [] ? ;
                         L1 = [1,2], L2 = [],    L3 = [] ? ; no
    ```
- Finding a sublist matching a given pattern:

```
| ?- Pattern = [_A,_,_A], append([_Pref,Pattern,_],[1,2,3,2,1,2]),
     length(_Pref, Index).           % obtain the index of the Pattern
Pattern = [2,3,2], Index = 1 ? ;     % Index is zero-based
Pattern = [2,1,2], Index = 3 ? ;  no
```

Finding list elements – BIP member/2

```
% member(E, L): E is an element of list L
member(Elem, [Elem|_]).
member(Elem, [_|Tail]) :-
    member(Elem, Tail).
```

```
member1(Elem, [Head|Tail]) :-
        (   Elem = Head
        ;   member1(Elem, Tail)
        ).
```

- Mode `member(+,+)` – checking membership

  ```
  | ?- member(2, [2,1,2]).  ⟹   yes                         BUT
  | ?- member(2, [2,1,2]), R=yes.  ⟹   R = yes ? ; R = yes ? ; no
  ```

- Mode `member(-,+)` – enumerating list elements:

  ```
  | ?- member(X, [1,2,3]).     ⟹   X = 1 ? ; X = 2 ? ; X = 3 ?  ; no
  | ?- member(X, [1,2,1]).     ⟹   X = 1 ? ; X = 2 ? ; X = 1 ?  ; no
  ```

- Finding common elements of lists – with both above modes:

  ```
  | ?- member(X, [1,2,3]),
       member(X, [5,4,3,2,3]).  ⟹   X = 2 ? ; X = 3 ? ; X = 3 ? ; no
  ```

- Mode `member(+,-)` – making a term an element of a list (infinite choice):

  ```
  | ?- member(1, L).           ⟹   L = [1|_A] ? ; L = [_A,1|_B] ? ;
                                    L = [_A,_B,1|_C] ? ; ...
  ```

- The search space of `member/2` is **finite**, if the 2nd argument is closed.

Reversing lists

- Naive solution (quadratic in the length of the list)

```
% nrev(L, R): List R is the reverse of list L.
nrev([], []).
nrev([X|L], R) :-
    nrev(L, RL),
    append(RL, [X], R).
```

- A solution which is linear in the length of the list

```
% reverse(L, R): List R is the reverse of list L.
reverse(L, R) :-  revapp(L, [], R).

% revapp(L1, L2, R): The reverse of L1 prepended to L2 gives R.
revapp([], R, R).
revapp([X|L1], L2, R) :-
    revapp(L1, [X|L2], R).
```

- In SICStus 4 `append/3` is a BIP, `reverse/2` is in library `lists`
- To load the library place this directive in your program file:

```
:- use_module(library(lists)).
```

append and revapp — building lists forth and back (ADVANCED)

- Prolog

```
app([], L, L).
app([X|L1], L2, [X|L3]) :-
    app(L1, L2, L3).
```

```
revapp([], L, L).
revapp([X|L1], L2, L3) :-
    revapp(L1, [X|L2], L3).
```

- C++

```
struct link   { link *next;
                char elem;
                link(char e): elem(e) {}   };
typedef link *list;
```

```
list app(list L1, list L2)
{ list L3, *lp = &L3;
  for (list p=L1; p; p=p->next)
  { list newl = new link(p->elem);
    *lp = newl; lp = &newl->next;
  }
  *lp = L2;  return L3;
}
```

```
list revapp(list L1, list L2)
{ list l = L2;
  for (list p=L1; p; p=p->next)
  { list newl = new link(p->elem);
    newl->next = l; l = newl;
  }
  return l;
}
```

Generalization of `member`: `select/3` – defined in library `lists`

```
% select(E, List, Rest):  Removing E from List results in list Rest.
select(E, [E|Rest], Rest).       % The head is removed, the tail remains.
select(E, [X|Tail], [X|Rest]):- % The head remains,
    select(E, Tail, Rest).       % the element is removed from the Tail.
```

Possible uses:

```
| ?- select(1, [2,1,3,1], L).          % Remove a given element
        L = [2,3,1] ? ; L = [2,1,3] ? ; no
| ?- select(X, [1,2,3], L).            % Remove an arbitrary element
        L=[2,3], X=1 ? ;  L=[1,3], X=2 ? ;   L=[1,2], X=3 ? ; no
| ?- select(3, L, [1,2]).              % Insert a given element!
        L = [3,1,2] ? ; L = [1,3,2] ? ; L = [1,2,3] ? ; no
| ?- select(3, [2|L], [1,2,7,3,2,1,8,9,4]).
        no                             % Can one remove 3 from [2|L]
                                       % to obtain [1,...]?
| ?- select(1, [X,2,X,3], L).
        L = [2,1,3], X = 1 ? ; L = [1,2,3], X = 1 ? ; no
```

- The search space of `select/3` is **finite**, if the 2nd **or** the 3rd arg. is closed.

Permutation of lists – two solutions (ADVANCED)

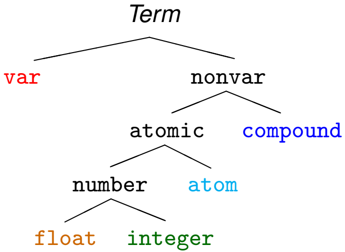
`perm(+List, ?Perm)`: The list `Perm` is a permutation of `List`

```
perm0([], []).
perm0(L, [H|P]) :-
    select(H, L, R),          % Select H from L as the head of the output, R remaining.
    perm0(R, P).              % Permute R to become P, the tail of the output list.

| ?- perm0([a,b,c], L).
                      L = [a,b,c] ? ; L = [a,c,b] ? ; L = [b,a,c] ? ;
                      L = [b,c,a] ? ; L = [c,a,b] ? ; L = [c,b,a] ? ; no
perm1([], []).
perm1([H|T], P) :-
    perm1(T, P1),             % Permute T, the tail of the input list, obtaining P1.
    select(H, P, P1).         % Insert H, the head of the input list, into an arbitrary
    % mode:+  -  +            % position within P1 to obtain the output list, P.
| ?- perm1([a,b,c], L).
                      L = [a,b,c] ? ; L = [b,a,c] ? ; L = [b,c,a] ? ;
                      L = [a,c,b] ? ; L = [c,a,b] ? ; L = [c,b,a] ? ; no
```

- `perm` is symmetric, so the two predicates have the same meaning (WHAT)
- But the second variant is much faster!

Contents

Principles of Prolog term ordering $\prec$



Different kinds ordered left-to-right:

`var` $\prec$ `float` $\prec$ `integer` $\prec$
$\prec$ `atom` $\prec$ `compound`

- Ordering of variables: system dependent
- Ordering of floats and integers: usual ($x \prec y \Leftrightarrow x < y$)
- Ordering of atoms: lexicographical (`abc`$\prec$`abcd`, `abcv`$\prec$`abcz`)
- Compound terms: $\mathtt{name}_a(a_1, \ldots, a_n) \prec \mathtt{name}_b(b_1, \ldots, b_m)$ iff
  1. $n < m$, e.g. `p(x,s(u,v,w))` $\prec$ `a(b,c,d)`, or
  2. $n = m$, and $\mathtt{name}_a \prec \mathtt{name}_b$ (lexicographically), e.g. `a(x,y)` $\prec$ `p(b,c)`, or
  3. $n = m$, $\mathtt{name}_a = \mathtt{name}_b$, and for the first i where $a_i \neq b_i$, $a_i \prec b_i$,
     e.g. `r(1,u+v,3,x)` $\prec$ `r(1,u+v,5,a)`

Built-in predicates for comparing Prolog terms

- Comparing two Prolog terms:

Goal	holds if
`Term1 == Term2`	`Term1` $\not\prec$ `Term2` $\wedge$ `Term2` $\not\prec$ `Term1`
`Term1 \== Term2`	`Term1` $\prec$ `Term2` $\vee$ `Term2` $\prec$ `Term1`
`Term1 @< Term2`	`Term1` $\prec$ `Term2`
`Term1 @=< Term2`	`Term2` $\not\prec$ `Term1`
`Term1 @> Term2`	`Term2` $\prec$ `Term1`
`Term1 @>= Term2`	`Term1` $\not\prec$ `Term2`

- The comparison predicates are not purely logical:

  `| ?- X @< 3, X = 4.` $\Longrightarrow$ `X = 4`

  `| ?- X = 4, X @< 3.` $\Longrightarrow$ `no`

  as they rely on the current instantiation of their arguments
- Comparison uses, of course, the canonical representation:

  `| ?- [1, 2, 3, 4] @< s(1,2,3).` $\Longrightarrow$ **yes**
- BIP `sort(L, S)` sorts (using `@<`) a list `L` of arbitrary Prolog terms, removing duplicates (w.r.t. `==`). Thus the result is a strictly increasing list `S`.

```
| ?- sort([1, 2.0, s(a,b), s(a,c), s, X, s(Y), t(a), s(a), 1, X], L).
L = [X,2.0,1,s,s(Y),s(a),t(a),s(a,b),s(a,c)] ?
```

Equality-like Prolog predicates – a summary

Recall: a Prolog term is *ground* if it contains no unbound variables

- `U = V`: `U` unifies with `V`
  No errors. May bind vars.

```
| ?- X = 1+2.       ⟹  X = 1+2
| ?- 3 = 1+2.       ⟹  no
```

- `U == V`: `U` is identical to `V`, i.e. `U=V` succeeds with no bindings
  No errors, no bindings.

```
| ?- X == 1+2.      ⟹  no
| ?- 3 == 1+2.      ⟹  no
| ?- +(X,Y)==X+Y    ⟹  yes
```

- `U =:= V`: The value of `U` is arithmetically equal to that of `V`.
  No bindings. Error if `U` or `V` is not a (ground) arithmetic expression.

```
| ?- X =:= 1+2.     ⟹  error
| ?- 1+2 =:= X.     ⟹  error
| ?- 2+1 =:= 1+2.   ⟹  yes
| ?- 3.0 =:= 1+2.   ⟹  yes
```

- `U is V`: `U` is unified with the value of `V`.
  Error if `V` is not a (ground) arithmetic expression.

```
| ?- X is 1+2.      ⟹  X = 3
| ?- 3.0 is 1+2.    ⟹  no
| ?- 1+2 is X.      ⟹  error
| ?- 3 is 1+2.      ⟹  yes
| ?- 1+2 is 1+2.    ⟹  no
```

Nonequality-like Prolog predicates – a summary

- Nonequality-like Prolog predicates **never** bind variables.

- U \= V: U does not unify with V. No errors.

```
| ?- X \= 1+2.            ⟹  no
| ?- X \= 1+2, X = 1.     ⟹  no
| ?- X = 1, X \= 1+2.     ⟹  yes
| ?- +(1,2) \= 1+2.       ⟹  no
```

- U \== V: U is not identical to V. No errors.

```
| ?- X \== 1+2.               ⟹  yes
| ?- X \== 1+2, X=1+2.        ⟹  yes
| ?- 3 \== 1+2.               ⟹  yes
| ?- +(1,2)\==1+2             ⟹  no
```

- U =\= V: The values of the arithmetic expressions U and V are different. Error if U or V is not a (ground) arithmetic expression.

```
| ?- X =\= 1+2.          ⟹  error
| ?- 1+2 =\= X.          ⟹  error
| ?- 2+1 =\= 1+2.        ⟹  no
| ?- 2.0 =\= 1+1.        ⟹  no
```

(Non)equality-like Prolog predicates – examples

		Unification		*Identical terms*		*Arithmetic*		
U	*V*	*U* = *V*	*U* \= *V*	*U* == *V*	*U* \== *V*	*U* =:= *V*	*U* =\= *V*	*U* is *V*
1	2	*no*	yes	*no*	yes	*no*	yes	*no*
a	b	*no*	yes	*no*	yes	error	error	error
1+2	+(1,2)	yes	*no*	yes	*no*	yes	*no*	*no*
1+2	2+1	*no*	yes	*no*	yes	yes	*no*	*no*
1+2	3	*no*	yes	*no*	yes	yes	*no*	*no*
3	1+2	*no*	yes	*no*	yes	yes	*no*	yes
X	1+2	X=1+2	*no*	*no*	yes	error	error	X=3
X	Y	X=Y	*no*	*no*	yes	error	error	error
X	X	yes	*no*	yes	*no*	error	error	error

Legend: yes – success; *no* – failure.

Contents

Higher order programming: using predicates as arguments

- Example: collect all nonzero elements of a list

```
% nonzero_elems(Xs, Ys): Ys is a list of all nonzero elements of Xs
nonzero_elems([], []).
nonzero_elems([X|Xs], Ys) :-
    (   0 \= X -> Ys = [X|Ys1]
    ;   Ys = Ys1
    ),
    nonzero_elems(Xs, Ys1).
```

- Generalize to a predicate where the condition is given as an argument

```
% include(Pred, Xs, Ys): Ys = list of elems of Xs that satisfy Pred
include(_Pred, [], []).
include(Pred, [X|Xs], Ys) :-
    (   call(Pred, X) -> Ys = [X|Ys1]
    ;   Ys = Ys1
    ),
    include(Pred, Xs, Ys1).
```

- Specialize `include` for collecting nonzero elements:

```
nonz(X) :- 0 \= X.
nonzero_elems(L, L1) :- include(nonz, L, L1).
```

Higher order predicates

- A higher order predicate (or meta-predicate) is a predicate with an argument which is interpreted as a goal, or a *partial goal*
- A partial goal is a goal with the last few arguments missing
  - e.g., a predicate name is a partial goal (hence variable name `Pred` is often used for partial goals)
- The BIP `call(PG, X)`, where `PG` is a partial goal, adds `X` as the last argument to `PG` and executes this new goal:
  - if `PG` is an atom ⇒ it calls `PG(X)`, e.g. `call(number, X)` ≡ `number(X)`
  - if `PG` is a compound `Pred(A`$_1$`,...,A`$_n$`)` ⇒ it calls `Pred(A`$_1$`,...,A`$_n$`,X)`, e.g. `call(\=(0), X)` ≡ `\=(0,X)` ≡ `0 \= X`
- Predicate `include(Pred, L, FL)` is in `library(lists)`

```
| ?- L=[1,2,a,X,b,0,3+4],
     include(number, L, Nums).  % Nums = { x ∈ L | number(x) }
Nums = [1,2,0] ? ; no
| ?- L=[0,2,0,3,-1,0],
     include(\=(0), L, NZs).    % NZs =  { x ∈ L | \=(0,x) }
NZs = [2,3,-1] ?
```

Calling predicates with additional arguments

- Recall: a callable term is a compound or atom.
- There is a group of built-in predicates `call/N`
  - `call(Goal)`: invokes `Goal`, where `Goal` is a callable term
  - `call(PG, A)`: Adds `A` as the last argument to `PG`, and invokes it.
  - `call(PG, A, B)`: Adds `A` and `B` as the last two args to `PG`, invokes it.
  - `call(PG, A`$_1$`, ..., A`$_n$`)`: Adds `A`$_1$`, ..., A`$_n$ as the last n arguments to `PG`, and invokes the goal so obtained.
- `PG` is a partial goal, to be extended with additional arguments before calling. It has to be a callable term.

```
even(X) :- X mod 2 =:= 0.

| ?- include(even, [1,3,2,9,6,4,0], FL).
                                      ⟹      FL = [2,6,4,0] ; no

divisible_by(N, X) :- X mod N =:= 0.

| ?- include(divisible_by(3), [1,3,2,9,6,4,0], FL).
                                      ⟹      FL = [3,9,6,0] ; no
```

- In descriptions we often abbreviate `call(PG, A`$_1$`, ..., A`$_n$`)` to `PG(A`$_1$`, ..., A`$_n$`)`

An important higher order predicate: `maplist/3`

- `maplist(:PG, ?L, ?ML)`: for each `X` element of `L` and the corresponding `Y` element of `ML`, `call(PG, X, Y)` holds, where `PG` is a partial goal requiring two additional arguments
- Annotation "`:`" (as in `:PG` above) marks a meta argument, i.e. a term to be interpreted as a goal or a partial goal

```
maplist(_PG, [], []).
maplist(PG, [X|Xs], [Y|Ys]) :-
    call(PG, X, Y),
    maplist(PG, Xs, Ys).
```

```
| ?- maplist(reverse, [[1,2],[3,4]], LL). ⟹  LL = [[2,1],[4,3]] ? ; no
```

```
square(X, Y) :- Y is X*X.

mult(N, X, NX) :- NX is N*X.
```

```
| ?- maplist(square, [1,2,3,4], L).  ⟹  L = [1,4,9,16] ? ; no
| ?- maplist(mult(2), [1,2,3,4], L). ⟹  L = [2,4,6,8] ? ; no
| ?- maplist(mult(-5), [1,2,3], L).  ⟹  L = [-5,-10,-15] ? ; no
```

Variants of `maplist`

In SICStus, `maplist` can also be used with 2 and 4 arguments

- `maplist(:Pred, +Xs)` is true if for each x element of `Xs`, `Pred`(x) holds.
- Example: check if a condition holds for all elements of a list

```
all_positive(Xs) :-          % all elements of Xs are positive
    maplist(<(0), Xs).       % ∀ X ∈ Xs, <(0, X), i.e. 0 < X holds
```

- `maplist(:Pred, ?Xs, ?Ys, ?Zs)` is true when `Xs`, `Ys`, and `Zs` are lists of equal length, and `Pred(X, Y, Z)` is true for corresponding elements `X` of `Xs`, `Y` of `Ys`, and `Z` of `Zs`. At least one of `Xs`, `Ys`, `Zs` has to be a closed list.
- Example: add two vectors

```
add_vectors(VA, VB, VC) :-
    maplist(plus, VA, VB, VC).          plus(A, B, C) :- C is A+B.
```

`| ?- add_vectors([10,20,30], [3,2,1], V).` $\Longrightarrow$ `V = [13,22,31] ? ; no`

- The implementation of `maplist/4` (easy to generalize :-):

```
maplist(_PG, [], [], []).
maplist(PG, [X|Xs], [Y|Ys], [Z|Zs]) :-
    call(PG, X, Y, Z), maplist(PG, Xs, Ys, Zs).
```

Another important higher order predicate: `scanlist` (SWI: `foldl`)

- Example: `plus(A, S0, S) :- S is S0+A.`

  ```
  | ?- scanlist(plus, [1,3,5], 0, Sum).    ⟹  Sum = 9 ? ; no
                    % 0+1+3+5 = 9
  ```

  This executes as: `plus(0, 1, S`$_1$`), plus(S`$_1$`, 3, S`$_2$`), plus(S`$_2$`, 5, Sum).`
- In general: `scanlist(acc, [E`$_1$`,E`$_2$`,...,E`$_n$`], S`$_0$`, S`$_n$`)` is expanded as:
  `acc(S`$_0$`, E`$_1$`, S`$_1$`), acc(S`$_1$`, E`$_2$`, S`$_2$`), ..., acc(S`$_{n-1}$`, E`$_n$`, S`$_n$`)`
- `scanlist(:PG, ?L, ?Init, ?Final)`:
  - `PG` represents the above accumulating predicate `acc`
  - `scanlist` applies the `acc` predicate repeatedly, on all elements of list `L`, left-to-right, where `Init` = `S`$_0$ and `Final` = `S`$_n$.
- For processing two lists (of the same length), use `scanlist/5`, e.g.

  ```
  prodsum(A, B, PS0, PS) :- PS is PS0 + A*B.

  scalar_product(As, Bs, SP) :- scanlist(prodsum, As, Bs, 0, SP).
  | ?- scalar_product([1,0,2], [3,4,5], SP).    ⟹  SP = 13 ? ; no
  ```
- In SICStus, there is also a `scanlist/6` predicate, for processing 3 lists

Contents

All solutions built-in predicates – introduction

- All solution BIPs are higher order predicates analogous to list comprehensions in Haskell, Python, etc.
- There are three such predicates: `findall/3` (the simplest), `bagof/3` and `setof/3`; having the same arguments, but somewhat different behavior
- Examples for `findall/3`:

```
| ?- findall(X, (member(X,   [1,7,8,3,2,4]), X > 3),  L).
%           {X |          X ∈  {1,7,8,3,2,4},  X > 3} = L
                 ⟹    L = [7,8,4] ? ; no
| ?- findall(X, (member(X,   [1,7,8,3,2,4]), X > 8),  L).
%           {X |          X ∈  {1,7,8,3,2,4},  X > 8} = L
                 ⟹    L = [] ? ; no
| ?- findall(X-Y, (between(1, 3, X), between(1, X, Y)),  L).
%           {X-Y |            1≤X≤3,           1≤Y≤X } = L
                 ⟹    L = [1-1,2-1,2-2,3-1,3-2,3-3] ? ; no
```

Recall: `between(+N, +M, ?X)` enumerates in `X` the integers `N`, `N+1`, ..., `M`. In SICStus, it requires loading `library(between)`.

Finding all solutions: the BIP `findall(?Templ, :Goal, ?L)`

Approximate meaning: `L` is a list of `Templ` terms for each solution of `Goal`

The execution of the BIP `findall/3` (procedural semantics):

- Interpret term `Goal` as a goal, and call it
- For each solution of `Goal`:
  - store a *copy* of `Templ` (copy ⟹ replace vars in `Templ` by new ones)
    Note that copying requires time proportional to the size of `Templ`
  - continue with failure (to enumerate further solutions)
- When there are no more solutions (`Goal` fails)
  - collect the stored `Templ` values into a list, unify it with `L`.
- When a solution contains (possibly multiple instances of) a variable (e.g. `A`), then each of these will be replaced by a single new variable (e.g. `_A`):

```
| ?- findall(T, member(T, [A-A,B-B,A]), L).
                                   ⟹ L= [_A-_A,_B-_B,_C] ? ; no
```

All solutions: the BIP `bagof(?Templ, :Goal, ?L)`

- Exactly the same arguments as in `findall/3`.
  `bagof/3` is the same as `findall/3`, except when there are unbound variables in `Goal` which do not occur in `Templ` (so called free variables)

  ```
  % emp(Er, Ee): employer Er employs employee Ee.
  emp(a,b).   emp(a,c).   emp(b,c).   emp(b,d).
  ```

  `| ?- findall(E, emp(R, E), Es).` % `Es` ≡ the list of all employees
  ⟹ `Es = [b,c,c,d] ? ; no`   i.e. `Es` = $\{E \mid \exists\, R.\ (R \text{ employs } E)\}$

- `bagof` does not treat free vars as existentially quantified. Instead it enumerates all possible values for the free vars (all employers) and for each such choice it builds a separate list of solutions:

  `| ?- bagof(E,emp(R,E),Es).` % `Es` ≡ list of `Es` employed by any possible `R`.
  ⟹ `R = a, Es = [b,c] ? ;`
  ⟹ `R = b, Es = [c,d] ? ; no`

- Use operator `^` to achieve existential quantification in `bagof`:

  `| ?- bagof(E, R^emp(R, E), Es).` % Collect `Es` for which ∃`R.emp(R, E)`
  ⟹ `Es = [b,c,c,d] ? ; no`

- `bagof` preserves variables (but it is slower than `findall` :-( ):

  `| ?- bagof(T, member(T, [A-A,B-B,A]), L).` ⟹ `L = [A-A,B-B,A] ? ; no`

All solutions: the BIP `setof/3`

- `setof(?Templ, :Goal, ?List)`
- The execution of the procedure:
  - same as: `bagof(Templ, Goal, L0), sort(L0, List)`
  - recall: `sort(+L, ?SL)` is a built-in predicate which sorts `L` using the `@<` built-in predicate removes duplicates and unifies the result with `SL`
- Example:

```
graph([a-b,a-c,b-c,c-d,b-d]).

% Graph has a node V.
has_node(Graph, V) :- member(A-B, Graph), ( V = A ; V = B).

% The set of nodes of G is Vs.
graph_nodes(G, Vs) :- setof(V, has_node(G, V), Vs).

| ?- graph(_G), graph_nodes(_G, Vs). ⟹  Vs = [a,b,c,d] ? ; no
```

Contents

Causes of inefficiency – preview

- Unnecessary choice points (ChPs) waste both time and space
  Recursive definitions often leave choice points behind on exit, e.g.:
  - ```
    % fact0(+N, ?F): F = N!.
    fact0(0, 1).
    fact0(N, F) :- N > 0, N1 is N-1, fact0(N1, F1), F is N*F1.
    ```
  - Remedy: use if-then-else or the cut BIP (coming soon)
  - ```
    % last0(L, E): The last element of L is E.
    last0([E], E).
    last0([_|L], E) :- last0(L, E).
    ```
  - Remedy: rewrite to make use of indexing (or cut, or if-then-else)
- General recursion, as opposed to tail recursion
  As an example, see the `fact0/2` predicate above
  Remedy: re-formulate to a tail recursive form, using accumulators

The cut – the BIP underlying if-then-else and negation

- The cut, denoted by `!`, is a BIP with no arguments, i.e. its functor is `!/0`.
- Execution: the cut always succeeds with these two side effects:
  - Restrict to the first solution of a goal:
    Remove all choice points created within the goal(s) preceding the `!`.
    ```
    % is_a_parent(+P): check if a given P is a parent.
    is_a_parent(P) :- has_parent(_, P), !.
    ```
  - Commit to the clause containting the cut:
    Remove the choice of any further clauses in the current predicate.
    ```
    fact1(0, F) :- !, F = 1.  % Assign output vars only after the cut,
                              % both for correctness and efficiency
    fact1(N, F) :- N > 0, N1 is N-1, fact1(N1, F1), F is N*F1.
    ```
- Definition: if `q :- ..., p, ....` then
  the parent goal of `p` is the goal matching the clause head `q`
- Effects of cut in the search tree: removes all choice points up to and including the node labelled with the parent goal of the cut.
- In the procedure box model: Fail port of cut $\Longrightarrow$ Fail port of parent goal

How does “cut” prune the search tree – an example

```
a(X, Y) :- b(X), c(X, Y).                b(s(1)).
a(X, Y) :- d(X, Y).                      b(s(2)).

c(s(X), Y) :- Y is X+10.                 d(s(3), 30).
c(s(X), Y) :- Y is X+20.                 d(t(4), 40).

a_cut(X, Y) :- b(X), !, c(X, Y).
a_cut(X, Y) :- d(X, Y).

test(Pred, X, Res) :-
    findall(X-Y, call(Pred, X, Y), Res).
```

Sample runs:

```
| ?- test(a,     s(_), Res).  ⟹  Res = [s(1)-11,s(1)-21,s(2)-12,
                                        s(2)-22,s(3)-30] ?
| ?- test(a,     t(_), Res).  ⟹  Res = [t(4)-40] ?
| ?- test(a_cut, s(_), Res).  ⟹  Res = [s(1)-11,s(1)-21] ?
| ?- test(a_cut, s(3), Res).  ⟹  Res = [s(3)-30] ?
| ?- test(a_cut, t(_), Res).  ⟹  Res = [t(4)-40] ?
```

Avoid leaving unnecessary choice points

- Add a cut if you know that remaining branches are doomed to fail. (These are so called green cuts, which do not remove solutions.)
- Example of a green cut:

```
% last1(L, E): The last element of L is E.
last1([E], E)    :- !.
last1([_|L], E) :- last1(L, E).
```

  In the absence of the cut, the goal `last1([1], X)` will return the answer `X = 1`, and leave a choice point. When this choice point is explored `last1([], X)` will be called which will always fail.
- Instead of a cut, one can use if-then-else:

```
last2([E|L], X) :- ( L == [] -> X = E
                   ; last2(L, X)
                   ).

fact2(N, F) :-     ( N == 0 -> F = 1
                   ; N > 0, N1 is N-1, fact2(N1, F1), F is N*F1
                   ).
```

Avoid leaving unnecessary choice points – indexing

- Recall a simple example predicate, summing a binary tree:

```
% tree_sum(+Tree, ?Sum):
% Sum is the sum of integers in the leaves of Tree.
tree_sum(leaf(Value), Value).          1st head arg's functor: leaf/1
tree_sum(node(Left, Right), S) :-      1st head arg's functor: node/2
        tree_sum(Left, S1), tree_sum(Right, S2), S is S1+S2.
```

- Indexing groups the clauses of a predicate based on the outermost functor of (usually) the first argument.
- The compiler generates code (using hashing) to select the subset of clauses that corresponds to this outermost functor.
- If the subset contains a single clause, no choicepoint is created. (This is the case in the above example.)

SICStus specific: avoid choice points in if-then-else (ADVANCED)

- Consider an if-then-else goal of the form: `( cond -> then ; else )`.
- Before `cond`, a ChP is normally created (removed at -> or before `else`).
- In **SICStus Prolog** no choice points are created, if `cond` only contains:
  - arithmetical comparisons (e.g., `<, =<, =:=`); and/or
  - built-in predicates checking the term type (e.g., `atom, number`); and/or
  - general comparison operators (e.g., `@<, @=<, ==`).
- Analogously, no ChPs are made for `head :- cond, !, then.`,
  if all arguments of `head` are distinct variables, and `cond` is just like above.
- Further improved variants of `fact2` and `last2` with no ChPs created:

```
fact3(N, F) :-     ( N =:= 0 -> F = 1    % used to be N = 0
                   ; N > 0, N1 is N-1, fact(N1, F1), F is N*F1
                   ).

last3([E|L], X) :- ( L == [] -> X = E    % used to be L = []
                   ; last3(L, X)
                   ).
```

Indexing – an introductory example

- A sample (meaningless) program to illustrate indexing.

```
p(0, a).            /* (1) */        q(1).
p(X, t) :- q(X).    /* (2) */        q(2).
p(s(0), b).         /* (3) */
p(s(1), c).         /* (4) */
p(9, z).            /* (5) */
```

- For the call `p(A, B)`, the compiler produces a case statement-like construct, to determine the list of applicable clauses:

(VAR)	if `A` is a variable:	(1) (2) (3) (4) (5)
(0/0)	if `A = 0` (`A`'s main functor is `0/0`):	(1) (2)
(s/1)	if `A`'s main functor is `s/1`:	(2) (3) (4)
(9/0)	if `A = 9`:	(2) (5)
(OTHER)	in all other cases:	(2)

- Example calls (do they create and leave a choice point?)
  - `p(1,    Y)` takes branch (OTHER), does not create a choice point.
  - `p(s(1), Y)` takes branch (s/1), creates a choice point, but removes it and exits without leaving a choice point.
  - `p(s(0), Y)` takes branch (s/1), and exits leaving a choice point.

Indexing

- Indexing improves the efficiency of Prolog execution by
  - speeding up the selection of clauses matching a particular call;
  - using a compile-time grouping of the clauses of the predicate.
- Most Prolog systems, including SICStus, use only the main (i.e. outermost) functor of the *first* argument for indexing, which is
  - `C/0`, if the argument is a constant (atom or number) `C`;
  - `R/N`, if the argument is a compound with name `R` and arity `N`;
  - undefined, if the argument is a variable.

Implementing indexing

- Compile-time: collect the set of (outermost) functors of nonvar terms occurring as first args, build the case statement (see prev. slide)
- Run-time: select the relevant clause list using the first arg. of the call. This is practically a constant time operation, as it uses *hashing*.
  - If the clause list is a singleton, *no choice point* is created.
  - Otherwise a choice point *is* created, which will be removed before entering the last branch.

Getting the most out of indexing

- Get deep indexing through helper predicates (rewrite `p/2` to `q/2`):

```
p(0, a).                      q(0, a).                  q_aux(0, b).
p(s(0), b).        ⟹          q(s(X), Y) :-             q_aux(1, c).
p(s(1), c).                         q_aux(X, Y).
p(9, z).                      q(9, z).
```

  Pred. `q(X,Y)` will not create choice points if `X` is ground.
- Indexing does not deal with arithmetic comparisons
  - E.g., `N = 0` and `N > 0` are not recognized as mutually exclusive.
- Indexing and lists
  - Putting the (input) list in the first argument makes indexing work.
  - Indexing distinguishes between `[]` and `[...|...]` (resp. functors: `'[]'/0` and `'.'/2`).
  - For proper lists, the order of the two clauses is not relevant
  - For use with open ended lists: put the clause for `[]` first, to avoid an infinite loop (an infinite choice may still remain)

Indexing list handling predicates

- Predicate `app/3` creates no choice points
  if the first argument is a proper list:

```
% app(L1, L2, L3): L1 ⊕ L2 = L3.              % 1st arg funct:
app([], L, L).                                % []/0
app([X|L1], L2, [X|L3]) :-                    % . /2
    app(L1, L2, L3).
```

- The same is true for `revapp/3`:

```
% revapp(L1, L2, L3):
% appending the reverse of L1 and L2 gives L3
revapp([], L, L).                             % []/0
revapp([X|L1], L2, L3) :-                     % . /2
    revapp(L1, [X|L2], L3).
```

Indexing list handling predicates, cont'd

- Getting the last element of a list: `last0/2` leaves a choice point.

```
% last0(L, E): The last element of L is E.
last0([H], H).                                 % . /2
last0([_|T], E) :-     last0(T, E).            % . /2
```

- The variant `last4/2` uses a helper predicate, creates no choice points:

```
last4([H|T], E) :-     last4(T, H, E).                     (*)

% last4(T, H, E): The last element of [H|T] is E.
last4([], E, E).                               % []/0
last4([H|T], _, E) :-  last4(T, H, E).         % . /2
```

- `member0/2` (as defined earlier) always leaves a choice point.

```
% member0(E, L): E is an element of L.
member0(E, [E|_T]).                            % VAR
member0(E, [_H|T]) :-  member0(E, T).          % VAR
```

- Write the head comment and the clauses of `member1/3`, so that `member1/2` leaves no choice point when the last element of a (proper) list is returned.

```
member1(E, [H|T]) :-  member1(T, H, E).                 % cf. (*)
% member1(T, H, E): ...
```

Tail recursion

- In general, recursion is expensive both in terms of time and space.
- The special case of tail recursion can be compiled to a loop. Conditions:
  1. the recursive call is the last to be executed in the clause body, i.e.:
     - it is textually the last subgoal in the body; or
     - the last subgoal is a disjunction/if-then-else, and the recursive call is the last in one of the branches
  2. no ChPs left in the predicate when the recursive call is reached
- Example

```
% all_pos(+L): all elements of number list L are positive.
all_pos([]).
all_pos([X|L]) :-
    X > 0, all_pos(L).
```

- *Tail recursion optimization, TRO*: the memory allocated by the clause is freed **before** the last call is executed.
- This optimization is performed not only for recursive calls but for the **last** calls in general (*last call optimization, LCO*).

Making a predicate tail recursive – accumulators

- Example: the sum of a list of numbers. The left recursive variant:

```
% sum0(+List, -Sum): the sum of the elements of List is Sum.
sum0([], 0).
sum0([X|L], Sum) :- sum0(L, Sum0), Sum is Sum0+X.
```

  Note that `sum0([a1,..., an], S)` $\Longrightarrow$ `S = 0+an+... +a1` (right to left)

- For TRO, define a helper pred, with an arg. storing the "sum so far":

```
% sum(+List, +Sum0, -Sum):
% (Σ List) + Sum0 = Sum, i.e.  Σ List = Sum-Sum0.
sum([], Sum, Sum).
sum([X|L], Sum0, Sum) :-
    Sum1 is Sum0+X,     % Increment the ``sum so far''
    sum(L, Sum1, Sum). % recurse with the tail and the new sum so far
```

- Arguments `Sum0` and `Sum` form an accumulator pair: `Sum0` is an intermediate while `Sum` is the final value of the accumulator. The initial value is supplied when defining `sum/2`:

```
% sumlist(+List, ?Sum): Σ List = Sum. Available from library(lists).
sumlist(List, Sum) :-  sum(List, 0, Sum).
```

  Note that `sumlist([a1,..., an], S)` $\Longrightarrow$ `S = 0+a1+... +an` (left to right)

Accumulators – making factorial tail-recursive

- Two arguments of a pred. forming an **accumulator** pair: the declarative equivalent of the imperative variable (i.e. a variable with a mutable state)
- The two parts: the state of the mutable quantity at pred. entry and exit.
- Example: making factorial tail-recursive. The mid-recursive version:

```
% fact0(N, F): F = N!.
fact0(N, F) :-     (    N =:= 0 ->  F = 1
                   ;    N > 0, N1 is N-1, fact0(N1, F1), F is F1*N
                   ).
```

`| ?- fact0(4, F).` $\Longrightarrow$ `F = 24` $\sim$ `1*1*2*3*4`

- Helper predicate: `fact(N, F0, F)`, `F0` is the product accumulated so far.

```
% fact(N, F0, F): F = F0*N!.
fact(N, F0, F) :-  (    N =:= 0 -> F = F0
                   ;    N > 0, F1 is F0*N, N1 is N-1, fact(N1, F1, F)
                   ).

fact(N, F) :-
     fact(N, 1, F).
```

`| ?- fact(4, F).` $\Longrightarrow$ `F = 24` $\sim$ `1*4*3*2*1`

Accumulating lists – higher order approaches (ADVANCED)

- Recap predicate `revapp/3`:

```
% revapp(L, R0, R): The reverse of L prepended to R0 gives R.
revapp0([], R0, R) :-     R = R0.
revapp0([X|L], R0, R) :- R1 = [X|R0], revapp0(L, R1, R).
```

- Introduce the list construction predicate `cons/3`

```
% L1 is a list constructed from the head X and tail L0.
cons(X, L0, L1) :-        L1 = [X|L0].
revapp1([], R0, R) :-     R = R0.
revapp1([X|L], R0, R) :- cons(X, R0, R1), revapp1(L, R1, R).
```

- A higher order (HO) solution (in SWI use `foldl` instead of `scanlist`):

```
revapp2(L, R0, R) :-      scanlist(cons, L, R0, R).
```

- Summing a list, HO solution (*% sum2(L, Sum): list L sums to Sum.*)

```
plus(X, S0, S1) :-        S1 is S0+X.
sum2(L, Sum) :-           scanlist(plus, L, 0, Sum).
```

- (ADV^2) Appending lists, HO sol. (*% app(L1, L2, L): L1 ⊕ L2 = L.*)

```
% decomp(X, C, B): List C can be decomposed to head X and tail B
decomp(X, C, B) :-      C = [X|B].
app(A, B, C) :-         scanlist(decomp, A, C, B).
```

Accumulating lists – avoiding append

- Example: calculate the list of leaf values of a tree. Without accumulators:

```
% tree_list0(+T, ?L): L is the list of the leaf values of tree T.
tree_list0(leaf(Value), [Value]).
tree_list0(node(Left, Right), L) :-
    tree_list0(Left, L1), tree_list0(Right, L2), append(L1, L2, L).
```

- Building the list of tree leaves using accumulators:

```
tree_list(Tree, L) :-
    tree_list(Tree, [], L). % Initialize the list accumulator to []
% tree_list(+Tree, +L0, L): The list of the
% leaf values of Tree prepended to L0 is L.
tree_list(leaf(Value), L0, L) :- L = [Value|L0].
tree_list(node(Left, Right), L0, L) :-
        tree_list(Right, L0, L1), tree_list(Left, L1, L).
```

```
| ?- tree_list(node(node(leaf(a),leaf(b)),leaf(c)), L). ⟹ L = [a,b,c]? ; no
```

- Note that one of the two recursive calls is tail-recursive.
- Also, there is no need to append the intermediate lists!

Accumulators for implementing imperative (mutable) variables

- Let $L = [x_1, \ldots,]$ be a number list. x_i is *left-visible* in L, iff $\forall j < i.(x_j < x_i)$
- Determine the count of left-visible elements in a list of positive integers:

Imperative, C-like algorithm

```
int viscnt(list L) {
  int MV = 0; // max visible
  int VC = 0; // visible cnt


loop:
  if (empty(L)) return VC;

  { int H = hd(L), L = tl(L);
    if (H > MV)
       { VC += 1; MV = H; }
    // else VC,MV unchanged
  }
  goto loop;
                    }
```

Prolog code

```
% List L has VC left-visible elements.
viscnt(L, VC) :- viscnt(L,
                        0,
                        0, VC).

% viscnt(L, MV, VC0, VC): L has VC-VC0
% left-visible elements which are > MV.
viscnt([], _, VC0, VC) :- VC = VC0.
viscnt(L0, MV0, VC0, VC) :-       % (1)
    L0 = [H|L1],
    (   H > MV0
    ->  VC1 is VC0+1, MV1 = H
    ;   VC1 = VC0, MV1 = MV0      % (2)
    ),
    viscnt(L1, MV1, VC1, VC).     % (3)
```

Mapping a C loop to a Prolog predicate

- Each C variable initialized before the loop and used in it becomes an input argument of the Prolog predicate
- Each C variable assigned to in the loop and used afterwards becomes an output argument of the Prolog predicate
- Each occurrence of a C variable is mapped to a Prolog variable, whenever the variable is assigned, a new Prolog variable is needed, e.g. `MV` is mapped to `MV0`, `MV1`, ...:
  - The initial values (`L0`,`MV0`, ...) are the args of the clause head[2] *(1)*
  - If a branch of if-then(-else) changes a variable, while others don't, then the Prolog code of latter branches has to state that the new Prolog variable is equal to the old one, *(2)*
  - At the end of the loop the Prolog predicate is called with arguments corresponding to the current values of the C variables, *(3)*

[2] References of the form *(n)* point to the previous slide.

Contents

Executable specifications – what are they?

- An executable specification is a piece of non-recursive Prolog code which is in a one-to-one correspondence with its specification
- Example 1: Finding a contiguous sublist with a given sum

```
% sublist_sum(+L, +Sum, ?SubL): SubL is a sublist of L summing to Sum.
| ?- sublist_sum([1,2,3], 3, SL). ⟹  SL = [1,2] ? ; SL = [3] ? ; no

:- use_module(library(lists)). % To import sumlist/2, append/2
sublist_sum(L, Sum, SubL) :-
    append([_,SubL,_], L),      % SubL is a sublist of L
    sumlist(SubL, Sum).         % Σ SubL = Sum
```

- Example 2: Finding elements occurring in pairs

```
% paired(+List, ?E, ?I):  E is an element of List equal to its
% right neighbour, occurring at (zero-based) index I.
| ?- paired([a,b,b,c,d,d], E, I). ⟹  E = b, I = 1 ? ;
                                  ⟹  E = d, I = 4 ? ; no

paired(L, E, I) :-
    append(Pref, [E,E|_], L),   % L starts with a sublist Pref,
                                % followed by two elements equal to E
    length(Pref, I).            % The length of Pref is I
```

Executable specification examples: plateau

- A list is a plateau, if its length is ≥ 2, and all its elements are the same. (Think of list elements as elevation values.)
  We assume that the list is ground (contains no variables). (*)
- Example 3: Checking if a list is a plateau. Four variants: N = 1,2,3,4

```
% plateauN(Pl, A): Pl is a plateau with elements equal to A.
```

1. Use `boring/2` (slide 96): `plateau1([A,A|Pl], A) :- boring(Pl, A).`
2. Use `maplist/2`: `plateau2([A,A|Pl], A) :- maplist(=(A), Pl).`
3. Use (double) negation: `Pl` has no element that differs from `A`

```
plateau3([A,A|Pl], A) :-    \+ ( member(X, Pl), \+ X = A ).
```

4. Use the `forall/2` library predicate (`library(aggregate)` in SICStus)

```
plateau4([A,A|Pl], A) :- forall( member(X, Pl),    X = A ).
```

- `forall(P, Q)` succeeds iff `Q` holds for each solution of `P`, defined as:
  `forall(P,Q) :- \+ (P, \+Q).` *% There is no solution of P for which Q fails.*
- **IMPORTANT**: Because of `\+`, `forall/2` will never instantiate vars, cf. (*)

```
| ?- length(L, 4), plateau2(L, 8). L = [8,8,8,8] ? ; no
| ?- length(L, 4), plateau4(L, 8). L = [8,8,_A,_B] ? ; no
```

Executable specification examples: the longest plateau prefix

- The maximal plateau prefix (MPP for short) of a list is its longest prefix that is a plateau. E.g. the MPP of `[1,1,1,2,1]` is `[1,1,1]`.
- Example 4: Given a list, obtain the length and the repeating element of its MPP. Fail if the list has no MPP (e.g. `[3,1,1,1,2,1]` has no MPP).

  % mpp(+L, ?Len, ?A): List L has an MPP of length Len, composed of A's

- Let's use `append/3` to split `L` into a `Pl` plateau prefix and `Suff` suffix:

  `append(Pl, Suff, L), plateauN(Pl, A),` <check `Pl` is maximal>

- `Pl` is maximal, if `Suff = []` or the head of `Suff` is not `A`:

  `(   Suff = [] -> true ; Suff = [X|_], X \= A   )`

- This can be simplified to: `\+ Suff = [A|_]` (it does not hold that the head of `Suff` is `A`)

```
mpp(L, Len, A) :-         % L has an MPP of length Len, composed of A's if
    Pl = [A,A|_],         % Pl's first two elems are the same, call them A
    append(Pl, Suff, L),  % Pl ⊕ Suff = L, Pl is a prefix of L followed by Suff
    forall(member(X,Pl),  % For each X element of Pl
           X = A),        %          X is equal to A      *** Pl is a plateau!
    \+ Suff = [A|_],      % Suff does not start with A   *** Pl is maximal!
    length(Pl, Len).      % The length of Pl is Len
```

Executable specification examples: maximal plateau sublist

- A contiguous sublist of a list is a maximal plateau sublist, if it is a plateau that cannot be extended neither leftwards nor rightwards
- Example 5: enumerate all maximal plateau sublists of a given list

```
% plateau(+L, ?I, ?Len, ?A): List L has a maximal plateau sublist that starts
% at (0-based) index I, has length Len, and is composed of A-s

| ?- plateau([1,1,1,2,1,4,4,3,7,7,7], I, Len, A).
I = 0, Len = 3, A = 1 ? ;
I = 5, Len = 2, A = 4 ? ;
I = 8, Len = 3, A = 7 ? ; no

plateau(L, I, Len, A) :-
    Pl = [A,A|_],                   % The first two elements of Pl are equal,
                                    % call them A
    append([Pref,Pl,Suff], L),      % Split L to Pref ⊕ Pl ⊕ Suff
    forall( member(X, Pl), X=A ),   % For each X element of Pl, X = A holds
    \+ Suff = [A|_],                % Suff does not start with A
    \+ last(Pref, A),               % Pref does not end with A
    length(Pl, Len),                % The length of Pl is Len
    length(Pref, I).                % The length of Pref is I
```

Contents

Additional slides

Subsequent slides were not presented in the class, these are included as further reading and for reference purposes

Building and decomposing compounds: the *univ* predicate

- BIP `=../2` (pronounce *univ*) is a standard op. (`xfx`, 700; just as =, ...)
- `Term =.. List` holds if
  - `Term` = $Fun(A_1, \ldots, A_n)$ and `List` = $[Fun, A_1, \ldots, A_n]$, where Fun is an atom and $A_1, \ldots, A_n$ are arbitrary terms; or
  - `Term` = C and `List` = $[C]$, where C is a constant. (Constants are viewed as compounds with 0 arguments.)
- Whenever you would like to use a var. as a compound name, use *univ*: `X = F(A1,...,An)` causes syntax error, use `X =.. [F,A1,...,An]` instead
- Call patterns for *univ*:
  - `+Term =.. ?List` decomposes `Term`
  - `-Term =.. +List` constructs `Term`
- Examples

`| ?- edge(a,b,10) =.. L.`	$\Longrightarrow$	`L = [edge,a,b,10]`
`| ?- Term =.. [edge,a,b,10].`	$\Longrightarrow$	`Term = edge(a,b,10)`
`| ?- apple =.. L.`	$\Longrightarrow$	`L = [apple]`
`| ?- Term =.. [1234].`	$\Longrightarrow$	`Term = 1234`
`| ?- Term =.. L.`	$\Longrightarrow$	**error**
`| ?- f(a,g(10,20)) =.. L.`	$\Longrightarrow$	`L = [f,a,g(10,20)]`
`| ?- Term =.. [/,X,2+X].`	$\Longrightarrow$	`Term = X/(2+X)`

Error handling in Prolog

- A BIP for catching exceptions (errors): `catch(:Goal, ?ETerm, :EGoal)`:
- Recall: "`:`" marks a meta argument, i.e. a term which is a goal
- BIP `catch/3` runs `Goal`
  - If no exception is raised (no error occurs) during the execution of `Goal`, `catch` ignores the remaining arguments
  - When an exception occurs, an exception term `E` is produced, which contains the details of the exception
    - If `E` unifies with the 2nd argument of `catch`, `ETerm`, it runs `EGoal`
    - Otherwise `catch` propagates the exception further outwards, giving a chance to surrounding `catch` goals
    - If the user code does not "catch" the exception, it is caught by the top level, displaying the error term in a readable form.

```
| ?- X is Y+1.
! Instantiation error in argument 2 of (is)/2
! goal:  _177 is _183+1
| ?- catch(X is Y+1, E, true).
E = error(instantiation_error,instantiation_error(_A is _B+1,2)) ? ; no
| ?- catch(X is Y+1, _, fail).
no
```

An interesting Prolog task

- A job interview question: construct an arithmetic expression containing integers 1, 3, 4, 6 each exactly once, using the four basic arithmetic operators +, -, *, /, 0 or more times, so that the expression evaluates to 24
- Let's write a Prolog program for solving this task:

```
:- use_module(library(lists), [permutation/2]).

% arith_expr(+L, +OpL, +Val, -Expr) :
% Expr is an arithmetic expression containing only operators present
% in the list OpL (operators may be used 0 or more times) and
% integers given in list L (each integer has to appear exactly once),
% so that the value of the expression is Val.
arith_expr(L, OpL, Val, Expr) :-
   permutation(L, PL),              % permute the list of integers into PL
   leaves_ops_expr(PL, OpL, Expr), % build Expr with PL as the leaves-list
   catch(Expr =:= Val, _, fail).    % check if Expr evaluates to Val, fail
                                    % if there is a division-by-0 error.
```

An interesting Prolog task, cont'd

```
% leaves_ops_expr(+L, +OpL, ?Expr): Expr is an arithmetic expression
% which uses operators from OpL (0 or more times each) whose leaves,
% read left-to-right, form the list L.
leaves_ops_expr(L, _OpL, Expr) :-
    L = [Expr].        % If L is a singleton, Expr is the only element
leaves_ops_expr(L, OpL, Expr) :-
    append(L1, L2, L),              % Split L to nonempty L1 and L2,
    L1 \= [], L2 \= [],
    leaves_ops_expr(L1, OpL, E1),   % generate E1 from L1 (using OpL),
    leaves_ops_expr(L2, OpL, E2),   % generate E2 from L2 (using OpL),
    member(Op, OpL),                % choose an operator Op from OpL,
    Expr =.. [Op,E1,E2].            % build the expression ‘E1 Op E2’

| ?- solve(66).
(3*4-1)*6
(4*3-1)*6
6*(3*4-1)
6*(4*3-1)
yes
```

A motivating symbolic processing example

- Polynomial: built from the atom '`x`' and numbers using ops '`+`' and '`*`'
- Calculate the value of a polynomial for a given substitution of `x`

```
% value_of(+Poly, +X, ?V): Poly has the value V, for x=X
```

<pre>value_of0(x, X, V) :- V = X.
value_of0(N, _, V) :-
    number(N), V = N.
 	
value_of0(P1+P2, X, V) :-
    value_of0(P1, X, V1),
    value_of0(P2, X, V2),
    V is V1+V2.
value_of0(Poly, X, V) :-
    Poly = *(P1,P2),
    value_of0(P1, X, V1),
    value_of0(P2, X, V2),
    PolyV = *(V1,V2),
    V is PolyV.</pre>
	<pre>value_of(x, X, V) :- !, V = X.
	value_of(N, _, V) :-
	    number(N), !, V = N.
	
	value_of(Poly, X, V) :-
	    Poly =.. [Func,P1,P2],
	    value_of(P1, X, V1),
	    value_of(P2, X, V2),
	    PolyV =.. [Func,V1,V2],
	    V is PolyV.</pre>

- Predicate `value_of` works for all binary functions supported by `is/2`.

```
| ?- value_of(exp(100,min(x,1/x)), 2, V).     ⟹     V = 10.0 ? ; no
```

Building and decomposing compounds: `functor/3`

- `functor(Term, Name, Arity)`:
  `Term` has the name `Name` and arity `Arity`, i.e.
  `Term` has the functor `Name/Arity`.
  (A constant `C` is considered to have the name `C` and arity 0.)
  - Call patterns:

    `functor(+Term, ?Name, ?Arity)` – decompose `Term`
    `functor(-Term, +Name, +Arity)` – construct a most general `Term` (*)
  - If `Term` is output (*), it is unified with the most general term with the given name and arity (with distinct new variables as arguments)
- Examples:

```
| ?- functor(edge(a,b,1), F, N).     ⟹   F = edge, N = 3
| ?- functor(E, edge, 3).            ⟹   E = edge(_A,_B,_C)
| ?- functor(apple, F, N).           ⟹   F = apple, N = 0
| ?- functor(Term, 122, 0).          ⟹   Term = 122
| ?- functor(Term, edge, N).         ⟹   error
| ?- functor(Term, 122, 1).          ⟹   error
| ?- functor([1,2,3], F, N).         ⟹   F = '.', N = 2
| ?- functor(Term, ., 2).            ⟹   Term = [_A|_B]
```

Building and decomposing compounds: `arg/3`

- `arg(N, Compound, A)`: the `N`th argument of `Compound` is `A`
  - Call pattern: `arg(+N, +Compound, ?A)`, where `N` $\geq$ `0` holds
  - Execution: The `N`th argument of `Compound` is **unified** with `A`.
    If `Compound` has less than `N` arguments, or `N = 0`, `arg/3` fails
  - Arguments are **unified** – `arg/3` can also be used for instantiating a variable argument of the structure (as in the second example below).
- Examples:

```
| ?- arg(3, edge(a, b, 23), Arg).  ⟹   Arg = 23
| ?- T=edge(_,_,_), arg(1, T, a),
     arg(2, T, b), arg(3, T, 23).  ⟹   T = edge(a,b,23)
| ?- arg(1, [1,2,3], A).           ⟹   A = 1
| ?- arg(2, [1,2,3], B).           ⟹   B = [2,3]
```

- Predicate *univ* can be implemented using `functor` and `arg`, and vice versa, for example:

```
Term =.. [F,A1,A2] ⟺ functor(Term, F, 2), arg(1,
Term, A1), arg(2, Term, A2)
```