The number of monounary algebras

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Abstract. In this note we give an asymptotic estimate for the number of monounary algebras of given size.

1. Introduction

A monounary algebra is an algebra with a single unary operation. The theory of monounary algebras is well-developed, for a recent monograph see [1]. Let $A = (A, f)$ be a monounary algebra. The function $f$ defines a directed graph on $A$. Let $G_A = (A, E)$, the vertex set is $A$ and the edges are $E = \{ (a, f(a)) \mid a \in A \}$. In $G_A$ every vertex has outdegree 1, and every graph $G$ with outdegree 1 defines a monounary algebra on its vertex set, where $f(a)$ is the single vertex such that $(a, f(a))$ is an edge in $G$. Hence, there is a bijection between monounary algebras and directed graphs, where each vertex has outdegree 1. At first, we investigate the number of connected monounary algebras.

Theorem 1.1. Let $C_n$ denote the number of connected monounary algebras of size $n$. Then there is an $\alpha > 1$ such that $\log_\alpha C_n \sim n$.

Proof. A directed graph corresponding to a connected monounary algebra is a directed cycle with a (possibly one-element) rooted tree at each vertex. If there is a loop in the graph we say that the length of the cycle is 1. Each edge of each rooted tree is directed towards the cycle. Hence, a connected graph of a monounary algebra of size $n$ is built up from a cycle of length $k$, where $1 \leq k \leq n$ and to each vertex of the cycle we glue a rooted tree such that the sum of the sizes of the rooted trees is $n$. Every monounary algebra of size $n$ can be obtained this way. Note that these graphs are not necessarily non-isomorphic. Naturally, each rooted tree is a monounary algebra. Let $T_n$ denote the number of rooted trees on $n$ vertices. By the above arguments we have

$$T_n \leq C_n \leq \sum_{k=1}^{n} \sum_{i_1 + \cdots + i_k = n} T_{i_1} T_{i_2} \cdots T_{i_k}$$

(1)

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Let $T_0 = C_0 = 1$. Let $\psi(x) = \sum_{n=0}^{\infty} T_n x^{n-1}$ and $\Gamma(x) = \sum_{n=0}^{\infty} C_n x^n$. Now $\Gamma$ is the generating function of the number of connected monounary algebras, and $x\psi(x) + 1$ is the generating function of the number of rooted trees. By formula (1) the following holds:

$$x\psi(x) + 1 \leq \Gamma(x) \leq 1 + \sum_{n=1}^{\infty} \left( \sum_{k=1}^{n} \sum_{i_1 + \cdots + i_k = n} T_{i_1} T_{i_2} \cdots T_{i_k} \right) x^n = \sum_{n=0}^{\infty} (x\psi(x))^n$$  \hspace{1cm} (2)

In [2] the function $\psi(x)$ is well analyzed. It is proved that there is a constant $c_T$ such that $T_n \sim c_T \alpha^n n^{-\frac{3}{2}}$, where $\alpha \approx 2.955765$. Moreover the power series $\psi(x)$ has radius of convergence $\frac{1}{\alpha}$ for the same $\alpha$, and $\psi(\frac{1}{\alpha}) = \alpha$, hence $\frac{1}{\alpha} \psi(\frac{1}{\alpha}) = 1$ holds. As $\sum_{n=0}^{\infty} (x\psi(x))^n = \frac{1}{1 - x\psi(x)}$ is strictly monotonically increasing in $\mathbb{R}^+$, the radius of convergence of this power series is the unique positive solution of the equation $x\psi(x) = 1$, which is $\frac{1}{\alpha}$. Therefore for each power series in $\Gamma(x)$ the radius of convergence is $\frac{1}{\alpha}$. This holds for $\Gamma(x)$, as well. Thus for the coefficients of $\Gamma(x)$ we have $\limsup_{n \to \infty} \sqrt[n]{C_n} = \alpha$. From $T_n \leq C_n$ we obtain $\liminf_{n \to \infty} \sqrt[n]{C_n} \geq \alpha$, thus $\lim_{n \to \infty} \sqrt[n]{C_n} = \alpha$, and $\log_\alpha C_n \sim n$ is gained.

**Theorem 1.2.** Let $M_n$ denote the number of monounary algebras of size $n$. Then $\log_\alpha M_n \sim n$.

**Proof.** A monounary algebra is the disjoint union of connected monounary algebras. Let $\mathcal{A}$ be a monounary algebra of size $n$ containing $\mu_i$ connected algebras of size $i$. Then $\sum_{i=1}^{n} i\mu_i = n$. Note that $\mu_i$ can be 0. The number of ways of picking $k$ indistinguishable objects of $p$ type is the coefficient of $x^k$ in the generating function $\frac{1}{(1-x)^p}$. Hence, the number of ways picking $\mu_i$ indistinguishable connected monounary algebras of $C_i$ type is the coefficient of $x^{i\mu_i}$ in $\frac{1}{(1-x)^{C_i}}$. Thus the generating function for the number of monounary algebras is

$$1 + \sum_{n=1}^{\infty} M_n x^n = \prod_{k=1}^{\infty} \frac{1}{(1-x)^{C_k}}.$$  \hspace{1cm} (3)

For $x > 0$ this series converges if and only if $\log \left( \prod_{k=1}^{\infty} \frac{1}{(1-x)^{C_k}} \right) = \sum_{k=1}^{\infty} -C_k \log(1-x^k)$ is convergent. If $x \in (0, \frac{1}{\alpha})$ then $x^k \in (0, \frac{1}{\alpha})$ for all $k \geq 1$. As $\log x$ is concave we have $\log(1-t) \geq -\log x$ for all $t \in (0, \frac{1}{\alpha})$ with $e = -\log(1 - \frac{1}{\alpha})$. Therefore $\sum_{k=1}^{\infty} -C_k \log(1-x^k) \leq \sum_{k=1}^{\infty} C_k c x^k = c(\Gamma(x) - 1)$. Hence this power series is convergent for $x \in (0, \frac{1}{\alpha})$. This yields $\lim_{n \to \infty} \sqrt[n]{M_n} \leq$
\( \alpha. \) The lower bound \( \liminf_{n \to \infty} \sqrt[n]{M_n} \geq \alpha \) can be derived from \( C_n \leq M_n. \) Thus \( \lim_{n \to \infty} \sqrt[n]{M_n} = \alpha \) and \( \log_{\alpha} M_n \sim n. \)

\[ \Box \]

References


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