FCS 2019 Midterm 1.

November 7^{th}, 2019

Each problem is worth 10 points. One needs 18 points to pass the test, 48 points on the two midterms together to obtain signature. Results without proper reasoning receive no credit. You can only use pen or pencil and paper. Usage of calculators, mobile phones or other electronic devices, furthermore notes is *forbidden.* It is also strictly forbidden to seek help from others or to give help to others. Violation of the rules results in disqualification from the midterm.

Write your name and Neptun code **IN PRINT** on each page you turn in!

Problems

1. How many permutations does the set $1, 2, \ldots, 10$ of numbers have such that numbers 1, 2 and 3 are in this (increasing) order but not necessarily on consecutive positions?
   The objects to be counted are generated in several steps so that each permutation to be counted is generated exactly once. (1 point)
   First the positions of numbers $1, 2, 3$ are determined (1 point)
   that can be done in $\binom{10}{3}$ ways. (2 point)
   The the possible orderings of numbers $4, 5, \ldots, 10$ is determined (1 point)
   that can be done in 7! ways. (2 point)
   These two decisions uniquely determine the permutation and each permutation to be counted is generetaed once. (1 point)
   The two decisions are independent of each other so by the product rule the number of possibilities is $\binom{10}{3} \cdot 7! = \frac{10!}{3!} = 10 \cdot 9 \cdot \ldots \cdot 4$. (2 point)
   Another way:
   The total number of permutations is 10!, (2 point)
   but this counts those permutations, as well where numbers $1, 2, 3$ are not in increasing order. (1 point)
   These three numbers have 3! possible order (1 point)
   on the other hand, clearly each of these orderings occur in exactly the same number of permutations. (3 point)
   Thus the number of permutations where the three numbers in the requested order is $\frac{10!}{3!}$. (3 point)

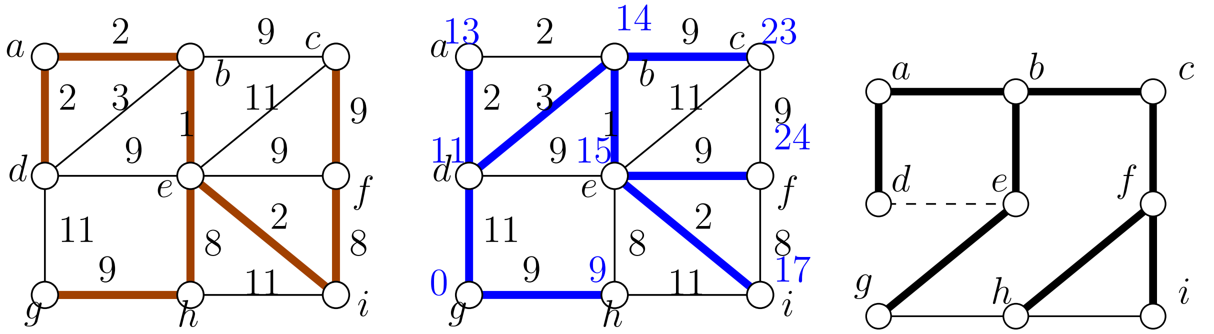
2. Vertex v is deleted from tree T. The degree sequence of the graph obtained is $1, 1, 1, 1, 1, 1, 1, 1, 1, 2, 2, 2, 3, 3$. Determine the degree of vertex v in tree T.
   There are 13 vertices left after the deletion, so the original graph had 14 vertices (2 points)
   and by the known property of trees 13 edges. (2 points)
   The sum of degrees after deletion is 20, so the graph obtained by the deletion has 10 edges. (3 points)
   Since exactly those edges are missing from tree T that are incident with v, (1 points)
   the degree of v in T is exactly $13 - 10 = 3$. (2 points)
   Another way:
   The graph obtained after the deletion has 13 vertices and 10 edges by that the sum of degrees is twice the number of edges (5 points)
   We learnt in class that an n-vertex forest with k components has $n - k$ edges, so $T \setminus \{v\}$ has $13 - 10 = 3$ components. (2 points)
   Vertex v is joined to each component by eyactly one edge, (1 points)
   hence $d(v) = 3$. (1 points)

3. The numbers besides the edges of graph G shown on the figure on the left represent the length of the edges. Give a subset of the edges of G such that from any vertex of G we can get to any other vertex of G using the chosen edges only, furthermore the total length of the chosen edges is minimal.
   Since each edge length is nonnegative, the solution will be such a spanning tree that the sum of the numbers on the edges is minimal. (3 points)

Such a spanning tree can be found by Kruskal's Algorithm, by considering the edges in increasing lenght order. (2 points)
The bold edges on the figure show the edges to be selected. (5 points)

4. The numbers besides the edges of graph G shown on the figure on the left represent the length of the edges. Is it true that vertex i is farther away by at least 7 from vertex g than vertex d is, that is $dist(g,i) \geq dist(g,d)+7$?
Observe that $dbei$ is a length 6 di-path in graph G. (3 points)
Thus, if a shortest gd-path is extended with this path such a gi-walk is obtained whose length is $dist(g,d)+6$. (3 points)
The shortest gi-path cannot be longer than this, so $dist(g,i) \leq dist(g,d)+6$. (3 points)
Thus, the answer to the question is NO. (1 points)
Another way:
Using Dijkstra's algorithm the distances of vertices of graph G for vertex g can be determined. (2 points)
Correctly running Dijkstra we get these distances (see middle picture) (7 points)
We conclude that the answer is NO. (1 points)

5. The tree on the picture on the right is a DFS-tree of graph G. It is known that gh and hi are edges of G. Is it possible that vertices d and e are connected by an edge in G?



In case of DFS there exists no cross edge. (3 points)
However, from whichever vertex the DFS is started, one of these edges would be cross edge. Starting from a, b, c, f, h respectively, i, edge de is cross edge, starting from d, e respectively, g, hi is cross. (6 points)
Thus the answer is NO, d and e cannot be neighbours in G. (1 points)
Another way:
In case of DFS there exists no cross edge. (3 points)
Since hi is not cross edge, the root of the DFS tree can only be either h or i. Since gh is not cross edge, the root of the DFS tree can only be either g or h. (3 points)
Thus the DFS was started from vertex h. (1 points)
However, in this case de is a cross edge, (2 points)
Thus the answer is NO, d and e cannot be neighbours in G. (1 points)

6. Let G be a graph that has an Eulerian walk. Furthermore, let T be such a spanning tree of G that the graph $G-T$ obtained by deleting the edges of T from G has an Eulerian circuit. Prove that G has a Hamiltonian path.
Graph $G-T$ has an Eulerian circuit, so all of its vertices have even degree. (2 points)
Since G has an Eulerian walk, so it has at most 2 vertices of odd degree. (1 points)
T has at least 2 leaves (2 points)
and every leaf v of T has odd degree in in G (even in $G-T$ plus 1 in T) (2 points)
So T must have exactly 2 leaves. (1 points)
Thus T is a spanning tree of G with exactly 2 leaves, so every non-leaf vertex in T has degree 2 in T, so T is a Hamiltonian path of G. (2 points)