REDUCTS OF THE RANDOM PARTIAL ORDER

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Abstract. We determine, up to the equivalence of first-order interdefinability, all structures which are first-order definable in the random partial order. It turns out that these structures fall into precisely five equivalence classes. We achieve this result by showing that there exist exactly five closed permutation groups which contain the automorphism group of the random partial order. Our classification lines up with previous similar classifications, such as the structures definable in the random graph or the order of the rationals; it also provides further evidence for a conjecture due to Simon Thomas which states that the number of structures definable in a homogeneous structure in a finite relational language is, up to first-order interdefinability, always finite. The method we employ is based on a Ramsey-theoretic analysis of functions acting on the random partial order, which allows us to find patterns in such functions and make them accessible to finite combinatorial arguments.

1. Reducts of homogeneous structures

The *random partial order* $\mathbb{P} := (P; \leq)$ is the unique countable partial order which is *universal* in the sense that it contains all countable partial orders as induced suborders and which is *homogeneous*, i.e., any isomorphism between two finite induced suborders of $\mathbb{P}$ extends to an automorphism of $\mathbb{P}$. Equivalently, $\mathbb{P}$ is the *Fraïssé limit* of the class of finite partial orders – confer the textbook [Hod97].

As the "generic order" representing all countable partial orders, the random partial order is of both theoretical and practical interest. The latter becomes in particular evident with the recent applications of homogeneous structures in theoretical computer science; see for example [BP11a, BP11b, BK09, Mac11]. It is therefore tempting to classify all structures which are first-order definable in $\mathbb{P}$, i.e., all relational structures on domain P all of whose relations can be defined from the relation $\leq$ by a first-order formula. Such structures have been called *reducts* of $\mathbb{P}$ in the literature [Tho91, Tho96]. It is the goal of the present paper to obtain such a classification *up to first-order interdefinability.* That is, we consider two reducts Γ, Γ' equivalent iff they are reducts of one another. We will show that up to this equivalence, there are precisely five reducts of $\mathbb{P}$.

Our result lines up with a number of previous classifications of reducts of similar generic structures up to first-order interdefinability. The first non-trivial classification of this kind was obtained by Cameron [Cam76] for the order of the rationals, i.e., the Fraïssé limit of the class of finite linear orders; he showed that this order has five reducts up to first-order interdefinability. Thomas [Tho91] proved that the random graph has five reducts up to first-order interdefinability as well, and later generalized this result by showing that for all $k \geq 2$,

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the random hypergraph with k-hyperedges has $2^k + 1$ reducts up to first-order interdefinability [Tho96]. Junker and Ziegler [JZ08] showed that the structure $(\mathbb{Q}; <, 0)$, i.e., the order of the rationals with an additional constant symbol, has 116 reducts up to interdefinability. Further examples include the random K_n-free graph for all $n \geq 3$ (2 reducts, see [Tho91]), the random tournament (5 reducts, see [Ben97]), and the random K_n-free graph with a fixed constant (13 reducts if $n = 3$ and 16 reducts if $n \geq 4$, see [Pon11]). A negative "result" is the random graph with a fixed constant, on which a subset of the authors of the present paper, together with another collaborator, gave up after having found 300 reducts. Obviously, the successful classifications have in common that the number of reducts is finite, and it is indeed an open conjecture of Thomas [Tho91] that all homogeneous structures in a finite relational language have only finitely many reducts up to first-order interdefinability.

The mentioned classifications have all been obtained by means of the automorphism groups of the reducts, and we will proceed likewise in the present paper. It is clear that if Γ is a reduct of a structure Δ, then the automorphism group $\mathrm{Aut}(\Gamma)$ of Γ is a permutation group containing $\mathrm{Aut}(\Delta)$, and also is a closed set with respect to the convergence topology on the space of all permutations on the domain of Δ. If Δ is ω-categorical, i.e., if Δ is up to isomorphism the only countable model of its first-order theory, then it follows from the theorem of Ryll-Nardzewski, Engeler and Svenonius (confer [Hod97]) that the converse is true as well: the closed permutation groups acting on the domain of Δ and containing $\mathrm{Aut}(\Delta)$ are precisely the automorphism groups of reducts of Δ; moreover, two reducts have equal automorphism groups if and only if they are first-order interdefinable. Since homogeneous structures in a finite language are ω-categorical, it is enough for us to determine all closed permutation groups that contain $\mathrm{Aut}(\mathbb{P})$ in order to obtain our classification.

Our approach to investigating the closed groups containing $\mathrm{Aut}(\mathbb{P})$ is based on a Ramsey-theoretic analysis of functions, and in particular permutations, on the domain P of $\mathbb{P} = (P; \leq)$; this allows us to find patterns of regular behavior with respect to the structure $\mathbb{P}$ in any arbitrary function acting on P. The method as we use it has been developed in [BPT11, BP11b, BP10, BP11a] and is a general powerful technique for dealing with functions on *ordered homogeneous Ramsey structures* in a finite language. But while this machinery has previously been used, for example, to re-derive and extend Thomas' classification of the reducts of the random graph, it is only in the present paper (and, at the same time, in [Pon11] for the reducts of K_n-free graphs with a constant) that it is applied to prove a new classification of reducts of a homogeneous structure up to first-order interdefinability.

Before stating our result, we remark that finer classifications of reducts of homogeneous structures, for example up to existential, existential positive, or primitive positive interdefinability, have also been considered in the literature, in particular in applications – see [BCP10, BPT11, BP10, BP11a].

2. The result

2.1. **The group formulation.** In a first formulation of our result, we will list the closed groups containing $\mathrm{Aut}(\mathbb{P})$ by means of sets of permutations generating them: we say that a set $\mathcal{S}$ of permutations on P *generates* a permutation α on P iff α is an element of the smallest closed permutation group $\langle \mathcal{S} \rangle$ that contains $\mathcal{S}$. Equivalently, writing id for the identity function on P, for every finite set $F \subseteq P$ there exist $n \geq 0$, $\beta_1, \ldots, \beta_n \in \mathcal{S}$, and $i_1, \ldots, i_n \in \{1, -1\}$ such that $\beta_1^{i_1} \circ \cdots \circ \beta_n^{i_n} \circ \mathrm{id}$ agrees with α on F. We also say that a permutation β generates α iff $\{\beta\}$ generates α.

If for $x, y \in P$ we define $x \geq y$ iff $y \leq x$, then the structure $(P; \geq)$ is isomorphic to $\mathbb{P}$ – it is, for example, easy to verify that it contains all finite partial orders and that it is homogeneous. Hence, there exists an isomorphism between the two structures, and we fix one such isomorphism $\updownarrow : P \to P$; so the function $\updownarrow$ simply reverses the order $\leq$ on P. It is easy to see that any two isomorphisms of this kind generate one another, and the exact choice of the permutation is thus irrelevant for our purposes.

Upwards-closed subsets of $\mathbb{P}$ are called *filters*, and downwards-closed subsets *ideals*. Note that the complement $P \setminus F$ of a filter $F \subseteq P$ is an ideal and vice-versa. A filter is called *irrational* iff both the filter and its corresponding ideal are non-empty and contain no maximal or minimal elements. Given an irrational filter $F \subseteq P$, we can define a partial order $\trianglelefteq$ on P by setting

$$x \trianglelefteq y \ \leftrightarrow (x, y \in F \wedge x \leq y) \vee (x, y \in P \setminus F \wedge x \leq y) \vee (x \in F \wedge y \in P \setminus F \wedge y \nleq x),$$

where $a \nleq b$ is short for $\neg(a \leq b)$. Then $(P; \trianglelefteq)$ and $\mathbb{P}$ are isomorphic. The easiest way to see this is by checking that $(P; \trianglelefteq)$ satisfies the following *extension property*, which determines $\mathbb{P}$ up to isomorphism and which we will use throughout the paper: for any finite set $S = \{s_1, \ldots, s_k\} \subseteq P$ and any conjunction $\phi(x, y_1, \ldots, y_k)$ of statements of the form $a \leq b$ or $a \nleq b$ which includes the conjunct $y_i \leq y_j$ ($y_i \nleq y_j$) iff $s_i \leq s_j$ ($s_i \nleq s_j$) holds in $\mathbb{P}$, if ϕ is satisfiable in any partial order, then ϕ is satisfiable in $\mathbb{P}$ by an assignment which sends y_i to s_i for all $1 \leq i \leq k$. In words, this extension property says that if we fix any finite set of elements $s_1, \ldots, s_k \in P$, and express properties of another imaginary element x in the language of partial orders using the constants $s_1, \ldots, s_k$, then an element enjoying these properties actually exists in $\mathbb{P}$ unless the properties are inconsistent with the theory of partial orders (for example, if we want that $x \leq s_i$ and $s_j \leq x$ for some distinct s_i, s_j with $s_i \leq s_j$).

Verifying the extension property for $(P; \trianglelefteq)$ (which is easy), one sees that $(P; \trianglelefteq)$ and $\mathbb{P}$ are indeed isomorphic; pick a permutation $\circlearrowright_F$ on P witnessing this. It is not hard to see that any two permutations obtained this way generate one another, even if they were defined by different irrational filters. We therefore also write $\circlearrowright$ for any $\circlearrowright_F$ when the filter F is not of particular interest.

Theorem 1. *The following five groups are precisely the closed permutation groups on P which contain* $\mathrm{Aut}(\mathbb{P})$*.*

(1) $\mathrm{Aut}(\mathbb{P})$*;*
(2) $\mathrm{Rev} := \langle \{\updownarrow\} \cup \mathrm{Aut}(\mathbb{P}) \rangle$*;*
(3) $\mathrm{Turn} := \langle \{\circlearrowright\} \cup \mathrm{Aut}(\mathbb{P}) \rangle$*;*
(4) $\mathrm{Max} := \langle \{\updownarrow, \circlearrowright\} \cup \mathrm{Aut}(\mathbb{P}) \rangle$*;*
(5) *The full symmetric group* Sym_P *of all permutations on P.*

2.2. **The reduct formulation.** We now turn to the relational formulation of our result; that is, we will specify five reducts of $\mathbb{P}$ such that any reduct of $\mathbb{P}$ is first-order interdefinable with one of the reducts of our list.

Define a binary relation $\perp$ on P by $\perp := \{(x, y) \in P^2 \ : \ x \nleq y \wedge y \nleq x\}$. We call the relation the *incomparability relation*, and refer to elements $x, y \in P$ as *incomparable* iff (x, y) is an element of $\perp$; in that case, we also write $x \perp y$. Elements $x, y \in P$ are *comparable* iff they are not incomparable.

For $x, y \in P$, write $x < y$ iff $x \leq y$ and $x \neq y$. Now define a ternary relation Cycl on P by

$$\begin{aligned}\mathrm{Cycl} := \{(x,y,z) \in P^3 \mid\ & (x < y < z) \vee (y < z < x) \vee (z < x < y) \vee \\ & (x < y \wedge x \perp z \wedge y \perp z) \vee \\ & (y < z \wedge y \perp x \wedge z \perp x) \vee \\ & (z < x \wedge z \perp y \wedge x \perp y).\end{aligned}$$

Finally, define a ternary relation Par on P by

$$\begin{aligned}\mathrm{Par} := \{(x,y,z) \in P^3 \mid\ & x, y, z \text{ distinct} \wedge \\ & \text{the number of incomparabilities on } (x,y,z) \text{ is even}\}.\end{aligned}$$

Theorem 2. *Let Γ be a reduct of $\mathbb{P}$. Then Γ is first-order interdefinable with precisely one of the following structures.*

(1) $\mathbb{P} = (P; \leq)$;
(2) $(P; \perp)$;
(3) $(P; \mathrm{Cycl})$;
(4) $(P; \mathrm{Par})$;
(5) $(P; =)$.

Moreover, for $1 \leq x \leq 5$, Γ is first-order interdefinable with structure (x) if and only if $\mathrm{Aut}(\Gamma)$ *equals group number (x) in Theorem 1.*

3. Ramsey theory

Our combinatorial method for proving Theorem 1 is to apply Ramsey theory in order to find patterns of regular behavior in arbitrary functions on $\mathbb{P}$, and follows [BPT11, BP11b, BP10, BP11a]. We make this more precise.

Definition 3. Let Δ be a structure. The *type* $\mathrm{tp}(a)$ of an n-tuple a of elements in Δ is the set of first-order formulas with free variables $x_1, \ldots, x_n$ that hold for a in Δ.

Definition 4. Let Δ, Λ be structures. A *type condition* between Δ and Λ is a pair (t, s), where t is a type of an n-tuple in Δ, and s is a type of an n-tuple in Λ, for some $n \geq 1$.

A function $f : \Delta \to \Lambda$ *satisfies* a type condition (t, s) between Δ and Λ iff for all n-tuples $a = (a_1, \ldots, a_n)$ of elements of Δ with $\mathrm{tp}(a) = t$ the n-tuple $f(a) := (f(a_1), \ldots, f(a_n))$ has type s in Λ. A *behavior* is a set of type conditions between structures Δ and Λ. A function from Δ to Λ *has behavior* B iff it satisfies all the type conditions of B.

Definition 5. Let Δ, Λ be structures. A function $f : \Delta \to \Lambda$ is *canonical* iff for all types t of n-tuples in Δ there exists a type s of an n-tuple in Λ such that f satisfies the type condition (t, s). In other words, n-tuples of equal type in Δ are sent to n-tuples of equal type in Λ under f, for all $n \geq 1$.

We remark that since $\mathbb{P}$ is homogeneous, every first-order formula is over $\mathbb{P}$ equivalent to a quantifier-free formula, and so the type of an n-tuple a in $\mathbb{P}$ is determined by which of its elements are equal and between which elements the relation $\leq$ holds. In particular, the type a only depends on its binary subtypes, i.e., the types of the pairs (a_i, a_j), where $1 \leq i, j \leq n$. Therefore, a function $f : \mathbb{P} \to \mathbb{P}$ is canonical iff it satisfies the condition of the definition for types of 2-tuples.

Roughly, our strategy is to make the functions we work with canonical, and thus easier to handle. To be able to do this, we must first enrich the structure $\mathbb{P}$ by a linear order in order to improve its combinatorial properties, as follows. We do not give the – in some cases fairly technical – definitions of all notions in this discourse, as they will not be needed later on; in any case, Proposition 6 that follows is used as a black box for this paper, and the reader interested in its proof is referred to [BPT11]. The class $\mathcal{C}$ of all finite structures $\mathcal{A} = (A; \leq', \prec')$ with two binary relations $\leq'$ and $\prec'$, where $\leq'$ is a partial order and $\prec'$ is a total order extending $\leq'$, is an *amalgamation class* [Hod97], and moreover a *Ramsey class* [Sok10, Theorem 1 (1)]. By the first property, it has a Fraïssé limit, i.e., there exists a unique countable structure which is homogeneous and whose class of finite induced substructures up to isomorphism equals $\mathcal{C}$. Checking the extension property, one sees that the partial order of this limit is just the random partial order, and by uniqueness of the dense linear order without endpoints its total order is isomorphic to the order of the rationals. Hence, there exists a linear order $\prec$ on P which is isomorphic to the order of the rationals, which extends $\leq$, and such that the structure $\mathbb{P}^+ := (P; \leq, \prec)$ is precisely the Fraïssé limit of the class $\mathcal{C}$. So $\mathbb{P}^+$ is a homogeneous structure in a finite language which has a linear order among its relations and which is *Ramsey*, i.e. its class of finite induced substructures up to isomorphism, which equals the class $\mathcal{C}$, is a Ramsey class. The following proposition is then a consequence of more general statements in [BPT11, BP11a] about such structures. To state it, let us extend the notion "generates" to non-permutations: for a set of functions $\mathcal{F} \subseteq P^P$ and $f \in P^P$, we say that f is *mon-generated* by $\mathcal{F}$ iff it is contained in the smallest transformation monoid on P which contains $\mathcal{F}$ and which is a closed set in the convergence topology on P^P. In other words, f is mon-generated by $\mathcal{F}$ iff for all finite $F \subseteq P$ there exist $n \geq 0$ and $f_1, \ldots, f_n \in \mathcal{F}$ such that $f_1 \circ \cdots \circ f_n \circ \mathrm{id}$ agrees with f on F. For a structure Δ and elements $c_1, \ldots, c_n$ of Δ, we write $(\Delta, c_1, \ldots, c_n)$ for the structure obtained by adding the constant symbols $c_1, \ldots, c_n$ to Δ.

Proposition 6. *Let $f : P \to P$ be a function, and let $c_1, \ldots, c_n, d_1, \ldots, d_m \in P$. Then $\{f\} \cup \mathrm{Aut}(\mathbb{P}^+)$ mon-generates a function which is canonical as a function from $(\mathbb{P}^+, c_1, \ldots, c_n)$ to $(\mathbb{P}^+, d_1, \ldots, d_m)$, and which is identical with f on $\{c_1, \ldots, c_n\}$.*

Having enriched $\mathbb{P}$ with the linear order $\prec$ and having taken advantage of Proposition 6, we pass to a suitable substructure in order to get rid of $\prec$.

Definition 7. Let $\mathcal{G}$ be a permutation group acting on a set D. Then for $n \geq 1$ and $a = (a_1, \ldots, a_n) \in D^n$, the set

$$\{(\alpha(a_1), \ldots, \alpha(a_n)) : \alpha \in \mathcal{G}\} \subseteq D^n$$

is called an n-orbit of $\mathcal{G}$. The 1-orbits are just called *orbits*. If Δ is a structure, then the n-orbits of Δ are defined as the n-orbits of $\mathrm{Aut}(\Delta)$.

By the theorem of Ryll-Nardzewski, Engeler and Svenonius, two n-tuples in an ω-categorical structure belong to the same n-orbit if and only if they have the same type.

Definition 8. Let Δ be a structure on domain D. A subset S of D is called a *skeleton* of Δ iff it induces a substructure of Δ which is isomorphic to Δ. Now let $\sqsubset$ be a linear order on D. Then a skeleton S is called *$\sqsubset$-clean* iff whenever $(a_1, a_2), (b_1, b_2) \in S^2$ lie in the same 2-orbit with respect to Δ, then either $(a_1, a_2), (b_1, b_2)$ or $(a_1, a_2), (b_2, b_1)$ lie in the same 2-orbit with respect to $(\Delta, \sqsubset)$.

Lemma 9. *Let $c_1, \ldots, c_n \in P$. Then $(\mathbb{P}, c_1, \ldots, c_n)$ has a skeleton which is $\prec$-clean.*

Proof. The skeleton can be selected inductively using the extension property of $\mathbb{P}^+$. $\square$

Lemma 10. *Let $f : P \to P$ be a function, and let $c_1, \ldots, c_n, d_1, \ldots, d_m \in P$. Then $\{f\} \cup \mathrm{Aut}(\mathbb{P})$ mon-generates a function $g : P \to P$ with the following properties.*

- *g agrees with f on $\{c_1, \ldots, c_n\}$;*
- *g is canonical as a function from $(\mathbb{P}, c_1, \ldots, c_n)$ to $(\mathbb{P}, d_1, \ldots, d_m)$.*

Proof. Let $S \subseteq P$ be a $\prec$-clean skeleton of $(\mathbb{P}, c_1, \ldots, c_n)$, and let h be the function guaranteed by Proposition 6. We then have that h, considered as a function from $(\mathbb{P}, c_1, \ldots, c_n)$ to $(\mathbb{P}, d_1, \ldots, d_m)$, is canonical on S, that is, it satisfies the definition of canonicity for tuples in S. Let $i : (P; \leq, c_1, \ldots, c_n) \to (S; \leq, c_1, \ldots, c_n)$ be an isomorphism, and set $g := h \circ i$. Then g is canonical as a function from $(\mathbb{P}, c_1, \ldots, c_n)$ to $(\mathbb{P}, d_1, \ldots, d_m)$, and agrees with f on $\{c_1, \ldots, c_n\}$. Since i preserves $\leq$ and its negation, it is mon-generated by $\mathrm{Aut}(\mathbb{P})$. Hence so is g, proving the lemma. □

4. Proof of the group theorem

4.1. **Ordering orbits.**

Notation 11. Let $c_1, \ldots, c_n \in P$. For $R_1, \ldots, R_n \in \{=, <, \perp, >\}$, we set

$$X_{R_1,\ldots,R_n} := \{x \in P : c_1 R_1 x \ \wedge \cdots \wedge \ c_n R_n x\}.$$

The constants $c_1, \ldots, c_n$ are not specified in the notation, but will always be clear from the context.

The following is easy to verify using the extension property and homogeneity of $\mathbb{P}$.

Fact 12. *Let $c_1, \ldots, c_n \in P$. The sets $X_{R_1,\ldots,R_n}$ are either empty, or equal to $\{c_i\}$ for some $1 \leq i \leq n$, or infinite and induce the random poset. The orbits of $(\mathbb{P}, c_1, \ldots, c_n)$ are precisely the non-empty sets $X_{R_1,\ldots,R_n}$.*

The most important situation for us will be when we fix $u, v \in P$ such that $u < v$. Then the structure $(\mathbb{P}, u, v)$ has eight orbits: two are the singletons $\{u\}$ and $\{v\}$, and the other six orbits are the infinite sets $X_{>,>}$, $X_{\perp,>}$, $X_{\perp,\perp}$, $X_{<,>}$, $X_{<,\perp}$ and $X_{<,<}$.

Definition 13. For disjoint subsets X, Y of P we write

- $X \leq Y$ iff there exist $x \in X$, $y \in Y$ such that $x \leq y$;
- $X \perp Y$ iff $x \perp y$ for all $x \in X$, $y \in Y$;
- $X < Y$ iff $x < y$ for all $x \in X$ and all $y \in Y$.

We call X, Y *incomparable* iff $X \perp Y$, and *comparable* otherwise (which is the case iff $X \leq Y$ or $Y \leq X$). We say that X, Y are *strictly comparable* iff $X < Y$ or $Y < X$.

Lemma 14. *Let $c_1, \ldots, c_n \in P$. The relation $\leq$ defines a partial order on the orbits of $(\mathbb{P}, c_1, \ldots, c_n)$.*

Proof. Reflexivity and transitivity follow from the respective properties of $\mathbb{P}$. To see that $X \leq Y$ and $Y \leq X$ imply $X = Y$, observe first that it follows from Fact 12 that X is convex, i.e., if $x, z \in X$ satisfy $x \leq z$ and $y \in P$ is so that $x \leq y$ and $y \leq z$, then $y \in X$. Now there exist $x, x' \in X$ and $y, y' \in Y$ such that $x \leq y$ and $x' \geq y'$. Since y, y' belong to the same orbit, they satisfy the same first-order formulas over $(\mathbb{P}, c_1, \ldots, c_n)$, and hence there exists $z \in X$ such that $z \geq y$. Since X is convex, we have $y \in X$, which is only possible if $X = Y$ since distinct orbits are disjoint. □

Let X, Y be infinite orbits of $(\mathbb{P}, c_1, \ldots, c_n)$. Then precisely one of the following cases holds.

- X and Y are strictly comparable;
- X and Y are incomparable;
- X and Y are comparable, but not strictly comparable.

In the third case, if $X \leq Y$, then there exist $x, x' \in X$ and $y, y' \in Y$ such that $x < y$ and $x' \perp y'$, and there are no $x'' \in X$ and $y'' \in Y$ such that $x'' > y''$.

Definition 15. If for two disjoint subsets X, Y of P we have $X \leq Y$, $Y \not\leq X$, and $X \not< Y$, or vice-versa, then we write $X \div Y$.

4.2. **Behaviors generating** Sym_P**.**

Lemma 16. *Let $\mathcal{G} \supseteq \mathrm{Aut}(\mathbb{P})$ be a closed group such that for all finite $A \subseteq P$ there is a function mon-generated by $\mathcal{G}$ which sends A to a chain or an antichain. Then $\mathcal{G} = \mathrm{Sym}_P$.*

Proof. Suppose first that for all finite $A \subseteq P$ there is a function mon-generated by $\mathcal{G}$ which sends A to an antichain. Let s, t be n-tuples of elements in P, for some $n \geq 1$. Let $g : P \to P$ and $h : P \to P$ be functions mon-generated by $\mathcal{G}$ such that $g(s)$ (the n-tuple obtained by applying g to every component of s) and $h(t)$ induce antichains in $\mathbb{P}$. By the homogeneity of $\mathbb{P}$, there exists an automorphism $\alpha \in \mathrm{Aut}(\mathbb{P})$ such that $\alpha(g(s)) = h(t)$. Also, since $\mathcal{G}$ contains the inverse of all of its functions, there exists a function $p : P \to P$ mon-generated by $\mathcal{G}$ such that $p(h(t)) = t$, and hence $p(\alpha(g(s))) = t$. Since $p \circ \alpha \circ g$ is mon-generated by $\mathcal{G}$, there exists $\beta \in \mathcal{G}$ which agrees with this function on s. Hence, $\beta(s) = t$, proving that $\mathcal{G}$ is n-transitive for all $n \geq 1$, and so $\mathcal{G} = \mathrm{Sym}_P$.

Now suppose that for all finite $A \subseteq P$ there is a function mon-generated by $\mathcal{G}$ which sends A to a chain. Let any finite $A \subseteq P$ be given, and let $B \subseteq P$ be so that $|B| = |A|$ and such that B induces an independent set in $\mathbb{P}$. Let $g : P \to P$ and $h : P \to P$ be functions mon-generated by $\mathcal{G}$ such that $g[A]$ and $h[B]$ induce chains in $\mathbb{P}$. There exists $\alpha \in \mathrm{Aut}(\mathbb{P})$ such that $\alpha[g[A]] = h[B]$. Let $p : P \to P$ be a function generated by $\mathcal{G}$ such that $p[h[B]] = B$. Then $p[\alpha[g[A]]] = B$, and hence we are back in the preceding case.

Finally, observe that one of the two cases must occur: for otherwise, there exist finite $A_1, A_2 \subseteq P$ such that A_1 cannot be set to an antichain, and A_2 cannot be sent to a chain by any function which is mon-generated by $\mathcal{G}$. But then $A_1 \cup A_2$ can neither be sent to a chain nor to an antichain by any such function, a contradiction. $\square$

Definition 17. Let $X, Y \subseteq P$ be disjoint, and let $f : P \to P$ be a function. We say that f

- *behaves like* id *on* X iff $x < x'$ implies $f(x) < f(x')$ and $x \perp x'$ implies $f(x) \perp f(x')$ for all $x, x' \in X$;
- *behaves like* $\updownarrow$ *on* X iff $x < x'$ implies $f(x) > f(x')$ and $x \perp x'$ implies $f(x) \perp f(x')$ for all $x, x' \in X$;
- *behaves like* id *between* X *and* Y iff $x < y$ implies $f(x) < f(y)$, $x > y$ implies $f(x) > f(y)$, and $x \perp y$ implies $f(x) \perp f(y)$ for all $x \in X, y \in Y$.

Lemma 18. *Let $\mathcal{G} \supseteq \mathrm{Aut}(\mathbb{P})$ be a closed group, and let $c_1, \ldots, c_n \in P$. Let $g : (\mathbb{P}, c_1, \ldots, c_n) \to \mathbb{P}$ be a canonical function mon-generated by $\mathcal{G}$. Then g behaves like* id *or like* $\updownarrow$ *on each infinite orbit X of $(\mathbb{P}, c_1, \ldots, c_n)$, or else $\mathcal{G} = \mathrm{Sym}_P$.*

Proof. Let X be an infinite orbit, and let $x, x' \in X$ such that $x \perp x'$. Then the type of (x, x') in $(\mathbb{P}, c_1, \ldots, c_n)$ equals the type of (x', x) in $(\mathbb{P}, c_1, \ldots, c_n)$. Hence, the type of $(g(x), g(x'))$ must equal the type of $(g(x'), g(x))$ in $\mathbb{P}$, which is only possible if $g(x) \perp g(x')$, and hence g preserves $\perp$ on X.

Now if $g(a) < g(a')$ for some $a, a' \in X$ with $a < a'$, then the same holds for all $a, a' \in X$ with $a < a'$, and g behaves like id on X. If $g(a') < g(a)$ for some $a, a' \in X$ with $a < a'$, then g behaves like $\updownarrow$ on X. Finally, if $g(a) \perp g(a')$ for some $a, a' \in X$ with $a < a'$, then g sends X to an antichain. Since X contains all finite partial orders, and by the homogeneity of $\mathbb{P}$, we can then refer to Lemma 16 to conclude that $\mathcal{G} = \mathrm{Sym}_P$. □

Lemma 19. *Let $\mathcal{G} \supseteq \mathrm{Aut}(\mathbb{P})$ be a closed group, and let $c_1, \ldots, c_n \in P$. Let $g : (\mathbb{P}, c_1, \ldots, c_n) \to \mathbb{P}$ be a canonical function mon-generated by $\mathcal{G}$. Then $g[X] \div g[Y]$ for all infinite orbits X, Y of $(\mathbb{P}, c_1, \ldots, c_n)$ with $X \div Y$, or else $\mathcal{G} = \mathrm{Sym}_P$.*

Proof. Suppose there are infinite orbits X, Y with $X \div Y$ but for which $g[X] \div g[Y]$ does not hold. Assume without loss of generality that $X \leq Y$. By Lemma 18, we may assume that g behaves like id or like $\updownarrow$ on X and on Y.

First consider the case where $g[X] < g[Y]$ or $g[Y] < g[X]$. Let $A \subseteq P$ be finite; we claim that $\mathcal{G}$ mon-generates a function which sends A to a chain. There is nothing to show if A is itself a chain, so assume that there exist x, y in A with $x \perp y$. Then using the extension property, one readily checks that there exists $\alpha \in \mathrm{Aut}(\mathbb{P})$ which sends the principal ideal of x in A into X and all other elements of A, and in particular y, into Y. Set $h := g \circ \alpha$. Then $h(x)$ and $h(y)$ are comparable, and h does not add any incomparabilities between elements of A. Hence, repeating this procedure and composing the functions, we obtain a function which sends A to a chain. Lemma 16 then implies $\mathcal{G} = \mathrm{Sym}_P$.

The other case is where $g[X] \perp g[Y]$. Then an isomorphic argument shows that we can map any finite subset A of P to an antichain via a function which is mon-generated by $\mathcal{G}$. Again, Lemma 16 yields $\mathcal{G} = \mathrm{Sym}_P$. □

Lemma 20. *Let $\mathcal{G} \supseteq \mathrm{Aut}(\mathbb{P})$ be a closed group, and let $c_1, \ldots, c_n \in P$. Let $g : (\mathbb{P}, c_1, \ldots, c_n) \to \mathbb{P}$ be a canonical function mon-generated by $\mathcal{G}$. Then g behaves like* id *on all infinite orbits of $(\mathbb{P}, c_1, \ldots, c_n)$, or it behaves like $\updownarrow$ on all infinite orbits of $(\mathbb{P}, c_1, \ldots, c_n)$, or else $\mathcal{G} = \mathrm{Sym}_P$.*

Proof. By Lemma 18, we may assume that g behaves like id or $\updownarrow$ on all infinite orbits. Suppose that the behavior of g is not the same on all infinite orbits. Consider the graph H on the infinite orbits of $(\mathbb{P}, c_1, \ldots, c_n)$ in which two orbits X, Y are adjacent if and only if $X \div Y$ holds. We claim that H connected. To see this, let X, Y be infinite orbits with $X < Y$. Pick $x, x' \in X$ and $y, y' \in Y$ such that $x < x'$ and $y' < y$. By the extension property, there exists $z \in P$ such that $x < z$, $z \perp x'$, $z \perp y'$, and $z < y$. Let Z be the orbit of z in $(\mathbb{P}, c_1, \ldots, c_n)$. Then $X \div Z$ and $Z \div Y$, and so there is a path from X to Y in H. Now if X, Y are infinite orbits which are incomparable, then there exists an infinite orbit Z with $X < Z$ and $Y < Z$, and so again there is a path from X to Y in H.

Since H is connected, there exist infinite orbits X, Y with $X \div Y$ such that g behaves like id on X and like $\updownarrow$ on Y. Assume that $X \leq Y$; the proof of the case $Y \leq X$ is dual. By Lemma 19, we may furthermore assume that $g[X] \div g[Y]$, or else we are done. This leaves us with two possibilities, $g[X] \leq g[Y]$ or $g[Y] \leq g[X]$.

The first case $g[X] \leq g[Y]$ splits into two subcases:

- For all $x \in X$, $y \in Y$, $x < y$ implies $g(x) < g(y)$ and $x \perp y$ implies $g(x) \perp g(y)$;
- For all $x \in X$, $y \in Y$, $x < y$ implies $g(x) \perp g(y)$ and $x \perp y$ implies $g(x) < g(y)$.

Let $x, x' \in X$ and $y, y' \in Y$ be so that $x < x'$, $x < y'$, $x' < y$, $y' < y$, and $x' \perp y'$. Then in the first subcase we can derive $g(x') < g(y)$, $g(y) < g(y')$, and $g(x') \perp g(y')$, a contradiction. In the second subcase, $g(x) < g(x')$, $g(x') < g(y')$, and $g(x) \perp g(y')$, again a contradiction.

In the second case $g[Y] \geq g[X]$ we have the following possibilities:

- For all $x \in X$, $y \in Y$, $x < y$ implies $g(x) > g(y)$ and $x \perp y$ implies $g(x) \perp g(y)$;
- For all $x \in X$, $y \in Y$, $x < y$ implies $g(x) \perp g(y)$ and $x \perp y$ implies $g(x) > g(y)$.

Let $x, x' \in X$ and $y, y' \in Y$ be as before. Then in the first subcase we can derive $g(x) < g(x')$, $g(y') < g(x)$, and $g(x') \perp g(y')$, a contradiction. In the second subcase, $g(y) < g(y')$, $g(y') < g(x')$, and $g(y) \perp g(x')$, again a contradiction. □

4.3. **Behaviors generating** Rev**.**

Lemma 21. *Let $\mathcal{G} \supseteq \mathrm{Aut}(\mathbb{P})$ be a closed group, and let $c_1, \ldots, c_n \in P$. Let $g : (\mathbb{P}, c_1, \ldots, c_n) \to \mathbb{P}$ be a canonical function mon-generated by $\mathcal{G}$. If g behaves like $\updownarrow$ on some infinite orbit of $(\mathbb{P}, c_1, \ldots, c_n)$, then $\mathcal{G} \supseteq \mathrm{Rev}$.*

Proof. Let X be the infinite orbit. Pick an isomorphism $i : (P; \leq) \to (X; \leq)$. Then given any finite $A \subseteq P$, there exists $\alpha \in \mathrm{Aut}(\mathbb{P})$ such that $\alpha \circ g \circ i$ agrees with $\updownarrow$ on A. Since g and i are generated by $\mathcal{G}$, there exists $\beta \in \mathcal{G}$ such that β agrees with $\updownarrow$ on A. Hence, $\updownarrow \in \mathcal{G}$. □

4.4. **Behaviors generating** Turn**.**

Lemma 22. *Let $\mathcal{G} \supseteq \mathrm{Aut}(\mathbb{P})$ be a closed group, and let $c_1, \ldots, c_n \in P$. Let $g : (\mathbb{P}, c_1, \ldots, c_n) \to \mathbb{P}$ be a canonical function mon-generated by $\mathcal{G}$ which behaves like* id *on all of its orbits. Then g behaves like* id *between all infinite orbits of $(\mathbb{P}, c_1, \ldots, c_n)$, or else $\mathcal{G} \supseteq \mathrm{Turn}$.*

Proof. Let infinite orbits X, Y be given.

We start with the case $X \div Y$. Say without loss of generality $X \leq Y$. By Lemma 19, we may assume that $g[X] \div g[Y]$, or else $\mathcal{G} = \mathrm{Sym}_P$. Hence $g[X] \leq g[Y]$ or $g[Y] \leq g[X]$. If $g[X] \leq g[Y]$, then either g behaves like id between X and Y and we are done, or $x < y \to g(x) \perp g(y)$ and $x \perp y \to g(x) < g(y)$ hold for all $x \in X$, $y \in Y$; the latter, however, is impossible, as for $x, x' \in X$ and $y \in Y$ with $x < x'$, $x < y$, and $x' \perp y$ we would have $g(x) < g(x') < g(y)$ and $g(x) \perp g(y)$. Now suppose $g[Y] \leq g[X]$. Then we have one of the following:

- For all $x \in X$, $y \in Y$, $x < y$ implies $g(x) > g(y)$ and $x \perp y$ implies $g(x) \perp g(y)$;
- For all $x \in X$, $y \in Y$, $x < y$ implies $g(x) \perp g(y)$ and $x \perp y$ implies $g(x) > g(y)$.

The first case is absurd since picking x, x', y as above yields $g(x) < g(x')$, $g(x) > g(y)$, and $g(x') \perp g(y)$. We claim that in the second case $\mathcal{G}$ contains $\circlearrowright$. Let $F \subseteq P$ be any irrational filter. Let $A \subseteq P$ be finite, and set $A_2 := A \cap F$, and $A_1 := A \setminus A_2$. Then there exists an automorphism α of $\mathbb{P}$ which sends A_2 into Y and A_1 into X. The composite $g \circ \alpha$ behaves like $\circlearrowright_F$ on A for what concerns comparabilities and incomparabilities, and hence there exists $\beta \in \mathrm{Aut}(\mathbb{P})$ such that $\beta \circ g \circ \alpha$ agrees with $\circlearrowright_F$ on A. By topological closure we infer $\circlearrowright_F \in \mathcal{G}$.

Now consider the case where X, Y are strictly comparable, say $X < Y$. Then we know from the proof of Lemma 20 that there exists an infinite orbit Z such that $X \leq Z \leq Y$, $X \div Z$ and $Z \div Y$. Let $x \in X$ and $y \in Y$ be arbitrary. There exists $z \in Z$ such that $x < z < y$. As g behaves like id between X and Z and between Z and Y, we have that $g(x) < g(z) < g(y)$, and hence g behaves like id between X and Y.

It remains to discuss the case $X \perp Y$. Suppose that $g[X]$ and $g[Y]$ are comparable, say $g[X] < g[Y]$. Then given any finite $A \subseteq P$ with incomparable elements x, y, using the extension property we can find $\alpha \in \mathrm{Aut}(\mathbb{P})$ which sends x into X, all elements of A which are incomparable with x into Y, and all other elements of A into infinite orbits which are comparable with both X and Y. Applying $g \circ \alpha$ then increases the number of comparabilities on A, and hence repeated applications of such functions will send A onto a chain, proving $\mathcal{G} = \mathrm{Sym}_P$. □

Lemma 23. *Let $\mathcal{G} \supseteq \mathrm{Aut}(\mathbb{P})$ be a closed group, and let $c_1, \ldots, c_n \in P$. Let $g : (\mathbb{P}, c_1, \ldots, c_n) \to \mathbb{P}$ be a canonical function mon-generated by $\mathcal{G}$ which behaves like* id *on all of its orbits. Then g behaves like* id *between all orbits of $(\mathbb{P}, c_1, \ldots, c_n)$, and hence is mon-generated by* $\mathrm{Aut}(\mathbb{P})$*, or else $\mathcal{G} \supseteq$* Turn.

Proof. Let $1 \leq i \leq n$, and let X be an infinite orbit which is incomparable with $\{c_i\}$. Suppose that $g[X]$ and $\{g(c_i)\}$ are strictly comparable, say $\{g(c_i)\} < g[X]$. Let Y be an infinite orbit such that $X \leq Y$, $X \div Y$, and $\{c_i\} < Y$. Let moreover Z be an infinite orbit such that $Z < \{c_i\}$, $Z \leq X$ and $Z \div X$. Then by the preceding lemma, we may assume that g behaves like id between X, Y and Z. We cannot have $g[Z] < \{g(c_i)\}$ as this would imply $g[Z] < g[X]$, contradicting the fact that g behaves like id between Z and X. Suppose that $g[Z] \perp \{g(c_i)\}$. Set $S := Z \cup X \cup Y \cup \{c_i\}$. Then it is easy to see that $(S; \leq)$ satisfies the extension property, and hence is isomorphic which $\mathbb{P}$; fix an isomorphism $i : (P; \leq, c_i) \to (S; \leq, c_i)$. This isomorphism is mon-generated by $\mathrm{Aut}(\mathbb{P})$ since it can be approximated by automorphisms of $\mathbb{P}$ on all finite subsets of P. The restriction of g to S is canonical as a function from $(S; \leq, c_i)$ to $\mathbb{P}$. Hence, the function $h := g \circ i$ is canonical as a function from $(\mathbb{P}, c_i)$ to $\mathbb{P}$, and has the same behavior as the restriction of g to S. Let $\alpha \in \mathrm{Aut}(\mathbb{P})$ be so that $\alpha(h(c_i)) = c_i$. Then $t := h \circ \alpha \circ h$ has the property that $t(x) > t(c_i)$ for all $x \neq c_i$, and that $t(x) \perp t(y)$ if and only if $x \perp y$, for all $x, y \in P \setminus \{c_i\}$. Hence, given any finite $A \subseteq P$ which is not a chain, we can pick $x \in A$ which is not comparable to all other elements of A, and find $\beta \in \mathrm{Aut}(\mathbb{P})$ which sends x to c_i; then $t \circ \beta$ strictly increases the number of comparabilities among the elements of A. Repeating this process and composing the functions, we find a function which is mon-generated by $\mathcal{G}$ and which maps A onto a chain. Hence, $\mathcal{G} = \mathrm{Sym}_P$.

Therefore, we may henceforth assume that g behaves like id between all $\{c_i\}$ and all infinite orbits X with $\{c_i\} \perp X$. Now suppose that there exists $1 \leq i \leq n$ and an infinite orbit X with $X < \{c_i\}$ such that $\{g(c_i)\} < g[X]$. Pick an infinite orbit Y which is incomparable with c_i, and which satisfies $X \leq Y$. Then $\{g(c_i)\} < g[Y]$ since g behaves like id between X and Y, a contradiction. Next suppose there exists $1 \leq i \leq n$ and an infinite orbit X with $X < \{c_i\}$ such that $\{g(c_i)\} \perp g[X]$. Then pick an infinite orbit Y as in the preceding case, and an infinite orbit Z with $\{c_i\} < Z$. Now given any finite $A \subseteq P$ which does not induce an antichain, we can pick $y \in A$ which is not minimal in A. Taking $\alpha \in \mathrm{Aut}(\mathbb{P})$ which sends y to c_i and A into $X \cup Y \cup Z \cup \{c_i\}$, we then have that application of $g \circ \alpha$ increases the number of incomparabilites of A. Repeated composition of such functions yields a function which sends A onto an antichain. Hence, $\mathcal{G} = \mathrm{Sym}_P$. The case where there exist $1 \leq i \leq n$ and an infinite orbit X with $\{c_i\} < X$ such that $\{g(c_i)\} \perp g[X]$ is dual.

We turn to the case where we have two distinct finite orbits $\{c_i\}$ and $\{c_j\}$. Suppose first that they are comparable, say $c_i < c_j$. Picking an infinite orbit Z with $\{c_i\} < Z < \{c_j\}$ then yields, by what we know already, $\{g(c_i)\} < g[Z] < \{g(c_j)\}$, so we are done. Finally, suppose that $c_i \perp c_j$. Then given any finite $A \subseteq P$ which has incomparable elements x, y, we can send x to c_i, y to c_j, and the rest of A to infinite orbits via some $\alpha \in \mathrm{Aut}(\mathbb{P})$. But then application of $g \circ \alpha$ increases the number comparabilities on A, and hence repeating the process yields a function which sends A to a chain. Hence, $\mathcal{G} = \mathrm{Sym}_P$. ☐

4.5. **Climbing up the group lattice.**

Proposition 24. *Let $\mathcal{G} \supsetneq \mathrm{Aut}(\mathbb{P})$ be a closed group. Then $\mathcal{G}$ contains either* Rev *or* Turn.

Proof. There exist $\pi \in \mathcal{G} \setminus \mathrm{Aut}(\mathbb{P})$ and elements $u, v \in P$ such that $\pi(u) \not\leq \pi(v)$. Let $g : (\mathbb{P}, u, v) \to \mathbb{P}$ be a canonical function mon-generated by $\mathcal{G}$ which agrees with π on $\{u, v\}$.

If g behaves like $\updownarrow$ on some infinite orbit of $(\mathbb{P}, u, v)$, then $\mathcal{G} \supseteq \mathrm{Rev}$ by Lemma 21. Otherwise Lemma 23 states that g is generated by $\mathrm{Aut}(\mathbb{P})$ or $\mathcal{G} \supseteq \mathrm{Turn}$. Since $g(u) \not\leq g(v)$, only the latter possibility can be the case. $\square$

Proposition 25. *Let* $\mathcal{G} \supsetneq \mathrm{Rev}$ *be a closed group. Then* $\mathcal{G}$ *contains* Turn.

Proof. Let $\pi \in \mathcal{G} \setminus \mathrm{Rev}$. Then there exists a finite tuple $c = (c_1, \ldots, c_n)$ of elements of P such that no function in Rev agrees with π on c. Let $g : (\mathbb{P}, c_1, \ldots, c_n) \to \mathbb{P}$ be canonical function which is mon-generated by $\mathcal{G}$ and which agrees with π on $\{c_1, \ldots, c_n\}$. By Lemma 20, we may assume that either g behaves like id on all infinite orbits, or it behaves like $\updownarrow$ on all infinite orbits of $(\mathbb{P}, c_1, \ldots, c_n)$. By composing g with $\updownarrow$, we may assume that it behaves like id on all infinite orbits. But then Lemma 23 implies that $\mathcal{G} \supseteq \mathrm{Turn}$, or that g is mon-generated by $\mathrm{Aut}(\mathbb{P})$. The latter is, of course, impossible. $\square$

Proposition 26. *Let* $\mathcal{G} \supsetneq \mathrm{Turn}$ *be a closed group. Then* $\mathcal{G}$ *contains* Rev.

Proof. By the results of [PPPS11]. $\square$

Proposition 27. *Let* $\mathcal{G} \supsetneq \mathrm{Max}$ *be a closed group. Then* $\mathcal{G}$ *is 3-transitive.*

Proof. By the results of [PPPS11]. $\square$

Proposition 28. *Let* $\mathcal{G}$ *be a* 3*-transitive closed group containing* Turn*. Then* $\mathcal{G} = \mathrm{Sym}_P$.

Proof. We prove by induction that $\mathcal{G}$ is n-transitive for all $n \geq 3$. Our claim holds for $n = 3$ by assumption. So let $n \geq 4$ and assume that $\mathcal{G}$ is $(n-1)$-transitive. We claim that every n-element subset of P can be mapped onto an antichain by a permutation in $\mathcal{G}$; n-transitivity then follows as in the proof of Lemma 16. We prove this claim in several steps, and will need the following partial orders.

For every natural number k with $1 \leq k \leq n$, let

- S_n^k be the n-element poset consisting of k independent points and a chain of $(n-k)$ elements below them;
- T_n^k be the dual of S_n^k;
- A_n^k be the n-element poset consisting of k independent points, an element below them, and an antichain of size $(n-k-1)$ independent from these points;
- B_n^k be the dual of A_n^k;
- C_k be the $k+1$-element poset consisting of k independent points and an element below them; that is, $C_k = A_{k+1}^k = S_{k+1}^k$.

Step 1: From anything to A_n^k or B_n^k for $k \geq \frac{n-1}{2}$.

We first show that any n-element set $A \subseteq P$ can me mapped to a copy of A_n^k or B_n^k, where $k \geq \frac{n-1}{2}$, via a function in $\mathcal{G}$. Let A be given, and write $A = A' \cup \{a\}$, where A' has $n-1$ elements. Then by induction hypothesis there exists $\pi \in \mathcal{G}$ which maps A' to an antichain. Let $F \subseteq P$ be an irrational filter which separates $\pi(a)$ from $\pi[A']$, i.e., for all $b \in \pi[A']$ we have $b \in F$ if and only if $\pi(a) \notin F$. Then one can check that either $\pi[A]$ or $(\circlearrowright_F \circ \pi)[A]$ induce A_n^k or B_n^k in $\mathbb{P}$ for some $k \geq \frac{n-1}{2}$.

Step 2: From A_n^k (T_n^k) to S_n^k (T_n^k) for $k \geq \frac{n-1}{2}$.

We now show that any copy of A_n^k in $\mathbb{P}$ can be mapped to a copy of S_n^k via a function in $\mathcal{G}$. The dual proof then shows that any copy of B_n^k can be mapped to a copy of T_n^k.

Let $\{x_1, \ldots, x_{n-1}\}$ and $\{y_1, \ldots, y_{n-1}\}$ be disjoint subsets of P inducing an antichain and a chain, respectively. By the $(n-1)$-transitivity of $\mathcal{G}$, the map $x_i \mapsto y_i$, $1 \leq i \leq n-1$, can be extended to a permutation $\pi \in \mathcal{G}$. Let X be the orbit of $(\mathbb{P}, x_1, \ldots, x_{n-1})$ such that $x \perp x_i$ for all $x \in X$ and all $1 \leq i \leq n-1$. By Lemma 10 there exists a canonical function $g : (\mathbb{P}, x_1, \ldots, x_{n-1}) \to (\mathbb{P}, y_1, \ldots, y_{n-1})$ mon-generated by $\mathcal{G}$ that agrees with π on $\{x_1, \ldots, x_{n-1}\}$. We may assume that g behaves like id or like $\updownarrow$ on X, by Lemma 18. If g behaves like $\updownarrow$ on X, then $\mathcal{G}$ contains $\updownarrow$ by Lemma 21; replacing g by $\updownarrow \circ g$ and replacing each y_i by $\updownarrow(y_i)$, we may assume that g behaves like id on X. Let $D \subseteq X$ be so that it induces C_k, and observe that $D' := D \cup \{x_1, \ldots, x_{n-k-1}\}$ induces a copy of A_n^k in $\mathbb{P}$. Since g is canonical, all elements of X, and in particular all elements of D are sent to the same orbit Y of $(\mathbb{P}, y_1, \ldots, y_{n-1})$. Thus, for all $1 \leq i \leq n-1$ we have that either $g[D] < \{y_i\}$, or $g[D] \perp \{y_i\}$, or $g[D] > \{y_i\}$. Let S be the set of those y_i for which the first relation holds, and set $E := g[D] \cup (\{y_1, \ldots, y_{n-1}\} \setminus S)$. Let $F \subseteq P$ be an irrational filter which separates E from S, i.e., F contains S but does not intersect E. Then $\circlearrowright_F [S] \perp \circlearrowright_F [E]$. Choose an irrational filter F' which contains $\circlearrowright_F [S]$ and which does not intersect $\circlearrowright_F [E]$. Then $\circlearrowright_{F'} \circ \circlearrowright_F [S] < \circlearrowright_{F'} \circ \circlearrowright_F [E]$. Set $h := \circlearrowright'_F \circ \circlearrowright_F \circ g$. Now for all $1 \leq i \leq n-1$ we have that either $h[D] > \{h(x_i)\}$ or $h[D] \perp \{h(x_i)\}$. Moreover, h behaves like id on D, and the $h(x_i)$ form a chain. Either there are at least $\frac{n-1}{2}$ elements among the $h(x_i)$ for which $h[D] > \{h(x_i)\}$, or there are at least $\frac{n-1}{2}$ of the $h(x_i)$ for which $h[D] \perp \{h(x_i)\}$. In the first case, observe that $k > \frac{n-1}{2}$ implies $\frac{n-1}{2} \geq n-k-1$. Hence, by relabeling the x_i, we may assume that $h[D] > \{h(x_i)\}$ for $1 \leq n-k-1$, and so h sends D' to a copy of S_n^k, finishing the proof. In the second case, pick an irrational filter $F'' \subseteq P$ which contains all $h(x_i)$ for which $h[D] \perp \{h(x_i)\}$, and which does not contain any element from $h[D]$. Then replacing h by $\circlearrowright_{F''} \circ h$ brings us back to the first case.

Step 3: From S_n^k (T_n^k) to an antichain when $k > \frac{n-1}{2}$.

We show that if $k > \frac{n-1}{2}$, then any copy of S_n^k in $\mathbb{P}$ can be mapped to an antichain by a permutation in $\mathcal{G}$. Clearly, the dual argument then shows the same for T_n^k. Let $\{u_1, \ldots, u_{n-1}\} \subseteq P$ be so that it induces a chain. By the $(n-1)$-transitivity of $\mathcal{G}$, there is some $\rho \in \mathcal{G}$ that maps $\{u_1, \ldots, u_{n-1}\}$ to an antichain $\{v_1, \ldots, v_{n-1}\}$. Let Z be the orbit of $(\mathbb{P}, u_1, \ldots, u_{n-1})$ that is above all the u_j. By Lemma 10 there exists a canonical function $f : (\mathbb{P}, u_1, \ldots, u_{n-1}) \to (\mathbb{P}, v_1, \ldots, v_{n-1})$ mon-generated by $\mathcal{G}$ that agrees with ρ on $\{u_1, \ldots, u_{n-1}\}$. All elements of Z are mapped to one and the same orbit O of $(\mathbb{P}, v_1, \ldots, v_{n-1})$. Now pick $z_1, \ldots, z_k \in Z$ which induce an antichain. By applying an appropriate instance of $\circlearrowright$ in a similar fashion as in Step 2, we may assume that O is incomparable with at least $\frac{n-1}{2}$ of the singletons $\{v_i\}$. Choose $(n-k)$ out of these v_i. This is possible, as $k > \frac{n-1}{2}$ and consequently $\frac{n-1}{2} \geq n-k$. By relabeling the u_i, we may assume that the chosen elements are $v_1, \ldots, v_{n-k}$. Then $f[\{z_1, \ldots, z_k\}] \cup \{v_1, \ldots, v_{n-k}\}$ is an antichain. Since $\{z_1, \ldots, z_k, u_1, \ldots, u_{n-k}\}$ induces a copy of S_n^k, we are done.

Step 4: From A_n^k to an antichain when $k = \frac{n-1}{2}$.

Assuming that $k = \frac{n-1}{2}$, we show that any copy of A_n^k in $\mathbb{P}$ can be mapped to an antichain by a function in $\mathcal{G}$. Note that this assumption implies that n is odd, so $n \geq 5$, and thus $k = \frac{n-1}{2} \geq 2$.

Let $\{x_1, \ldots, x_{k-1}\} \subseteq P$ induce an antichain. Let $s \in P$ be a point below all the x_i, and let $\{y_1, \ldots, y_k\} \subseteq P$ induce an antichain whose elements are incomparable with all the x_i and s. The set $A := \{s, x_1, \ldots, x_{k-1}, y_1, \ldots, y_k\}$ induces a copy of A_{n-1}^{k-1}. By the $(n-1)$-transitivity of $\mathcal{G}$ there exists $\varphi \in \mathcal{G}$ which maps A to an antichain $\{z_1, \ldots, z_{n-1}\} \subseteq P$. Without loss of generality, we write $\varphi(s) = z_{n-1}$, $\varphi(x_i) = z_i$ for $1 \leq i \leq k-1$, and $\varphi(y_i) = z_{k+i}$ for $1 \leq i \leq k$. By Lemma 10 there exists a canonical function $h : (\mathbb{P}, s, x_1, \ldots, x_{k-1}, y_1, \ldots, y_k) \to$

$(\mathbb{P}, z_1, \ldots, z_{n-1})$ mon-generated by $\mathcal{G}$ which agrees with φ on A. Let U be the orbit of $(\mathbb{P}, s, x_1, \ldots, x_{k-1}, y_1, \ldots, y_k)$ whose elements are larger than s and incomparable to all other elements of A. Since h is canonical, $h[U]$ is contained in an orbit V of $(\mathbb{P}, z_1, \ldots, z_{n-1})$.

Assume that the elements of the orbit V do not satisfy the same relations with all the z_i for $1 \leq i \leq n-2$. Then there is a partition $R \cup S = \{z_1, \ldots, z_{n-2}\}$, with both R and S non-empty, such that the elements of V are incomparable with the elements of R and strictly comparable with the elements of S. By applying an appropriate instance of $\circlearrowright$ we may assume that $|R| \geq k$. Pick any $R' \subseteq R$ of size k, any $S' \subseteq S$ of size 1, and a k-element antichain $W \subseteq U$. Then $h^{-1}[R'] \cup h^{-1}[S'] \cup W$ induces an antichain of size n whose image I under h induces either A_n^k or B_n^k. In the second case, let $F \subseteq P$ be an irrational filter which separates the largest element of I from its other elements. Then $\circlearrowright_F$ sends I to a copy of A_n^k. Thus in either case, $\mathcal{G}$ contains a function which sends an n-element antichain to a copy of A_n^k. Since $\mathcal{G}$ contains the inverse of all of its functions, it also maps a copy of A_n^k to an antichain.

Finally, assume that V satisfies the same relations with all the z_i for $1 \leq i \leq n-2$. By applying an appropriate instance of $\circlearrowright$ we may assume that V is incomparable with all the z_i for $1 \leq i \leq n-2$. Let $W \subseteq U$ induce a $(k-1)$-element antichain, and consider $R := W \cup \{x_1, y_1, \ldots, y_k, s\}$; then R induces a copy of A_n^k. If V is incomparable with z_{n-1}, then $h[R]$ is an antichain and we are done. So assume that V and z_{n-1} are comparable. Then $h[R]$ induces A_n^{k-1} or B_n^{k-1}. Let $F \subseteq P$ be an irrational filter that separates $h(s)$ from the other elements of $h[R]$. Then $\circlearrowright_F \circ h[R]$ induces B_n^{n-k+1} or A_n^{n-k+1}. By Steps 2 and 3, both A_n^{n-k+1} and B_n^{n-k+1} can be mapped to an antichain by permutations from $\mathcal{G}$, finishing the proof. □

5. Relational descriptions of the groups

Proposition 29. $\mathrm{Aut}(P; \perp) = \langle \{\updownarrow\} \cup \mathrm{Aut}(\mathbb{P}) \rangle$.

Proof. By definition, the function $\updownarrow$ preserves the incomparability relation and its negation, so the inclusion $\supseteq$ is trivial. For the other direction, let $f \in \mathrm{Aut}(P; \perp)$. We claim that f is either an automorphism of $\mathbb{P}$, or satisfies itself the definition of $\updownarrow$ (i.e., $f(b) \leq f(a)$ iff $a \leq b$ for all $a, b \in P$). Suppose that f is not an automorphism of $\mathbb{P}$, and pick $a \leq b$ such that $f(a) \not\leq f(b)$. Since f preserves comparability, we then have $f(b) \leq f(a)$. To prove our claim, since f preserves $\perp$ it suffices to show that likewise $f(d) \leq f(c)$ for all $c \leq d$.

We first observe that if $e \leq b$ and $e \perp a$, then $f(e) \geq f(b)$. For if we had $f(e) \leq f(b)$, then it would follow that $f(e) \leq f(b) \leq f(a)$, a contradiction since f preserves $\perp$. Hence, $f(e) \not\leq f(b)$, and so $f(e) \geq f(b)$ since f preserves comparability.

Next let $r, s \in P$ so that $r \leq s$, $r \leq b$, and $s \perp b$; we show $f(r) \geq f(s)$. Since $f(r)$ and $f(s)$ are comparable, it is enough to rule out $f(r) \leq f(s)$. By our previous observation, we have $f(b) \leq f(r)$, so $f(r) \leq f(s)$ would imply $f(b) \leq f(s)$, contradicting the fact that f preserves $\perp$.

Now let $u, v \in P$ be so that $u \leq v$ and such that both u and v are incomparable with both a and b. Then using the extension property, we can pick $r, s \in P$ as above and such that $u \leq s$ and $v \perp s$. By the preceding paragraph, $f(r) \geq f(s)$, and applying the above once again with (u, v) taking the role of (r, s) and (r, s) the role of (a, b), we conclude $f(v) \geq f(u)$.

Finally, given arbitrary $c, d \in P$ with $c \leq d$, we use the extension property to pick $u, v \in P$ incomparable with all of a, b, c, d, and apply the above twice to infer $f(c) \geq f(d)$. □

Proposition 30. $\mathrm{Aut}(P; \mathrm{Cycl}) = \langle \{\circlearrowright\} \cup \mathrm{Aut}(\mathbb{P}) \rangle$.

Proof. By the results of [PPPS11]. ☐

Proposition 31. $\mathrm{Aut}(P; \mathrm{Par}) = \langle \{\updownarrow, \circlearrowright\} \cup \mathrm{Aut}(\mathbb{P}) \rangle$.

Proof. By the results of [PPPS11]. ☐

References

[BCP10] Manuel Bodirsky, Hubie Chen, and Michael Pinsker. The reducts of equality up to primitive positive interdefinability. *Journal of Symbolic Logic*, 75(4):1249–1292, 2010.

[Ben97] James H. Bennett. *The reducts of some infinite homogeneous graphs and tournaments*. PhD thesis, Rutgers university, 1997.

[BK09] Manuel Bodirsky and Jan Kára. The complexity of temporal constraint satisfaction problems. *Journal of the ACM*, 57(2), 2009. An extended abstract appeared in the proceedings of STOC'08.

[BP10] Manuel Bodirsky and Michael Pinsker. Minimal functions on the random graph. Preprint, arXiv:1003.4030, 2010.

[BP11a] Manuel Bodirsky and Michael Pinsker. Reducts of Ramsey structures. In *Model Theoretic Methods in Finite Combinatorics*, volume 558 of *Contemporary Mathematics*. American Mathematical Society, 2011. 31 pages. In print; preprint available from `arxiv.org/abs/1105.6073`.

[BP11b] Manuel Bodirsky and Michael Pinsker. Schaefer's theorem for graphs. In *Proceedings of STOC'11*, pages 655–664, 2011. Preprint of full journal version available from `arxiv.org/abs/1011.2894`.

[BPT11] Manuel Bodirsky, Michael Pinsker, and Todor Tsankov. Decidability of definability. In *Proceedings of LICS'11*, pages 321–328, 2011.

[Cam76] Peter J. Cameron. Transitivity of permutation groups on unordered sets. *Mathematische Zeitschrift*, 148:127–139, 1976.

[Fra88] Roland Fraïssé. *Theory of relations*, volume 118 of *Studies in Logic and Foundations of Mathematics*. Elsevier, 1988.

[Hod97] Wilfrid Hodges. *A shorter model theory*. Cambridge University Press, Cambridge, 1997.

[JZ08] Markus Junker and Martin Ziegler. The 116 reducts of $(\mathbb{Q}, <, a)$. *Journal of Symbolic Logic*, 74(3):861–884, 2008.

[Mac11] Dugald Macpherson. A survey of homogeneous structures. *Discrete Mathematics*, 311(15):1599–1634, 2011.

[Pon11] András Pongrácz. Reducts of the Henson graphs with a constant. Preprint, 2011.

[PPPS11] Péter Pál Pach, Michael Pinsker, András Pongrácz, and Csaba Szabó. A new transformation of partially ordered sets. 2011. Preprint.

[Sok10] Miodrag Sokić. *Ramsey property of posets and related structures*. PhD thesis, University of Toronto, 2010.

[Tho91] Simon Thomas. Reducts of the random graph. *Journal of Symbolic Logic*, 56(1):176–181, 1991.

[Tho96] Simon Thomas. Reducts of random hypergraphs. *Annals of Pure and Applied Logic*, 80(2):165–193, 1996.

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