Semantic and Declarative Technologies

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Course information

- Course layout
  - Introduction to Logic — Weeks 1–2
  - Declarative Programming
    - Prolog – Programming in Logic — Weeks 3–7
    - Constraint Programming — Weeks 8–10
  - Semantic Technologies
    - Logics for the Semantic Web — Weeks 11–13
- Requirements
  - 2 assignments (150 points each) — 300 points
  - 2 tests (mid-term and final, 200 points each) — 400 points total
  - many small exercises + class activity — 300 points total
- Course webpage: `http://cs.bme.hu/~szeredi/ait`
- Course rules: `http://cs.bme.hu/~szeredi/ait/course-rules.pdf`

Part I

Introduction to Logic

Foundations of logic – overview

- Main theme of the course:
  - How to use mathematical logic in
    - programming
    - intelligent web search
- We start with a brief introduction to Logic
  - Propositional Logic:
    - Syntax and semantics
    - The notion of consequence
    - The resolution inference algorithm
    - Bonus: solving various logic puzzles
  - First Order Logic (FOL)
    - Syntax and Model oriented semantics
    - The notion of consequence for FOL
    - The resolution inference algorithm for FOL

Contents

Atomic and compound propositions

- Consider the sentence: *It is raining and I'm staying at home*
- How many propositions (statements) are there in this sentence?
- There are three:
  - two atomic propositions: $A =$ *"It is raining"*, $B =$ *"I'm staying at home"*
  - and the whole sentence is a compound proposition $C = A \wedge B$
  - read the symbol $\wedge$ as "and"
  - C is called a conjunction
- Atomic proposition: anything, to which a truth value can be assigned
- Truth values: true and false, often represented by integers 1 and 0
- The term propositional formula (or proposition for short) refers to both atomic and compound propositions

Conjunction

- Knowing the truth values of A and B can you tell the truth value of $A \wedge B$?
  Think of $A =$ *"It is raining"*, $B =$ *"I'm staying at home"*

A	B	$A \wedge B$
false	false	false
false	true	false
true	false	false
true	true	true

A	B	$A \wedge B$
0	0	0
0	1	0
1	0	0
1	1	1

  In brief: $A \wedge B$ is true if and only if (iff) ... both A and B are true
- Is the $\wedge$ operator commutative? I.e. $A \wedge B \stackrel{?}{=} B \wedge A$. Why?
  Because $0 \wedge 1 = 1 \wedge 0$
- Is $\wedge$ associative? I.e. $(A_1 \wedge A_2) \wedge A_3 \stackrel{?}{=} A_1 \wedge (A_2 \wedge A_3)$. Why?
  Because both sides are 1 iff each of A_1, A_2, A_3 is 1.
- n-fold conjunction: $C_n = A_1 \wedge A_2 \wedge \cdots \wedge A_n$. When is C_n 1?
  When *all* A_is are 1.
- What is the truth value of an empty conjunction C_0 (C_n with $n = 0$)?
  Hint: Describe the relationship between C_{n-1} and C_n, use this for $n = 1$
  $C_n = C_{n-1} \wedge A_n$, $C_1 = A_1$, hence $A_1 = C_0 \wedge A_1$. This is true iff $C_0 = 1$.

Disjunction and negation

- Another example: *It is not raining or (else) I'm staying at home*
- The two atomic propositions are the same as earlier:
  $A =$ *"It is raining"*, $B =$ *"I'm staying at home"*
- *"It is not raining"* converts to $\neg A$, where $\neg$ denotes negation, read as "it's not the case that ..."
- The whole sentence can be formalised as $\neg A \vee B$
- Read the symbol $\vee$ as "or"; $U \vee V$ is called a disjunction
- The truth tables for disjunction and negation (with 0 – 1 values only):

A	B	$A \vee B$
0	0	0
0	1	1
1	0	1
1	1	1

A	$\neg A$
0	1
1	0

Implication

- Example: *If it is raining, then drive slower than 100 km/h*
- I obey this sign provided that *If it is raining, then I drive slowly. . .*
- This is an implication, formally written as $A \rightarrow B$,
  the premise: $A =$ *"It is raining", conclusion:* $B =$ *"I drive slowly . . . "*
- When it is not raining, does it matter whether I drive slowly?
- The truth table for implication:

A	B	$A \rightarrow B$
0	0	1
0	1	1
1	0	0
1	1	1

- Express implication using disjunction and negation: $A \rightarrow B = \neg A \vee B$
- $A \rightarrow B$ evaluates to 0 iff $A = 1, B = 0$

Equivalence and exclusive or

- Example 1: *I use an umbrella if and only if it is raining*
- This is an equivalence, formally written as $A \leftrightarrow B$ or $A \equiv B$,
  $A =$ *"I use an umbrella"*, $B =$ *"It is raining"*,
- Example 2: *We either go to movies or have dinner (but not both)*
- This is an exclusive or (XOR), formally written as $A\, xor\, B$ or $A \oplus B$,
  $A =$ *"we go to movies"*, $B =$ *"we have a dinner"*,
- The truth tables for equivalence and exclusive or:

A	B	$A \equiv B$
0	0	1
0	1	0
1	0	0
1	1	1

A	B	$A \oplus B$
0	0	0
0	1	1
1	0	1
1	1	0

- Express equivalence using exclusive or, and the other way round:
  $(A \equiv B) = \neg(A \oplus B)$, $(A \oplus B) = \neg(A \equiv B)$

Normal forms

- A proposition has lots of equivalent formulations:
  $A \rightarrow B \equiv \neg A \vee B \equiv \neg(A \wedge \neg B)$
- To design an efficient reasoning algorithm, it makes sense to use one of normal forms (NF), such as:
  - DNF (Disjunctive Normal Form) or CNF (Conjunctive NF)
- Both allow only three operations: $\wedge, \vee$, and $\neg$
- In both NFs '$\neg$' can only be used in front of atomic propositions.
  A formula is called a literal if it is either A or $\neg A$, where A is atomic.
- A DNF takes the form $C_1 \vee \ldots \vee C_n$, $n \geq 0$, where each C_i is a conjunction of literals $L_{i1} \wedge \ldots \wedge L_{im_i}$
- A CNF takes the form $D_1 \wedge \ldots \wedge D_n$, $n \geq 0$, where each D_i is a disjunction of literals $L_{i1} \vee \ldots \vee L_{im_i}$
- Produce the CNF and DNF of $A \oplus B$ (exclusive or)!
- Notice that the DNF can be easily derived from a truth table

Models and tautologies

- Recall some algebraic formulas from high school:

  $x^2 - 3x + 2 = 0$ equation – true for *some* values of x

  $x^2 - 4 = (x - 2)(x + 2)$ identity – true for *all* values of x

- Consider a propositional formula with n atomic propositions, e.g.

$$((A \wedge B) \rightarrow C) \quad \equiv \quad (A \rightarrow (B \rightarrow C))$$

- Here $n = 3$, so there are $2^n = 8$ *valuations* for atomic propositions:
  (A, B, C) can be $(0, 0, 0); (0, 0, 1); (0, 1, 0); \ldots ; (1, 1, 0); (1, 1, 1)$
- Each such valuation is called a *model* or a *universe*
- A model satisfies a propositional formula, if the formula is true when the atomic propositions take the 0–1 values specified by the model.
  E.g. the model $(0, 0, 0)$ satisfies the above equivalence
- A formula is called a tautology if all models satisfy the formula
  (cf. the above algebraic identity being true for all possible values of x)

Some important tautologies

- Show that this formula is a tautology:

$$((A \wedge B) \rightarrow C) \quad \equiv \quad (A \rightarrow (B \rightarrow C)) \qquad (1)$$

  - Let us find all the models in which the left hand side evaluates to 0: There is only one such model $(A, B, C) = (1, 1, 0)$
  - Let us find all the models in which the right hand side evaluates to 0: There is only one such model $(A, B, C) = (1, 1, 0)$
  - Hence the above formula is a tautology
- Show that the following formulas are tautologies:

$$\neg\neg U \quad \equiv \quad U$$
$$\neg(U \wedge V) \quad \equiv \quad \neg U \vee \neg V \qquad (2)$$
$$\neg(U \vee V) \quad \equiv \quad \neg U \wedge \neg V \qquad (3)$$

(2) and (3) are called De Morgan's laws.

Contents

An automated inference system: resolution

- The *first order resolution* inference algorithm was devised by Alan Robinson around 1964
- Let us first introduce its simplified form for propositional logic
- Resolution uses the *conjunctive normal form* (CNF), also called *clausal form* (recall):
  - a CNF is a conjunction of *clauses*: $Cl_1 \wedge \ldots \wedge Cl_n$
  - a clause is a disjunction of *literals*: $L_1 \vee \ldots \vee L_k$
  - a literal is either A or $\neg A$, where A is an atomic proposition
- Conjunction is commutative and associative, and duplicate conjuncts can be elminated, therefore CNF is normally viewed as a set of clauses.
- Similarly, a clause is represented by a set of literals.

A sample translation to clausal form

- Example: Transform to clausal form: $((A \wedge B) \rightarrow D) \wedge (C \rightarrow (A \wedge B))$ (*)
  - replace all connectives by equivalents using only $\neg, \wedge, \vee$
  - move negations inside using De Morgan Laws
  - apply distributivity repeatedly to eliminate $\wedge$s inside $\vee$s: $U \vee (V \wedge W) = (U \vee V) \wedge (U \vee W)$
  - transform $\wedge$ and $\vee$ operators to sets, elminating duplicates
- If a clause (a disjunction) contains both U and $\neg U$ then it is *meaningless* (it carries no information as $(U \vee \neg U) \equiv$ *true*), therefore it can be removed
- Simplified notation (used in first Prolog versions)
  - literals written as signed atomic propositions, e.g. `-A`, `+B` (for `¬A`, `B`)
  - clauses written as sequences of literals followed by a full stop, e.g. `-A -B +D.` for $\neg A \vee \neg B \vee D$
- The CNF of (*): $(\neg A \vee \neg B \vee D) \wedge (\neg C \vee A) \wedge (\neg C \vee B)$
  The CNF in set notation: $\{\{\neg A, \neg B, D\}, \{\neg C, A\}, \{\neg C, B\}\}$
  The CNF in simplified notation: `-A -B +D. -C +A. -C +B.`

The resolution inference rule – introduction

- Consider these two clauses: `+A -B -C.` (1)
  `+A +D +B.` (2)
- Literal # 2 in clause (1) is `-B`, while literal # 3 in clause (2) is `+B`. These literals are *opposite*, i.e. one is the negation of the other.
- Given two clauses containing opposite literals, the resolution rule infers a new clause, called the resolvent, containing the union of all literals of the two clauses, except the opposite literals.
- In the example the resolvent clause is `+A -C +D.` (3)
  Note that there is only one `+A` as $\mathtt{A} \vee \mathtt{A} = \mathtt{A}$.
- Resolution is sound, i.e. (3) follows from (1) and (2). This is due to the *resolution principle*:

$$(\neg U \vee V) \wedge (U \vee W) \rightarrow (V \vee W) \qquad (4)$$

- Proof: Assume the LHS is true. U is either true or false.
  - If U is true V has to be true, as the first disj. is true.
  - If U is false W has to be true, as the second disj. is true.

  In either case the RHS is true.

The resolution inference rule – full definition

- Input: two clauses $C = \mathtt{L}_1\ \mathtt{L}_2\ \ldots\ \mathtt{L}_n.$
  $D = \mathtt{M}_1\ \mathtt{M}_2\ \ldots\ \mathtt{M}_k.$

  where $\mathtt{L}_i$ = `+X` and $\mathtt{M}_j$ = `-X`, or $\mathtt{L}_i$ = `-X` and $\mathtt{M}_j$ = `+X`.
- Let $C' = C \setminus \{\mathtt{L}_i\}$, $D' = D \setminus \{\mathtt{M}_j\}$, where $\setminus$ denotes set difference.

  (The set difference $S_1 \setminus S_2$ is obtained by removing all elements of S_2 – if present – from S_1)

  Thus $C' = \mathtt{L}_1\ \ldots\ \mathtt{L}_{i-1}\ \mathtt{L}_{i+1}\ \ldots\ \mathtt{L}_n.$
  $D' = \mathtt{M}_1\ \ldots\ \mathtt{M}_{j-1}\ \mathtt{M}_{j+1}\ \ldots\ \mathtt{M}_k.$
- Resolution of C and D yields the clause $E = C' \cup D'$ (meaning $C' \vee D'$), called the $resolvent_{ij}(C, D)$, or simply $resolvent(C, D)$;

  $E = \mathtt{L}_1\ \ldots\ \mathtt{L}_{i-1}\ \mathtt{L}_{i+1}\ \ldots\ \mathtt{L}_n\ \mathtt{M}_1\ \ldots\ \mathtt{M}_{j-1}\ \mathtt{M}_{j+1}\ \ldots\ \mathtt{M}_k.$
  (with duplicates removed)

The resolution rule – remarks

- Informally: the resolution rule can be interpreted as viewing the clauses as arithmetic formulas, to be summed up and removing *exactly one* pair of "summands" `+X -X`
  - Example: *resolvent*(`+A-B-C`, `+B+D`) =`+A-C+D`
  - Remark: this analogy does not work, if there is a literal which occurs in both clauses,
    e.g. *resolvent*(`+A-B-C`, `+B+D+A`) =`+A-C+D` (only one `+A` is kept)
- The case of having two or more "summands" with opposite signs also breaks the analogy
  - Here only one pair of such summands is removed
  - Example: $resolvent_{21}$(`+A-B-C`, `+B+D+C`) =`+A-C+D+C`= 1 (true), or $resolvent_{33}$(`+A-B-C`, `+B+D+C`) =`+A-B+B+D`= 1
  - Thus resolution does not produce a meaningful clause in this case

Example: solving an inspector Craig puzzle using resolution

- The puzzle below is cited from "What Is The Name Of This Book?" by Raymond M. Smullyan, chapter "From the cases of Inspector Craig"
- Puzzles in this chapter involve suspects of a crime, named A, B, etc. Some of them are guilty, some innocent.
- Example:
  An enormous amount of loot had been stolen from a store. The criminal (or criminals) took the heist away in a car. Three well-known criminals A, B, C were brought to Scotland Yard for questioning. The following facts were ascertained:
  1. No one other than A, B, C was involved in the robbery.
  2. C never works without A (and possibly others) as an accomplice.
  3. B does not know how to drive.

  Is A innocent or guilty?

Inspector Craig puzzle – solution

- Let's recall the facts
  1. No one other than A, B, C was involved in the robbery.
  2. C never works without A (and possibly others) as an accomplice.
  3. B does not know how to drive.
- Transform each statement into a formula involving the letters A, B, C as atomic propositions. Proposition A stands for "A is guilty", and so on.
  1. A is guilty or B is guilty or C is guilty: $A \vee B \vee C$
  2. If C is guilty then A is guilty: $C \rightarrow A$
  3. It cannot be the case that only B is guilty: $B \rightarrow (A \vee C)$
- Transform each propositional formula into conjunctive normal form (CNF), then show the clauses in simplified form:
  1. $A \vee B \vee C$ (already in CNF), clause: `+A +B +C.`
  2. $C \rightarrow A$, CNF: $\neg C \vee A$, clause: `-C +A.`
  3. $B \rightarrow (A \vee C)$, CNF: $\neg B \vee A \vee C$, clause: `-B +A +C.`

  (Note that in general a single formula can give rise to multiple clauses.)

Inspector Craig puzzle – resolution proof

- Collect the clauses, give each a reference number and perform a resolution proof:

```
(1)    +A +B +C.
(2)    -C +A.
(3)    -B +A +C.     resolve (1) lit 2 with (3) lit 1 => (4)
(4)    +A +C.        resolve (4) lit 2 with (2) lit 1 => (5)
(5)    +A.
```

- We deduced that `A` is true, so the solution of the puzzle is: A is guilty
- Notice that `+A` occurs in each of the above clauses
- As clauses are disjunctions, `A` being true means that all clauses are true
- Hence the statements of the puzzle impose no restrictions on propositions B and C (all 4 combinations allowed)

Removing trivial consequences

Consider this set of clauses: $CS = \{$ `-B+C+D`, `+A+C`, `-A-B`, `+A-B+C` $\}$

- Find a clause in CS that is a consequence of another clause in CS.
- Hint: of these formulas, which implies which other? $U \vee V$, U, V?

  (If we know $U \vee V$ is true, can U be false?) Yes, it can.

  (If we know U is true, can $U \vee V$ be false?) No
- Hence U implies $U \vee V$, and similarly V implies $U \vee V$
- Viewing clauses as sets, if $C \subseteq D$, then $C \rightarrow D$ ("subset" $\rightarrow$ "whole set")
- `+A+C` $\rightarrow$ `+A-B+C`, so `+A-B+C` is a trivial consequence of `+A+C`

Trivial consequences

- A clause $C \vee D$ ($D \neq$ empty) is said to be a trivial consequence of C
- Is it of interest to obtain the set of all consequences of CS?
- No, we get marred by trivial consequences, e.g. `-A-B-C`, `-A-B+C`, ...
- It makes more sense to construct a maximal set of *non-trivial* consequences, i.e. a set MCS which contains all consequences of CS, except those that are a trivial consequence of a clause already in MCS
- Removing a trivial consequence is valid because $(C \wedge (C \vee D)) \equiv C$

Maximal set of non-trivial consequences (ADVANCED)

For the mathematically minded, here is a precise definition of the *maximal set of non-trivial consequences*

- For a set of clauses CS, its maximal set of consequences is MCS iff:
  - *each clause in MCS is a consequence of CS:*

    for each $C \in MCS$, $CS \rightarrow C$
  - *there are no trivial consequences in MCS:*

    for each $C_1, C_2 \in MCS$, C_2 is not a trivial consequence of C_1
  - *MCS contains all non-trivial consequences:*

    for each clause C such that $CS \rightarrow C$ holds, either $C \in MCS$ holds, or else C is a trivial consequence of a $C' \in MCS$.

Constructing *MCS* – continuing the example

- The set of input clauses:

```
(1) -B+C+D
(2) +A+C
(3) -A-B
(4) +A-B+C
```

- Remove (4), as it is implied by (2)
- Resolve (2) with (3) adding a new clause:

```
(5) -B+C
```

- Remove (1), as it is implied by (5)
- As no removal or resolution step can be applied, exit with the following maximal set of (non-trivial) consequences:

```
(2) +A+C
(3) -A-B
(5) -B+C
```


A saturation algorithm for obtaining *MCS*

Given a set of clauses CS_0, you can obtain its maximal set of consequences by performing the following algorithm:

1. set CS to CS_0
2. if CS contains an empty clause, exit with CS_0 being inconsistent
3. if there are $C_1, C_2 \in CS$ such that C_2 is a trivial consequence of C_1, then remove C_2 from CS, and repeat step 3
4. if there are $C_1, C_2 \in CS$ such that C_1 resolved with C_2 yields C_3 where $C_3 \not\equiv true$ and $C_3 \notin CS$, then add C_3 to CS, and continue at step 3
5. (the conditions of both steps 3 and 4 failed) exit with $MCS = CS$

Contents

First Order Logic – An example

- Consider an island inhabited by at least one person
  - Some people (possibly none) are optimistic.
  - A person may have another person as a friend. There is no information on the number of friends a person may have, this could be 0, 1, or more. Also, friendship may not be mutual.
- We know the following facts
  - (a) If someone has a non-optimistic friend, then they are optimistic.
  - (b) There is at least one person, who has a friend.
- Try convincing yourself that the following statement must hold:
  - (c) There is an optimistic person on the island.
- Describe statements (a), (b) and (c) formally. Use the following notation:
  - Let $hasF(x, y)$ denote that x has y as their friend
  - Let $opt(x)$ mean that x is optimistic
  - Use the quantifiers $\exists$ and $\forall$ as in the example (which states that each optimistic person has a friend): $\forall x.(opt(x) \rightarrow \exists y.hasF(x, y))$

First Order Logic – Proving a consequence

- **Recall:** (a) If someone has a non-optimistic friend, then they are optimistic.
  (b) There is at least one person, who has a friend.
- From (a) and (b), can you deduce (c), i.e. someone is optimistic?
  - (b) states that there is a person (say `p1`) who has a friend (say `p2`)
  - Do case-based reasoning: `p2` is either optimistic or not
    - Case 1: `p2` is optimistic. This implies that (c) is true
    - Case 2: `p2` is not optimistic. As `p1` has (the non-optimistic) `p2` as a friend, because of (a), `p1` is optimistic. Thus (c) is true again.
  - Having shown that both possible cases lead to (c) being true, we have proven that statement (c) holds on the island.
- Thus (c) is a semantic consequence of $\{(a),(b)\}$: $\{(a),(b)\} \models (c)$
- This proof works for any island (math-speak: model)
- A model for this example consists of
  - a set Δ containing the inhabitants of the island
  - the interpretation of the 1-argument predicate `opt/1` $\subseteq \Delta$
  - the interpretation of the 2-argument predicate `hasF/2` $\subseteq \Delta \times \Delta$
- A model has all information needed to check the truth of a FOL formula

First Order Logic (FOL)

First Order Logic

- Includes Propositional Logic as a special case
  - All connectives of Propositional Logic can be used in FOL
- Views of logic
  - Syntax (What are the well-formed statements)
  - Proofs (How can one obtain true statements?)
  - Semantics (What is the meaning of statements and their components?)
  - Pragmatics (How to use all this?)

Contents

First Order Logic – Syntax

Building blocks of FOL

- Symbols:
  - logical symbols: propositional connectives $\vee$, $\neg$, ...; quantifiers $\forall$ $\exists$, punctuation etc.– these have a *fixed meaning*
  - non-logical symbols such as *hasF* – these have *arbitrary meaning*

  An analogy with programming languages:
  logical symbols – keywords, non-logical symbols – identifiers
- Terms represent individual objects in our universe, e.g. if *f(x)* and *m(x)* denote the father and the mother of *x*, and *s()* denotes an individual named Susan, then *m(f(s()))* refers to Susan's father's mother, i.e. the paternal grandmother of Susan
- Formulas state truths, e.g. *hasF(m(f(s())),m(s()))* – meaning Susan's paternal grandmother has Susan's mother as a friend.

The alphabet of FOL

What symbols are used in FOL formulas?

- logical symbols
  - punctuation symbols: (,) .
  - logic connectives:
    $\wedge$ (conjunction), $\vee$ (disjunction), $\neg$ (negation),
    $\exists$ (existential quantifier symbol – "exists such ... that ..."),
    $\forall$ (universal quantifier symbol – "for all ... holds that ..."),
    $=$ (equality predicate)
  - variable symbols: $x_1, \ldots, x_i, \ldots$
- non-logical symbols
  - function symbols: $f, g, h, \ldots$, (including the special case of) constant (nullary function) symbols: $a, b, c, \ldots$
  - predicate symbols: $p, q, r, \ldots$
  - each function and predicate symbol has a fixed arity (# of args) ≥ 0
  - a signature (cf. declaring vars in a program) specifies a set of function and predicate symbols, together with their arities, e.g. functions: $f/1$ ($f(x)$ denotes the father of x), $m/1$ ("mother of"), predicates: $hasF/2$, $opt/1$

Syntax of FOL, computer scientists style

⟨term⟩	::=	⟨var symbol⟩	|
		⟨function symbol⟩(⟨arglist⟩)	
⟨arglist⟩	::=		| % *empty*
		⟨term⟩, ...	% *comma sep. list*
⟨atomic frm⟩	::=	⟨pred symbol⟩(⟨arglist⟩)	|
		⟨term⟩ = ⟨term⟩	
⟨formula⟩	::=	⟨atomic frm⟩	|
		(¬ ⟨formula⟩)	|
		(⟨formula⟩ ∧ ⟨formula⟩)	|
		(⟨formula⟩ ∨ ⟨formula⟩)	|
		∃ ⟨var symbol⟩ . (⟨formula⟩)	|
		∀ ⟨var symbol⟩ . (⟨formula⟩)	

Mathematicians often

- insert/delete parentheses and/or dots (in quantified formulas)
- omit empty function arguments (), e.g. $s \equiv$ (is a shorthand for) $s()$
- use multiple vars after a single quantifier, e.g. $\forall x, y.(\ldots) \equiv \forall x.(\forall y.(\ldots))$

Syntax of FOL, mathematician style (ADVANCED)

- A term is a text (a sequence of symbols) to name an object of the universe of discourse
  - A variable symbol is a term
  - If $t_1, \ldots, t_n$ are terms and f is a function symbol of arity n, then $f(t_1, \ldots, t_n)$ is a term
  - A term of FOL is obtained by applying the above two rules a finite number of times
- A well formed FOL formula (wff) is a text describing a statement
  - If $t_1, \ldots, t_n$ are terms and p is a predicate symbol of arity n, then $p(t_1, \ldots, t_n)$ is an atomic formula
  - If t_1 and t_2 are terms, then $t_1 = t_2$ is also an atomic formula.
  - If α and β are wffs, x is a variable symbol, then $(\neg\alpha)$, $(\alpha\wedge\beta)$, $(\alpha\vee\beta)$, $(\exists x.\alpha)$, $(\forall x.\alpha)$ are wffs, too.
  - A well formed formula is obtained by applying the above rules a finite number of times

Syntax of FOL, contd.

- Abbreviations – adding further propositional operations, as syntactic sugar:
  - $(\alpha \rightarrow \beta)$ is an abbreviation of: $(\neg\alpha \vee \beta)$
  - $(\alpha \equiv \beta)$ is an abbreviation of: $((\alpha \rightarrow \beta) \wedge (\beta \rightarrow \alpha))$
  - note that formulas $(\alpha \vee \beta)$ and $(\exists x.\alpha)$ could have been defined as abbreviations, using De Morgan's laws (extended to quantifiers):
    - $(\alpha \vee \beta) \equiv \neg(\neg\alpha \wedge \neg\beta)$
    - $(\exists x.\alpha) \equiv \neg(\forall x.\neg\alpha)$
- The scope of variables
  - An occurrence of variable x is bound if it appears inside a formula $\exists x.\alpha$ or $\forall x.\alpha$
  - A variable occurrence x is free if it is not bound
- A formula is a sentence (also called a closed formula) if it contains bound variables only
- Propositional Logic is a special case of FOL where all predicate symbols have arity 0 (and so no variables and no function symbols are allowed)

Some further practice

- Formalize in FOL the statements below, using the signature:
  function symbols: $f/1$ and $m/1$ (for father and mother), $s/0$ for Susan;
  predicate symbols $hasF/2$ (has friend), and $opt/1$ (optimist).
  1. Someone is an optimist. (recall)
  2. Everyone is an optimist.
  3. Everyone has a friend.
  4. There is someone who is befriended with their father's mother.
  5. Someone is not an optimist.
  6. Everyone is a friend of themselves.
  7. If x's father or mother is an optimist, so is x, for any x
  8. If x has a non-optimist friend, then x is an optimist, for any x. (recall)
  9. Anyone whose all friends are optimists is bound to have a friend.
  10. Susan is an optimist.
  11. Susan's maternal grandmother has Susan's paternal grandmother as a friend.
- Try finding subsets of the above FOL sentences so that another sentence above is a consequence of the given subset

Contents

Clauses in First Order Logic

- A FOL *clause* is
  - a set of literals (disjuncts),
  - each being a plain or negated *atomic* formula.
  - All variables are universally quantified.
- An example: one's female parent is their mother.
  `-hasParent(X,Y) -female(Y) +hasMother(X,Y).`
  $\equiv \forall \mathtt{X},\mathtt{Y}.((\mathtt{hasParent}(\mathtt{X},\mathtt{Y}) \wedge \mathtt{female}(\mathtt{Y})) \rightarrow \mathtt{hasMother}(\mathtt{X},\mathtt{Y}))$
- An arbitrary FOL statement can be transformed to a set of clauses:
  - do propositional transformations
    - express $\rightarrow$, $\equiv$ etc, using $\neg$, $\wedge$, and $\vee$
    - bring $\neg$ in front of atomic formulas
    - convert to CNF
  - bring quantifiers to the front of the formula
  - get rid of $\exists$ quantifiers by introducing so called Skolem functions (not relevant in Logic Programming, not discussed further)

A sample transformation to CNF

- Example: if x has a non-optimist friend, then x is an optimist
- FOL formula: $\forall x.(\exists y.(hasF(x,y) \wedge \neg opt(y)) \rightarrow opt(x))$
- Eliminate implication ($U \rightarrow V \equiv \neg U \vee V$):
  $\forall x.(\neg(\exists y.(hasF(x,y) \wedge \neg opt(y))) \vee opt(x))$
- Bring negation inside (use $\neg\exists u.W \equiv \forall u.\neg W$, and also De Morgan rules):
  $\forall x.(\forall y.(\neg hasF(x,y) \vee opt(y)) \vee opt(x))$
- Bring $\forall, \exists$ outside $\forall x.(\forall y.(\varphi_1(x,y)) \ldots \varphi_2(x)) \equiv \forall x,y.(\varphi_1(x,y) \ldots \varphi_2(x))$
  $\forall x,y.(\neg hasF(x,y) \vee opt(y) \vee opt(x))$
- Conjunctive Normal Form (CNF): $\neg hasF(x,y) \vee opt(y) \vee opt(x)$
- Simplified CNF: `-hasF(X,Y) +opt(Y) +opt(X).`
  (Note the use of capitalized identifiers for variables.)

How to read the clausal form?

- A general clause: $-A_1 \ \ldots\ -A_m \ +B_1 \ldots\ +B_n$., $m \geq 0, n \geq 0$
- It can be read as: the *conjunction* of negative literals implies the *disjunction* of positive literals:
  $(A_1 \wedge \ldots \wedge A_m) \rightarrow (B_1 \vee \ldots \vee B_n)$
- An empty conjunction (denoted by ■) is true, and an empty disjunction (denoted by □) is false, because *true* $\wedge A \equiv A$ and *false* $\vee A \equiv A$.
- Example: `-hasF(X,Y) +opt(Y) +opt(X).` can be read as
  One of a pair of friends has to be *opt*: $hasF(X, Y) \rightarrow opt(Y) \vee opt(X)$
- Alternative readings:
  Having a non-*opt* friend implies being *opt*: $hasF(X, Y) \wedge \neg opt(Y) \rightarrow opt(X)$
  A friend of a non-*opt* is an *opt*: $hasF(X, Y) \wedge \neg opt(X) \rightarrow opt(Y)$
  Two non-*opt*s cannot be friends: $\neg opt(X) \wedge \neg opt(Y) \rightarrow \neg hasF(X, Y)$
  Two non-*opt*s befriended is a contradiction:
  $\neg opt(X) \wedge \neg opt(Y) \wedge hasF(X, Y) \rightarrow \square$
- In general: you can place any subset of literals into the RHS disjunction and the remaining literals, each negated, into the LHS conjunction.

From propositional resolution to FOL resolution

Assume we have the following clauses:

```
-opt(s).                   % s is non-optimistic.                       (1)
-opt(m).                   % m is non-optimistic.                       (2)
-hasF(s,m)+opt(m)+opt(s).  % if s has m as a friend, either m or s is opt  (3')
```

- Given (1)–(3'), can you deduce something using resolution?
- Yes, one can deduce `-hasF(s,m)` using (propositional) resolution.
- What if w consider this FOL clause instead of (3'):
  `-hasF(X,Y)+opt(Y)+opt(X)  % if X has Y as a friend, either Y or X is opt` (3)
- Obviously, (3') is a special case of (3), i.e. (3') follows from (3).
- Substitutions for variables `X` and `Y` are obtained through *unification*, a two-way pattern matching algorithm.
- Unification is an essential component of FOL resolution.

FOL resolution – a small example

FOL resolution combines prop. resolution with minimal specialization, e.g.

```
-hasF(X,Y) +opt(Y) +opt(X).    (1)
-opt(s).                       (2)
-opt(m).                       (3)
```

Perform a FOL resolution step between literals (1)#2 and (3)#1:

- find a minimal substitution that makes the (unsigned) atomic formulas `opt(Y)` and `opt(m)` the same: $\sigma = \{\mathtt{Y} \leftarrow \mathtt{m}\}$
- apply σ to the *whole* (1) and (3), resulting in opposite literals:
  (1'): `-hasF(X,m)+opt(m)+opt(X)` and (3'): `-opt(m)`
- perform propositional resolution, producing:
  `-hasF(X,m) +opt(X).` (4)
  (Is this a valid statement? Yes: "if Mary is *x*'s friend, then *x* is an optimist")
- Next, resolve (4)#2 and (2)#1, $\sigma = \{\mathtt{X} \leftarrow \mathtt{s}\}$ producing:
  `-hasF(s,m).` (5)
- Each time we use a clause, we rename all its vars systematically
- Similar two-step deductions result in: `-hasF(s,s)`, `-hasF(m,s)`, `-hasF(m,m)`

Unification – making two terms the same

- Propositional resolution requires the presence of literals `+A` and `-B` in the two clauses, where `A` should be identical to `B`
- FOL resolution has a weaker requirement: `A` and `B` should be *unifiable*: there should be a substitution σ of variables with terms, such that $\mathtt{A}\sigma = \mathtt{B}\sigma$ ($\mathtt{A}\sigma$ denotes the formula obtained from `A` by applying substitution σ)
- A substitution replaces all occurrences of certain variables with arbitrary terms (possibly other variables)
  - $\sigma = \{\mathtt{X} \leftarrow \mathtt{b},\ \mathtt{Y} \leftarrow \mathtt{Z}\}$, $\mathtt{A} = \mathtt{hasF(X,Y)}$, $\mathtt{A}\sigma = \mathtt{hasF(b,Z)}$
  - $\sigma = \{\mathtt{X} \leftarrow \mathtt{a}\}$, $\mathtt{A} = \mathtt{hasF(m(X),X)}$, $\mathtt{A}\sigma = \mathtt{hasF(m(a),a)}$
- Example unification: formulas `A=hasF(a, X)` and `B=hasF(Y, b)` are unifiable using the substitution $\sigma = mgu(\mathtt{A},\mathtt{B}) = \{\mathtt{X} \leftarrow \mathtt{b},\ \mathtt{Y} \leftarrow \mathtt{a}\}$
- If there are multiple substitutions σ for which $\mathtt{A}\sigma = \mathtt{B}\sigma$, resolution uses the most general unifier, hence the abbreviation *mgu*
- Example: atomic formulas `p(X, X)` and `p(U, V)` are unifiable using the substitution $\sigma = \{\mathtt{X} \leftarrow \mathtt{U},\ \mathtt{V} \leftarrow \mathtt{U}\}$ – `U` is not substituted further
  - $\sigma' = \{\mathtt{X} \leftarrow \mathtt{a},\ \mathtt{V} \leftarrow \mathtt{a}, \mathtt{U} \leftarrow \mathtt{a}\}$ is also a unifier, but not a *mgu*
  - The *mgu* is unique, except for variable renaming:
    $\sigma_1 = \{\mathtt{X} \leftarrow \mathtt{V},\ \mathtt{U} \leftarrow \mathtt{V}\}$ and $\sigma_2 = \{\mathtt{V} \leftarrow \mathtt{X},\ \mathtt{U} \leftarrow \mathtt{X}\}$ are also *mgu*'s

FOL resolution – an example

- In Prop. Logic: $+a \quad -b$
  $+b \quad -c \Rightarrow +a \quad -c$
- In FOL: $+a(x,0) - b(x,2)$
  $+b(1,y) - c(y) \Rightarrow +a(1,0) - c(2)$
- Detailed steps:
  1. find subst. $\sigma = mgu(b(x,2), b(1,y)) = \{x \leftarrow 1, y \leftarrow 2\}$
     (not all variables are necessarily substituted)
  2. apply substitution σ to both clauses
     (vars are universally quantified – substitution is a valid inference):
     $$+a(1,0) - b(1,2)$$
     $$+b(1,2) - c(2)$$
  3. finally, apply propositional resolution, to obtain the resolvent:
     $$\Rightarrow +a(1,0) - c(2)$$

The resolution inference rule for FOL

- Resolution takes two clauses as input:
  $$C = L_1 \ldots L_n \text{ and } D = M_1 \ldots M_k$$
  where literals $L_i = \pm A$ and $M_j = \pm B$ have opposite signs, and their atomic formulas are unifiable: $\sigma = mgu(A, B)$
- Under the above conditions the resolution inference rule can be applied to C and D and results in the new clause
  $$(L_1 \ldots L_{i-1}\, L_{i+1} \ldots L_n\, M_1 \ \ldots M_{j-1}\, M_{j+1} \ldots M_n)\sigma$$
  obtained by
  - taking the union of the literals of clauses C and D
  - removing the literals L_i and M_j (the ones we resolve upon)
  - applying the substitution σ to the remaining literals
- As specialization (substitution of univ. quantified vars) and propositional resolution are sound operations, FOL resolution is also sound

The factoring inference rule for FOL (ADVANCED)

- For full FOL the resolution rule is not enough to obtain a complete proof system, one needs one more simple rule factoring– if there are two literals in a clause that are unifiable, you can replace them by a single literal, their unified form.
- The factoring deduction rule:
  - example in Propositional Logic: $+a+a - b \Rightarrow +a - b$
    this is „automatic", as clauses are considered sets of literals.
  - example in FOL: $+a(x,2)+a(1,y) - b(x,y) \Rightarrow +a(1,2) - b(1,2)$
  - in general: factoring takes a clause with two unifiable literals and produces a clause with these two literals merged:
    $L_1 \ldots L_n \Rightarrow (L_1 \ldots L_{j-1}\, L_{j+1} \ldots L_n)\sigma$ where $\sigma = mgu(L_i, L_j)$
- For the subset of FOL used in Prolog, this rule is not required, hence it is not discussed further.
- Ancestor resolution (see later) is alternative to factoring, when implementing a complete FOL theorem prover using Prolog technology.

Resolution: Susan's puzzle

Recall some formulas from slide 37:

7. If x's father or mother is an optimist, so is x, for any x
8. If x has a non-optimist friend, then x is an optimist, for any x.
11. Susan's maternal grandmother has her paternal grandmother as a friend.

Let us consider a variant of the above example:
We use the `hasP/2` (has parent) pred. instead of father and mother functions
Also, we simplify 11 to this: Susan's mother has her father as a friend.

We will now prove that Susan is bound to be an optimist, using resolution

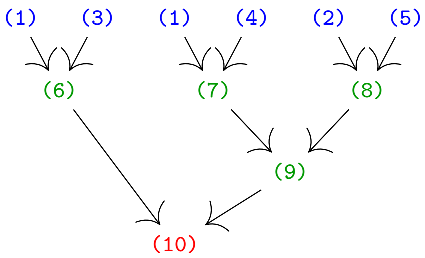
- x is an optimist if x has a parent who is an optimist.
  `+opt(X) -hasP(X, P) -opt(P).` (1)
- x is an optimist if x has a friend who is not an optimist.
  `+opt(X) -hasF(X, F) +opt(F).` (2)
- Susan's (`s`'s) parents are `m` and `f`, and `m` has `f` as her friend.
  `+hasP(s, m).` (3)
  `+hasP(s, f).` (4)
  `+hasF(m, f).` (5)

A resolution proof of the "optimist" example

- Initial clauses: (1) +opt(X) -hasP(X, P) -opt(P).
  (2) +opt(X) -hasF(X, F) +opt(F).
  (3) +hasP(s, m). (4) +hasP(s, f). (5) +hasF(m, f).
- A possible resolution proof that (1), . . . , (5) implies opt(s):

(1) + (3)	(6) +opt(s) -opt(m).	% s is opt if m is opt
(1) + (4)	(7) +opt(s) -opt(f).	% s is opt if f is opt
(2) + (5)	(8) +opt(m) +opt(f).	% m is opt if f is not opt
(7) + (8)	(9) +opt(s) +opt(m).	% s is opt if m is not opt
(6) + (9)	(10) +opt(s).	% s is optimistic

- The proof as a tree:



Can we deduce all the consequences of some clauses?

- Can a clause be resolved with itself?
- Answer: it depends. . . on what kind of logic we use:
  - In propositional logic: **no**, as a clause . . . +a. . . -a. . . is meaningless
  - In FOL: **yes**, see e.g. clause (1) from the previous slide

  (1) +opt(X) -hasP(X, P) -opt(P).
- Resolve (1) with a copy of itself (1') using literals (1)#3 and (1')#1

  (1') +opt(Y) -hasP(Y, Q) -opt(Q).
- Write down the resolvent, and read out its meaning in plain English.

  (2) +opt(X) -hasP(X, P) -hasP(P, Q) -opt(Q).

  "X is an optimist if X has an optimist grandparent (Q)."
- One can keep resolving the output of the previous step with (1), obtaining clauses that describe valid and useful consequences of (1):
  "If X has an optimist great-grandparent, than X is an optimist.". . .
  "If X has an optimist n^{th} ancestor, than X is an optimist."
- One can thus infer infinitely many clauses from {(1)}
- Inferring all consequences of a set of clauses is not a viable task in FOL
- That is why we focus on indirect proofs

Indirect resolution proofs

- Given a premise U and a consequence V, to prove $(U \rightarrow V)$ indirectly:
  - we assume $\neg(U \rightarrow V)$, i.e. $U \wedge \neg V$
  - we show that this leads to contradiction, i.e. $U \wedge \neg V \rightarrow$ false
- What is the truth value of an empty clause (empty disjunction)? false
- The indirect resolution proof of $(U \rightarrow V)$ consists of the following steps:
  - convert both U and $\neg V$ to (two) sets of clauses
  - take the union of the two sets and perform resolution (aiming at getting the shortest clauses possible)
  - when an empty clause is reached, the proof is completed
- To prove that clauses (1), . . . , (5) from page 49 imply opt(s) :
  - add ¬ opt(s) ≡ -opt(s) as clause (10) (this is called the goal clause)
  - deduce an empty clause from the set { (1), . . . , (5), (10)} using resolution

The indirect resolution proof of the "optimist" example

- Initial clauses (so called program clauses: (1)–(5), goal clause: (10))

  (1) +opt(X) -hasP(X, P) -opt(P).
  (2) +opt(X) -hasF(X, F) +opt(F).
  (3) +hasP(s, m).
  (4) +hasP(s, f).
  (5) +hasF(m, f).

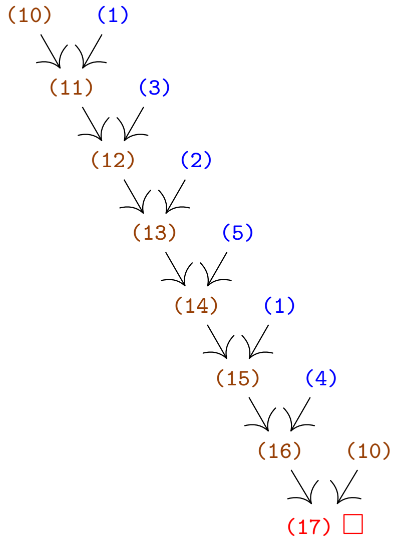
  (10) -opt(s).
- A possible resolution proof that (1), . . . , (5), (10) lead to contradiction:

	(10) -opt(s).	% s is non-opt
(10) + (1)	(11) -hasP(s, U) -opt(U).	% all parents of s are non-opt
(11) + (3)	(12) -opt(m).	% m is non-opt
(12) + (2)	(13) -hasF(m, V) +opt(V).	% all friends of m are opt
(13) + (5)	(14) +opt(f).	% f is opt
(14) + (1)	(15) +opt(Y) -hasP(Y, f).	% all children of f are opt
(15) + (4)	(16) +opt(s).	% s is opt
(16) + (10)	(17) □	% contradiction

(Recall that □ denotes an empty clause, i.e. an empty disjunction ≡ false)

The structure of the indirect "optimist" proof

- A linear resolution step is when a goal clause is resolved with a program clause, producing a new goal clause
- All steps in this proof, except the last, are linear
- The last step is an example of a so called ancestor resolution, as (16) is resolved with one of its ancestors in the proof tree, (10)
- In Prolog, only linear resolution steps are allowed



Can we find out who are "optimists"?

- Let's try to prove that $S = \exists z.opt(z)$ (there are some optimists), assuming clauses (1)–(5)
  Negate S: $\neg S \equiv \neg\exists z.opt(z) \equiv \forall z.\neg opt(z) \Longrightarrow$ clause `-opt(Z)`
- Initial clauses:

```
(1) +opt(X) -hasP(X, P) -opt(P).        (4) +hasP(s, f).
(2) +opt(X) -hasF(X, F) +opt(F).        (5) +hasF(m, f).
(3) +hasP(s, m).                        (10) -opt(Z).
```

- A resolution proof showing that (1), . . . , (5), (10) is contradictory:

	(10) -opt(Z).	% Z is non-opt
(10) + (1)	(11) -hasP(Z, U) -opt(U).	% all parents of Z are non-opt
(11) + (3)	+hasP(s, m).	% Z = s
	(12) -opt(m).	% m is non-opt
(12) + (2)	(13) -hasF(m, V) +opt(V).	% all friends of m are opt
(13) + (5)	(14) +opt(f).	% f is opt
(14) + (1)	(15) +opt(Y) -hasP(Y, f).	% all children of f are opt
(15) + (4)	(16) +opt(s).	% s is opt
(16) + (10)	(17) □	% contradiction

- We claimed (indirectly) that $\neg\exists z.opt(z)$, and the inference above constructed a counterexample: `Z = s` in the step resulting in clause (12)

Finding out who is an "optimist" using the answer literal

- We add a special `-answer(Z)` literal to the goal clause
- This literal cannot take part in reasoning, it just stores the answer
- Initial clauses:

```
(1) +opt(X) -hasP(X, P) -opt(P).        (4)  +hasP(s, f).
(2) +opt(X) -hasF(X, F) +opt(F).        (5)  +hasF(m, f).
(3) +hasP(s, m).                        (10) -opt(Z)-answer(Z).
```

- The proof:

	(10) -opt(Z)	-answer(Z).
(10) + (1)	(11) -hasP(Z, U) -opt(U)	-answer(Z).
(11) + (3)	(12) -opt(m)	-answer(s).
(12) + (2)	(13) -hasF(m, V) +opt(V)	-answer(s).
(13) + (5)	(14) +opt(f)	-answer(s).
(14) + (1)	(15) +opt(Y) -hasP(Y, f)	-answer(s).
(15) + (4)	(16) +opt(s)	-answer(s).
(16) + (10)	(17)	-answer(s).

- The proof ends when only the answer literal is left (cf. empty clause)
- The argument of the answer literal shows the answer: `s`
- Using alternative proofs multiple answers can be obtained

From resolution to Prolog

- The base resolution algorithm leaves several things open:
  - how are the two clauses to resolve upon selected?
  - how are the literals selected?
- Moving towards Prolog, we now view
  - a Conjunctive NF as a sequence (rather than a set) of clauses
  - a clause as a sequence of literals
- To make reasoning faster, we only allow a subset of FOL clauses: those with at most one positive literal (Definite or Horn clauses)
- The four kinds of Horn clauses:

Rule: exactly 1 pos lit, $\geq$ 1 neg lits	(1) +opt(X)-hasP(X, P)-opt(P).
Fact: exactly 1 pos lit, no neg lits	(3) +hasP(s, m).
Goal: no pos lits, $\geq$ 1 neg lits	(10) -opt(Z).
Empty: no pos lits, no neg lits	(17) □.

  (An empty clause can only occur as the final goal clause)
- Positive literals are written first (and are called the clause head)
- In our Susan example the only non-Horn clause was:

  (2) +opt(X)-hasF(X, F)+opt(F).

From resolution to Prolog (ctd.)

```
(o1) +opt(X) -hasP(X, P) -opt(P).        (p1) +hasP(s, f).
(o2) +opt(gm).                           (p2) +hasP(s, m).
                                         (p3) +hasP(m, gm).
```

- Rules and facts form procedure (boolean function) definitions, with head and body parts. (A fact has an empty body.)
- Rules and facts are grouped into procedures, based on their functors of form `F/N`, where `F` is the name of the clause head, and `N` is the # of args. E.g. procedure `opt/1` contains (o1)–(o2), proc. `hasP/2` contains (p1)–(p3).
- A goal clause can be viewed as a sequence of procedure calls. The literal `-opt(s)` is a call of the `opt` procedure, shown above
- In this example `s` acts as the *actual*, and `X` as the *formal* parameter of the procedure, *unification* is the means for parameter passing
- A resolution step can be viewed as a macro expansion: replace `-opt(s)` by the body of the above rule with subst. $\{X \leftarrow s\}$: `-hasP(s, P) -opt(P)`
- If multiple clause heads match a call, a so called choice point is created, choices are explored top-to-bottom via backtracking

Example: proving that Susan is an optimist, initial steps

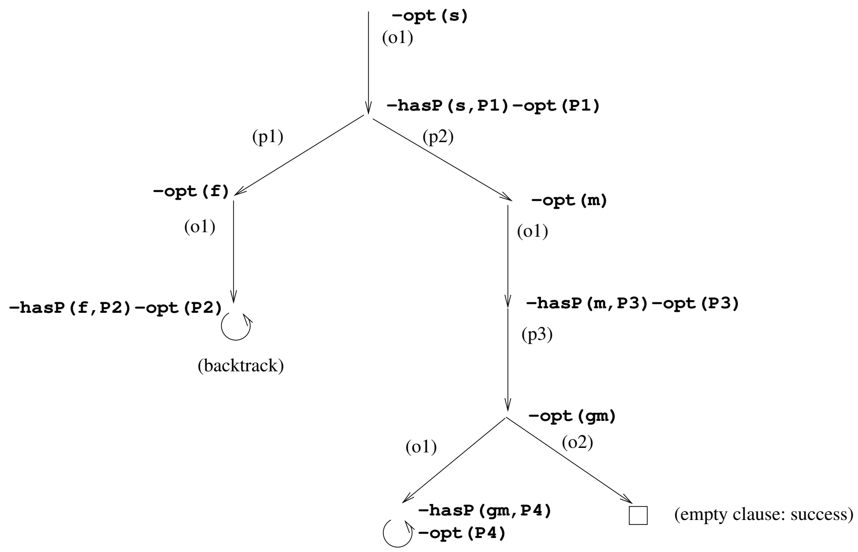
```
(o1) +opt(X) -hasP(X, P) -opt(P).        (p1) +hasP(s, f).
(o2) +opt(gm).                           (p2) +hasP(s, m).
                                         (p3) +hasP(m, gm).
```

Proving that Susan (s) is optimist, goal: (g1) `-opt(s)`

- **Step 1**, matching clause heads: (o1)
- resolve (g1) with copy 1 of (o1), subst. $\{X_1 \leftarrow s\}$ new goal clause:
  (g2) `-hasP(s, P1) -opt(P1).`
- **Step 2**, matching clauses: (p1), (p2); create CHoice Point 1, storing the goal (g2) and the list of choices: `[p1,p2]`
- resolve (g2) with (p1), subst. $\{P_1 \leftarrow f\}$ new goal clause: (g3) `-opt(f).`
- **Step 3**, single matching clause head: (o1), no CHP created
- resolve (g3) with copy 2 of (o1), subst. $\{X_2 \leftarrow f\}$ new goal clause:
  (g4) `-hasP(f, P2) -opt(P2).`
- **Step 4**, no matching clauses, backtrack to CHP 1, remove branch `p1`, leaving `[p2]`[1]. Go back to (g2), resolve it now with (p2), subst. $\{P_1 \leftarrow m\}$, new goal clause: (g5) `-opt(m).`

[1] As this is the last choice, CHP 1 is removed here.

Graphical representation of the resolution search tree



Prolog as a resolution theorem prover

- Recall the two kinds of clauses: the premises (program clauses) and the goal clause (the negation of the conclusion to be proved)
- Prolog execution uses the following indirect resolution algorithm:
  1. If the goal clause is empty, exit with success (of the indirect proof)
  2. Otherwise, find all program clauses whose first literal can be resolved with the first goal literal, scanning top to bottom
  3. If there are > 1 such clauses, create a choice point storing this list of applicable clauses and the current goal clause
  4. If there are ≥ 1 such clauses, resolve the goal clause with the first applicable program clause, make the resolvent the new goal clause, and go to step (1)
  5. If there are no such clauses, backtrack:
     - if no choice points are left, exit with failure (of the indirect proof)
     - consider the latest choice point (choice points form a stack), restore the goal clause from the choice point, resolve it with the next applicable clause and continue at step (1).

Prolog as a resolution theorem prover

- The Prolog programming language is based on resolution with these constraints (recap) :
  - only DEFINITE Clauses are allowed
  - the indirect, goal oriented resolution approach is used
  - resolution is applied in a *LINEAR* manner: start with the goal, resolve it with a rule or fact, and repeat this for the resolvent
  - the SELECTION of literals is restricted: only the first literals in both clauses can be used for resolution
- Prolog is thus based on SLD resolution – Selective Linear resolution on Definite clauses

Performing queries using resolution – practice

- Consider the program

```
+hP(a, b).                         (1)
+hP(b, c).                         (2)
+hP(b, d).                         (3)
+hP(d, e).                         (4)

+hGP(Ch, GP) -hP(Ch, P) -hP(P, GP). (5)
```

- Execute the following goals using SLD resolution:

```
-hGP(a,  GP).                      (11)
-hGP(b,  GP).                      (12)
-hGP(d,  GP).                      (13)
-hGP(Ch, e).                       (14)
-hGP(Ch, b).                       (15)
-hGP(Ch, GP).                      (16)
```


Further limitations of Prolog

- Equality can not be used in positive literals (clause heads), e.g. these formulas cannot be converted to Prolog:

$$\forall x.(x = s() \leftarrow opt(x)) \qquad \text{(only Susan can be optimistic)}$$

$$\forall x, y.(x + y = y + x) \qquad \text{(addition is commutative)}$$

- Consequence: function symbols become data constructors, e.g.

```
| ?- X = 1+2*3.          X = 1+2*3 ?
| ?- X is 1+2*3.         X = 7 ?      % is is a built-in for arithmetic
| ?- X = 1+2*3, Y+Z = X. X = 1+2*3, Y = 1, Z = 2*3 ?
```

- Prolog unification does not do the occurs check:
  - FOL resolution prescribes a variable x cannot be unified with a term α, if x occurs in α.
  - This costly check is practically useless in Prolog and by default is not performed by Prolog systems. (However, there is a built-in predicate `unify_with_occurs_check`, to perform this).

Contents

The notion of a model in First Order Logic

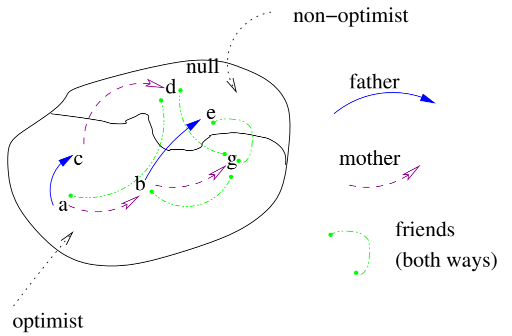
- A FOL formula V is said to be a semantic consequence of U $\Leftrightarrow$ on any “island” on which U holds, V holds as well.
- The mathematical counterpart of the “island” is the notion of model.
- A model for the Susan example consists of
  - a base set Δ (the inhabitants of the island), plus:
  - functions that correspond to symbols m and f: $m^I, f^I \in \Delta \to \Delta$
  - a 1-argument relation corresponding to symbol opt: $opt^I \subseteq \Delta$
  - a 2-argument relation corresponding to symbol $hasF$: $hasF^I \subseteq \Delta \times \Delta$
- The notation frequently used for a model in FOL is this:
  - a model $\mathcal{I}$ is a pair consisting of the base set and a mapping I:

$$\mathcal{I} = \langle \Delta, I \rangle$$

  - here I is a mapping between (function and predicate) symbols and their corresponding functions/relations, e.g. in our example:

$$I = \{m \mapsto m^I, f \mapsto f^I, opt \mapsto opt^I, hasF \mapsto hasF^I\}$$

A sample model



$\mathcal{I}_{sample} = \langle \Delta, I \rangle$,
$\Delta = \{a, b, c, d, e, g, null\}$
(null represents undefined function values)
$opt^I = \{a, b, c, g\}$
$f^I = \{a \mapsto c, b \mapsto e, c \mapsto null, d \mapsto null, \ldots, null \mapsto null\}$
$m^I = \{a \mapsto b, b \mapsto g, c \mapsto d, d \mapsto null, \ldots, null \mapsto null\}$
$hasF^I = \{\langle a, d \rangle, \langle d, a \rangle, \langle b, g \rangle, \langle g, b \rangle, \langle d, g \rangle, \langle g, d \rangle, \langle e, g \rangle, \langle g, e \rangle\}$

- Which of the following statements are true in the model $\mathcal{I}$?

(1) $\exists x.opt(x)$
(2) $\exists x.\neg opt(x)$
(3) $\forall x.opt(x)$
(4) $\exists x.hasF(m(x), f(x))$
(5) $\exists x.opt(f(f(x)))$
(6) $\exists x.opt(m(m(x)))$
(7) $\forall x.\neg hasF(x, x)$

Semantics of FOL

- The syntax defines which symbol sequences are well formed formulas
- Semantics determines the truth of well formed formulas in a given model of discourse (model theoretical or Tarski-style semantics)
- Let us fix a signature, i.e. the sets of function and predicate symbols and their arities
- $\mathcal{I} = \langle \Delta, I \rangle$ is called an interpretation or model for a given signature iff
  - Δ is an arbitrary non-empty set (domain)
  - I is a mapping (normally applied as a superscript), which maps each
    - function symbol f with arity n to an n-ary function on the domain Δ:
      $f^I \in \underbrace{\Delta \times \ldots \times \Delta}_{n} \mapsto \Delta$ (f^I denotes the function that corresponds to the function symbol f)
    - predicate symbol p with arity n to an n-ary relation on the domain Δ:
      $p^I \subseteq \underbrace{\Delta \times \ldots \times \Delta}_{n}$ (p^I denotes the relation that corresponds to the predicate symbol p)

Formal semantics of FOL (ADVANCED)

- Given a model $\mathcal{I} = \langle \Delta, I \rangle$ for a given signature, the semantics of FOL is concerned with mapping each term to an element of Δ and mapping each sentence to a truth value.
- The semantic mapping is defined recursively, thus it has to be concerned with non-closed formulas (those with free variables).
- To assign a meaning to non-closed formulas we need a so called variable assignment, or valuation:
  - a variable assignment is a function φ which maps each variable symbol to an element of the domain: $\varphi(x_i) \in \Delta$ for all i
  - Notation: $\varphi[x \mapsto d]$ is an assignment which maps all variables distinct from x to the same value as φ, while it maps x to $d \in \Delta$.
- Semantics of terms: given a model $\mathcal{I} = \langle \Delta, I \rangle$ and a valuation φ we map an arbitrary term t to its meaning $t^{\varphi,\mathcal{I}}$ using the following recursive definition:
  - If $t = x$ is a variable, then $t^{\varphi,\mathcal{I}} = \varphi(x)$,
  - If $t = f(t_1, \ldots, t_n)$, where $t_1, \ldots, t_n$ are terms and f is a function symbol of arity n, then $t^{\varphi,\mathcal{I}} = f^{\mathcal{I}}(t_1^{\varphi,\mathcal{I}}, \ldots, t_n^{\varphi,\mathcal{I}})$

Formal semantics of FOL, ctd. (ADVANCED)

- Semantics of formulas: let us fix a signature and a corresponding model $\mathcal{I} = \langle \Delta, I \rangle$
- Formula α is said to hold in model $\mathcal{I}$ under valuation φ ($\mathcal{I} \models_\varphi \alpha$):
  - $\mathcal{I} \models_\varphi p(t_1, \ldots, t_n)$ iff $\langle d_1, \ldots, d_n \rangle \in \mathcal{P}$, where $\mathcal{P} = p^\mathcal{I}$ and $d_i = t_i^{\varphi,\mathcal{I}}$.
  - $\mathcal{I} \models_\varphi t_1 = t_2$ iff $t_1^{\varphi,\mathcal{I}} = t_2^{\varphi,\mathcal{I}}$
  - $\mathcal{I} \models_\varphi \neg\alpha$ iff $\mathcal{I} \models_\varphi \alpha$ is not the case.
  - $\mathcal{I} \models_\varphi \alpha \wedge \beta$ iff both $\mathcal{I} \models_\varphi \alpha$ and $\mathcal{I} \models_\varphi \beta$ are true.
  - $\mathcal{I} \models_\varphi \alpha \vee \beta$ iff at least one of $\mathcal{I} \models_\varphi \alpha$ and $\mathcal{I} \models_\varphi \beta$ is true.
  - $\mathcal{I} \models_\varphi \forall x.\alpha$ iff for all $d \in \Delta$ it holds that $\mathcal{I} \models_{\varphi[x \mapsto d]} \alpha$.
  - $\mathcal{I} \models_\varphi \exists x.\alpha$ iff there exists $d \in \Delta$ such that $\mathcal{I} \models_{\varphi[x \mapsto d]} \alpha$.
- It can be shown that $\mathcal{I} \models_\varphi \alpha$ depends on $\varphi(x)$ only if x is free in α.
  - Thus $\mathcal{I} \models_\varphi \alpha$ does not depend on φ, if α is a sentence,
  - the notation $\mathcal{I} \models \alpha$ is used for sentences α, read as $\mathcal{I}$ satisfies α, or $\mathcal{I}$ is a model of α.

Semantic consequence: three versions of $\models$

- Given a model $\mathcal{I}$ and a sentence (a formula with no free vars) α we now know how to decide if $\mathcal{I} \models \alpha$ (α holds in $\mathcal{I}$, $\mathcal{I}$ is a model of α)
  - e.g., $\mathcal{I}_{sample} \models \exists x.opt(x)$ holds (see page 66) (1)
  - but $\mathcal{I}_{sample} \models \forall x.opt(x)$ does not hold, i.e.: $\mathcal{I}_{sample} \not\models \forall x.opt(x)$
- We can naturally extend this notation to a set of sentences S: $\mathcal{I} \models S$ ($\mathcal{I}$ is a model of S) iff for each $\alpha \in S$, $\mathcal{I} \models \alpha$
  - e.g., $\mathcal{I}_{sample} \models \{\exists x.opt(x), \exists x.\neg opt(x)\}$ (2)
- We now overload even further the symbol "$\models$": $S \models \alpha$ (sentence α is a semantic consequence of the set of sentences S) iff
  - all models of S are also models of α, i.e.
  - for all models $\mathcal{I}$, if $\mathcal{I} \models S$ then $\mathcal{I} \models \alpha$ also holds
  - e.g. $\{\forall x.opt(x)\} \models \exists x.opt(x)$ (3)
- Note that the first two versions speak about sentences being true in a model: $\mathcal{I} \models \alpha$, $\mathcal{I} \models \{\alpha_1, \ldots\}$, see (1) and (2)
- The third version relates sentences only, a set of premises and a conclusion: $\{\alpha_1, \ldots\} \models \alpha$, see (3)

Syntactic consequence

- The notion of syntactic consequence relies on a choice of a proof system, characterised by a set of inference rules.
- An example of a proof system is FOL resolution which includes two inference rules: resolution and factoring.
- $S \vdash \alpha$: sentence α is a syntactic consequence of a set of sentences S (wrt. a given proof system)
  iff there exists a proof of α from S in the given proof system.
- A proof of α from S is a list of sentences $\alpha_1, \ldots, \alpha_n$, such that $\alpha = \alpha_n$, and for all i
  - either $\alpha_i \in S$;
  - or α_i can be deduced by an inference rule of the given proof system from a subset of $\{\alpha_1, \ldots, \alpha_{i-1}\}$

Properties of proof systems

- Important properties of a proof system:
  - Soundness: if $U \vdash V$ then $U \models V$ (what we prove is true)
  - Completeness: if $U \models V$ then $U \vdash V$ (what is true can be proven)
- As an example, resolution can be shown to be a sound and complete proof system.
- Gödel was the first to prove the completeness of a proof system, i.e. that the two kinds of consequence – semantic and syntactic – are the same:

$$\models \equiv \vdash$$

see the logo of the Association for Logic Programming (ALP): `https://www.cs.nmsu.edu/ALP`

Association for Logic Programming

Problems and limitations of first order logic

- FOL is not powerful enough
  - The set of non-negative integers has the property that every integer can be reached by incrementing 0 by 1 a finite number of times.
  - This property cannot be formulated in FOL, and FOL axiomatisations of arithmetic (e.g. by Peano) have non-standard models: in these models there are integers that cannot be reached from 0 by a finite number of increases by 1.
- Gödel's incompleteness theorem: there is a formula that is true, but cannot be proven (equivalent to stating "I am not provable")
- FOL is too powerful, as it is not decidable
  - FOL is semi-decidable: there is an algorithm that will determine if $S \models \alpha$ holds, but no algorithm can be developed for checking if $S \not\models \alpha$ holds
  - In the past $\sim$ 30 years numerous subsets of FOL have been shown to be (fully) decidable: for these sublanguages there are algorithms that return a yes/no answer to the question: $S \models \alpha$?
  - We will learn about some decidable sublanguages of FOL in the final part (Part IV) of the course