ON FREE ALGEBRAS IN THE VARIETIES OF ITERATED SEMIDIRECT PRODUCTS OF MEET-SEMILATTICES

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ABSTRACT. We present a description of the finitely generated free algebras in the varieties of iterated semidirect products of semilattices. Asymptotical bounds for the free spectra of these varieties are given.

1. INTRODUCTION

Semidirect products and iterated semidirect products of semmilatices are thorougly investigated in [1]. Among others, it is shown that each variety of iterated meet semilattices is finitely generated and non-finitely based. In this paper we extend the results of [1] on these varieties. We present a new characterization for the word problem of these varieties, and give an asymptotic estimate for their free spectra. We do it via finding a normal form for the elements of the free algebras in each variety.

Let $\mathbf{A}$ be an m-element finite algebra. Let $\mathcal{V}$ denote the variety generated by $\mathbf{A}$, and denote by $\mathbf{F}_{\mathcal{V}}(n)$ the free algebra in $\mathcal{V}$ generated by n elements. The free spectrum of a variety $\mathcal{V}$ is the sequence of cardinalities $|\mathbf{F}_{\mathcal{V}}(n)|$, $n = 0, 1, 2, \ldots$. We can think of the free spectrum as the number of n-ary operations over $\mathbf{A}$. The p_n sequence of the variety is the number of essentially n-ary term operations over $\mathbf{A}$. It is known that the size of the n-generated free algebra ($|\mathbf{F}_{\mathcal{V}}(n)|$) in $\mathcal{V}$ is at most m^{m^n}. If $m \geq 2$, then $|\mathbf{F}_{\mathcal{V}}(n)| \geq n$. For example, the free spectrum of Boolean algebras is $|\mathbf{F}_{\mathcal{V}}(n)| = 2^{2^n}$. The first important question about free spectra is the following: Within the above bounds what are the possible sequences? For example, if $\mathbf{G}$ is a finite group, then the size of the n-generated relatively free group in the variety generated by $\mathbf{G}$ is exponential in n if $\mathbf{G}$ is nilpotent, and doubly-exponential if $\mathbf{G}$ is not nilpotent ([5] and [8]).

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There are very few results on the free spectra of semigroup varieties. For a basic reference on the general properties of p_n sequences for semigroups see [3]. A full description of finite semigroups for which the p_n sequence is bounded by a polynomial is presented in [4]. Among others, free spectra of surjective semigroups were considered in [3], bands in [9] and combinatorial 0-simple semigroups in [7].

2. Preliminaries

Let $t = t(x_1, \ldots, x_n)$ be an n-ary term. Then a term operation $t^{\mathbf{A}}$ is said to be *essentially n-ary* if it depends on all of its variables. That is, if for all $1 \leq i \leq n$ there exist $a_1, \ldots, a_{i-1}, a, b, a_{i+1}, \ldots, a_n \in A$ such that

$$t(a_1, \ldots, a_{i-1}, a, a_{i+1}, \ldots, a_n) \neq t(a_1, \ldots, a_{i-1}, b, a_{i+1}, \ldots, a_n).$$

The content of t for some term t is the set of variables occurring in t. We denote it by $c(t)$. For $n \geq 1$, denote the number of essentially n-ary term operations over $\mathbf{A}$ by $p_n(\mathbf{A})$. For the free spectrum of a variety we have

$$|\mathbf{F}_{\mathcal{V}}(n)| = \sum_{k=0}^{n} \binom{n}{k} p_k(\mathbf{A}) \tag{1}$$

Our main reference is going to be the book of J. Almeida ([1]), where detailed discussion of semidirect products of semigroups can be found. In this paper we only list the properties of iterated semidirect products of semilattices which are necessary for us. A semilattice is a commutative idempotent semigroup. The variety of semilattices will be denoted by $\mathcal{SL}$. The variety generated by semidirect products of two semilattices will be denoted by $\mathcal{SL}^2$, and $\mathcal{SL}^t$ will denote in general the variety generated by the t-times iterated semidirect product of semilattices. For every t the variety $\mathcal{SL}^t$ is locally finite and generated by $\mathbf{F}_{\mathcal{SL}^t}(2t)$, the $2t$ generated free algebra of the variety. Since the variety of semilattices $\mathcal{SL}$ is contained in each variety $\mathcal{SL}^t$, a term containing n variables necessary determines an essentially n-ary term operation. Let $\mathcal{SL}^t(n)$ be the set of the n-ary terms in $\mathcal{SL}^t$. We denote by $p_n(t)$ the number of essentially n-ary term operations in the variety $\mathcal{SL}^t$, thus $|\mathcal{SL}^t(n)| = p_n(t)$.

3. Recurrence formula

In this Section we present a new characterization of the word problem for the variety $\mathcal{SL}^t$, then a recurrence formula is given for the number of essentially n-ary terms.

At first recall the identity basis of $\mathcal{SL}^t$ from [1]. Let $X = \{x_1, x_2, \dots\}$ be a countable set of variables and X^+ (X^*) be the free semigroup (free monoid) over X.

$$u_{t-1} \dots u_1 x_i x_j = u_{t-1} \dots u_1 x_j x_i,$$
$$u_{t-1} \dots u_1 x_i^2 = u_{t-1} \dots u_1 x_i,$$

where $c(w)$ denotes the content of w for some $w \in X^*$ and $x_i, x_j \in c(u_1) \subseteq \dots \subseteq c(u_{t-1}), u_j \in X^+$). We say that

$$(u =) w_0, w_1, \dots, w_r (= v) \tag{2}$$

is a deduction of an identity $u = v$ from a set Σ of identities if for each $j \in \{0, \dots, r-1\}$ there exist factorizations

$$w_j = a_j(\varphi_j u_j) b_j \text{ and } w_{j+1} = a_j(\varphi_j v_j) b_j, \tag{3}$$

where each $\varphi_j \colon X^+ \to X^+$ is a substitution of the variables and one of the identities $u_j = v_j$ or $v_j = u_j$ belongs to Σ. The deduction is *left absorbing* if each a_j of the occurring prefixes in (3) is the empty word. We say the deduction (2) *involves no substitutions*, if all homomorphism φ_j are the identity function. Lemma 10.3.4. and Theorem 10.3.6. in [1] contains the following result.

Theorem 3.1. *For each $t \geq 2$ Σ_{t-1} is the identity basis for $\mathcal{SL}^t$. Moreover, $\mathcal{SL}^t \models u = v$ for $u, v \in X^*$ if and only if there exists a deduction of $u = v$ from Σ_{t-1} which is left absorbing and involves no substitutions.*

That is, if $\mathcal{SL}^t \models u = v$, then there exists a deduction $u = w_0, w_1, \dots, w_r = v$ such that each $w_j = w_{j+1}$ of the deduction is one of the following identities

$$u_{t-1} \dots u_1 xyw = u_{t-1} \dots u_1 yxw \tag{4}$$
$$u_{t-1} \dots u_1 x^2 w = u_{t-1} \dots u_1 xw, \tag{5}$$

where $x, y \in X$, $w \in X^*$, $u_j \in X^+$ ($j \in \{1, \dots, t-1\}$) and $x, y \in c(u_1) \subseteq \dots \subseteq c(u_{t-1})$. We call a step of the form (4) or (5) an *elementary step on level* t. From now on, let $u \sim_t v$ denote $\mathcal{SL}^t \models u = v$ for two terms $u, v \in X^*$. Note that $u \sim_t v$ and $c(v) = c(u) \subseteq c(w)$ yields $wu \sim_{t+1} wv$. Moreover, from $u \sim_t v$ follows $uw \sim_t vw$ for any terms $u, v, w \in X^*$, $t \geq 2$. Now, let us introduce a notation which we will use frequently throughout Sections 3 and 4.

Notation 3.2. Let $u \in X^+$ be a term. Let m_u be the last occurring variable. Let f_u be the prefix of u before the first occurence of m_u and let b_u be the suffix of u after the first occurence of m_u, i.e. $u = f_u m_u b_u$, where $c\,(f_u) = c\,(u) \setminus \{\, m_u \,\}$. Note that b_u is the empty term if a variable occurs only at the end of u, and f_u is the empty term if u contains only one variable.

Theorem 3.3. *Let $t \geq 2$ and u and v be two essentially n-ary terms over the set $X_n = \{\, x_1, \ldots, x_n \,\}$. According to Notation 3.2: $u = f_u m_u b_u$, $v = f_v m_v b_v$. Then $u \sim_t v$ if and only if*

(i) $m_u = m_v$,
(ii) $f_u \sim_t f_v$,
(iii) $b_u \sim_{t-1} b_v$.

Proof. Assume first that conditions (i),(ii) and (iii) hold. We prove that $u \sim_t v$. By (iii) there exist a deduction $b_u = w_0,\, w_1,\, \ldots,\, w_r = b_v$, such that every $w_i \sim_{t-1} w_{i+1}$ is an elementary step on level $t-1$. Note that $f_u m_u w_i \sim_t f_u m_u w_{i+1}$ is an elementary step on level t, since $c(f_u m_u) = X_n$. Then $f_u m_u b_u \sim_t f_u m_u b_v$ by the deduction $f_u m_u b_u = f_u m_u w_0,\, f_u m_u w_1,\, \ldots,\, f_u m_u w_r = f_u m_u b_v$. From (ii) we have $f_u \sim_t f_v$, therefore $f_u m_u b_v \sim_t f_v m_u b_v$. These two deductions together prove $f_u m_u b_u \sim_t f_v m_u b_v$. Finally, by (i) we have $m_u = m_v$, hence $u = f_u m_u b_u \sim_t f_v m_v b_v = v$.

For the other direction we prove that if $u = f_u m_u b_u \sim_t f_v m_v b_v = v$ by one elementary step on level t, then (i), (ii) and (iii) hold. Then by induction on the length of the deduction it follows that if $u = f_u m_u b_u \sim_t f_v m_v b_v = v$ then (i), (ii) and (iii) hold.

Assume first that we use an elementary step of form (4). Now, $u = u_{t-1} u_{t-2} \ldots u_1 xyw$ and $v = u_{t-1} u_{t-2} \ldots u_2 u_1 yxw$ for some terms $w \in X_n^*$ and $u_j \in X_n^+$ such that $x, y \in c(u_1) \subseteq c(u_2) \subseteq \cdots \subseteq c(u_{t-2}) \subseteq c(u_{t-1})$. We distinguish two cases according to whether or not $c(u_{t-1}) = X_n$.

Case 1. $c\,(u_{t-1}) = X_n$. This implies that m_u occurs in u_{t-1}, and therefore $f_u m_u$ is a prefix of u_{t-1}. Similarly $f_v m_v$ is a prefix of v_{t-1}, hence (i) and (ii) hold. For some term $s \in X_n^*$ we have $u_{t-1} = f_u m_u s = f_v m_v s$. Then the deduction $b_u = (s u_{t-2}) u_{t-3} \ldots u_1 xyw$, $(s u_{t-2}) u_{t-3} \ldots u_1 yxw = b_v$ shows $b_u \sim_{t-1} b_v$, and so (iii) holds.

Case 2. $c\,(u_{t-1}) \neq X_n$. Thus $c(u_{t-1} u_{t-2} \ldots u_2 u_1 xy) \neq X_n$, either, hence the last occurring variable in both u and v appears in w. Now, $u = u_{t-1} u_{t-2} \ldots u_1 xyw$, $v = u_{t-1} u_{t-2} \ldots u_1 yxw$, hence $m_u = m_v$ and $b_u = b_v$, proving (ii) and (iii). Moreover, there exists a term $s \in X_n^*$ such that $f_u = u_{t-1} u_{t-2} \ldots u_2 u_1 xys$ and $f_v = u_{t-1} u_{t-2} \ldots u_2 u_1 yxs$. Then $u_{t-1} u_{t-2} \ldots u_2 u_1 xys \sim_t u_{t-1} u_{t-2} \ldots u_2 u_1 yxs$ is an elementary step. Thus $f_u \sim_t f_v$ and (iii) holds.

The case where we use an elementary step of form (5) can be handled similarly. Induction on the length of the deduction showing $u \sim_t v$ finishes the proof, as each property (i), (ii) and (iii) is preserved by an elementary step. □

In other words, by Theorem 3.3 every n-ary term over $\mathcal{SL}^t$ can be represented as a triple. This triple consists of an $(n-1)$-ary term over $\mathcal{SL}^t$, a variable and an at most n-ary (possibly empty) term of $\mathcal{SL}^{t-1}$. This is the key observation for proving a recurrence formula for $p_n(t)$.

Theorem 3.4. *The following recurrence formula holds for the number of essentially n-ary term operations:*

$$p_n(t) = np_{n-1}(t) \sum_{k=0}^{n} \binom{n}{k} p_k(t-1). \tag{6}$$

Proof. In the variety $\mathcal{SL}^t$ every term containing n variables determines an essentially n-ary term operation. Let u be an essentially n-ary term over $\mathcal{SL}^t$. By Theorem 3.3 we can assign a triple f_u, m_u, b_u to u bijectively, where f_u is an $(n-1)$-ary term over $\mathcal{SL}^t$, m_u is a variable and b_u is an at most n-ary (possible empty) term of $\mathcal{SL}^{t-1}$. We count the number of such triples. We have n many choices for m_u and $p_{n-1}(t)$ many choices for f_u. The number of the at most n-ary terms over $\mathcal{SL}^{t-1}$ is the size of the n-generated free algebra in $\mathcal{SL}^{t-1}$. ccording to formula (1) in Section 2 we have $\sum_{k=0}^{n} \binom{n}{k} p_k(t-1)$ many choices for b_u. Thus the recurrence formula (6) is gained. □

4. Normal form

In Section 4 a normal form for the elements of the free algebra in the varieties $\mathcal{SL}^t$ is presented. The length of this normal form is polynomial in the number of variables. Additionally, one can easily calculate the product of these normal forms and obtain the result in normal form.

Construction 4.1. By Theorem 3.3 every n-ary term over $\mathcal{SL}^t$ can be represented as a triple. This triple consists of an $(n-1)$-ary term over $\mathcal{SL}^t$, a variable and an at most n-ary (possible empty) term of $\mathcal{SL}^{t-1}$. Let us assign this triple to the term. If we multiply these elements from left to right we obtain the original word. Now, we iterate this process for the first and the third parts, simoultaneously. After finitely many steps we arrive at terms of $\mathcal{SL}^1$ and unary terms of $\mathcal{SL}^s$ for some $s \leq t$. Connecting all the noted terms with the elements of the corresponding triple, we get a rooted tree, as it is illustrated on Figure 1.

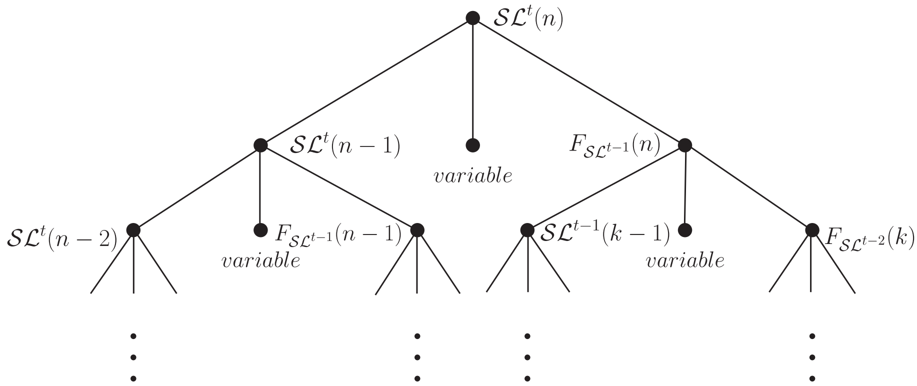

FIGURE 1

According to Theorem 3.3 this tree uniquely determines the original term, and the scheme of the tree only depends on the equivalence class of the original term. There are three kinds of leaves on the tree: unary terms of $\mathcal{SL}^s$ for some s, arbitrary terms of $\mathcal{SL}^1$ and variables. In the first two cases we assign to the leaf the shortest normal form of the term written on the leaf itself. That is, in the case of a unary term x_i^k of $\mathcal{SL}^s$ we assign x_i^l to the leaf, where $l = \min\{k, s\}$. While in the case of an arbitrary term w of $\mathcal{SL}^1$ the term $x_{i_1}x_{i_2}\cdots x_{i_r}$ is assigned, where the variables occurring in w are in increasing order according to their indices. We define the normal form of the term by writing the terms assigned to the leaves next to each other from left to right.

Figure 2 illustrates an example. It shows how the normal form of $x_1^3x_2x_1x_3x_2^2x_3x_1$ in $\mathcal{SL}^2$ is computed. The normal form is $x_1^2x_2x_1x_3x_1x_2x_3$. The variety is indicated in the upper right corner of the terms.

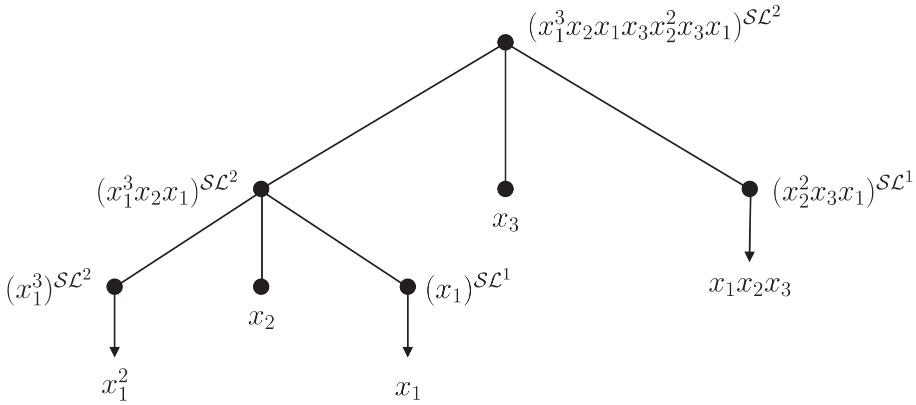

FIGURE 2

Let us denote the normal form of a term w in the variety $\mathcal{SL}^t$ by $\varphi_t(w)$. The following algorithm computes φ_t recursively.

Algorithm 4.2. Let w be an n-ary term.

(1) If $t = 1$, then let $\varphi_1(w) = x_{i_1} x_{i_2} \dots x_{i_k}$, where the variables occurring in w are in increasing order according to their indices.
(2) If $n = 1$ and $w = x_i^k$, then let $l = \min\{t, k\}$ and $\varphi_t\left(x_i^k\right) = x_i^l$.
(3) Otherwise let $\varphi_t(w)$ be the concatenation of the terms $\varphi_t(f_w)$, m_w and $\varphi_{t-1}(b_w)$, i.e. $\varphi_t(w) = \varphi_t(f_w)\, m_w \varphi_{t-1}(b_w)$.

Note that steps (1), (2) and (3) are invoked as many times as the number of vertices the tree has in Construction 4.1. Moreover each step takes linear time (in the length of the term).

Now, we show that we assigned a unique normal form to every term, and distinct terms have distinct normal forms.

Proposition 4.3. *Let u, v be n-ary terms. Then $u \sim_t v$ if and only if the normal form of u and v in $\mathcal{SL}^t$ are the same, that is, $\varphi_t(u) = \varphi_t(v)$.*

Proof. We prove the proposition by induction on t and n. If $t = 1$ or $n = 1$ then the proposition holds. Assume that $n \geq 2$ and $t \geq 2$.

Let $u \sim_t v$. By Theorem 3.3 we have $f_u \sim_t f_v$, $m_u = m_v$ and $b_u \sim_{t-1} b_v$. By the induction hypothesis $f_u \sim_t f_v$ implies $\varphi_t(f_u) = \varphi_t(f_v)$, and $\varphi_{t-1}(b_u) = \varphi_{t-1}(b_v)$ follows from $b_u \sim_{t-1} b_v$. From step (3) of Algorithm 4.2 we have $\varphi_t(u) = \varphi_t(f_u)\, m_u \varphi_{t-1}(b_u) = \varphi_t(f_v)\, m_v \varphi_{t-1}(b_v) = \varphi_t(v)$.

Now, assume that $\varphi_t(u) = \varphi_t(v)$. From step (3) of Algorithm 4.2 we have $m_u = m_v$, thus $\varphi_t(f_u) = \varphi_t(f_v)$ and $\varphi_{t-1}(b_u) = \varphi_{t-1}(b_v)$. By the induction hypothesis we have $f_u \sim_t f_v$ and $b_u \sim_{t-1} b_v$. From Theorem 3.3 $u \sim_t v$ follows. □

Proposition 4.4. *Let u be an n-ary term in the free algebra of $\mathcal{SL}^t$. Then $\varphi_t(u)$ is a shortest element in the equivalence class of u.*

Proof. We prove the proposition by induction on n and t. The statement holds if $n = 1$ or $t = 1$. Assume that $n \geq 2$ and $t \geq 2$, and let v be in the equivalence class of u. By Theorem 3.3 we have $f_u \sim_t f_v$, $m_u = m_v$ and $b_u \sim_{t-1} b_v$. By the induction hypothesis $\varphi_t(f_u)$ is in the equivalence class of f_u and $\varphi_t(f_u)$ is shorter than f_v. Similarly, $\varphi_{t-1}(b_u)$ is in the equivalence class of b_u and $\varphi_{t-1}(b_u)$ is shorter than b_v. By Theorem 3.3, $\varphi_t(u) = \varphi_t(f_u)\, m_u \varphi_{t-1}(b_u)$ is in the equivalence class of $u = f_u m_u b_u$ and is shorter than $v = f_v m_v b_v$. □

Finally, we give an upper bound on the length of the normal form and on the time demand of Algorithm 4.2 for computing the normal form of the product of two normal forms.

Proposition 4.5. *The normal form of an n-ary term in $\mathcal{SL}^t$ has length at most $\binom{n+t}{t} - 1$. Given two n-ary normal forms in $\mathcal{SL}^t$ the normal form of their product can be calculated in $O\left(n^{2t-1}\right)$ time.*

Proof. Let $M(n,t)$ denote the maximal length of the normal form of an n-ary term in the variety $\mathcal{SL}^t$. From Proposition 4.4 and Algorithm 4.2 we obtain $M(n,t) = M(n-1,t) + 1 + M(n,t-1)$ with initial values $M(1,t) = t$ and $M(n,1) = n$. This recurrence formula has the solution $M(n,t) = \binom{n+t}{t} - 1 = O(n^t)$.

Let $L(n,t)$ denote the number of leaves on the tree of the normal form in Construction 4.1. Again, a recurrence formula can be obtained: $L(n,t) = L(n-1,t) + L(n,t-1) + 1$ with initial values $L(n,1) = L(1,t) = 1$. This recurrence formula has the solution $L(n,t) = 2\binom{n+t-2}{t-1} - 1 = O(n^{t-1})$. Every non-leaf vertex of the tree is a parent of a leaf, thus the tree in Construction 4.1 has exactly $2L(n,t) = 4\binom{n+t-2}{t-1} - 2 = O(n^{t-1})$ vertices. The number of non-leaf vertices is equal to the number of steps invoked during Algorithm 4.2. Let u and v be two n-ary normal forms in $\mathcal{SL}^t$, then their lengths are at most $O\left(n^t\right)$. Steps (1), (2) and (3) are invoked $O(n^{t-1})$ times, and each time computing the arguments for the next recursive step takes linear time in the length of the term, i.e. $O\left(n^t\right)$ time. Thus Algorithm 4.2 takes $O\left(n^{2t-1}\right)$ time to run on uv. $\square$

5. Explicit formula

The aim of this section is to find explicit formulae for the p_n sequences and the free spectra of the varieties $\mathcal{SL}^t$. The size of the free monoids in the smallest varieties can be determined with high accuracy.

Proposition 5.1. *For the number of n-ary terms in $\mathcal{SL}^1$ and $\mathcal{SL}^2$ we have*

$$p_n(1) = 1 \quad \textit{and} \quad p_n(2) = n! \cdot 2^{\binom{n+1}{2}} \tag{7}$$

Proof. As any element of the free semilattice is determined by the set of its variables, we have $p_n(1) = 1$, and clearly $p_1(2) = 2$ holds. By iterated use of (6) we get

$$
\begin{aligned}
p_n(2) &= n\left(\sum_{k=0}^{n}\binom{n}{k}p_k(1)\right)p_{n-1}(2) = \\
&= n\left(\sum_{k=0}^{n}\binom{n}{k}p_k(1)\right)(n-1)\left(\sum_{k=0}^{n-1}\binom{n-1}{k}p_k(1)\right)p_{n-2}(2) = \\
&= n\left(\sum_{k=0}^{n}\binom{n}{k}p_k(1)\right)(n-1)\left(\sum_{k=0}^{n-1}\binom{n-1}{k}p_k(1)\right)\cdots 1\cdot\left(\sum_{k=0}^{1}\binom{1}{k}\right)p_0(2) = \\
&= n!\prod_{i=1}^{n}\left(\sum_{k=0}^{i}\binom{i}{k}p_k(1)\right) = n!\prod_{i=1}^{n}\left(\sum_{k=0}^{i}\binom{i}{k}\right) = n!\prod_{i=1}^{n}2^i = n!\cdot 2^{\binom{n+1}{2}}
\end{aligned}
$$

□

Corollary 5.2. $|F_{\mathcal{SL}^1}(n)| = 2^n - 1$ *and* $|F_{\mathcal{SL}^2}(n)| = n!\cdot 2^{\binom{n+1}{2}} + O(n!\cdot 2^{\binom{n}{2}})$.

Proof. By formulae (1) and (7) we get

$$
|F_{\mathcal{SL}^1}(n)| = \sum_{i=1}^{n}\binom{n}{i}p_n(1) = \sum_{i=1}^{n}\binom{n}{i} = 2^n - 1.
$$

For $t = 2$ the same arguments yield

$$
\begin{aligned}
|F_{\mathcal{SL}^2}(n)| &= \sum_{i=1}^{n}\binom{n}{i}i!\cdot 2^{\binom{i+1}{2}} = \\
&= n!\cdot 2^{\binom{n+1}{2}} + n(n-1)!\cdot 2^{\binom{n}{2}} + \binom{n}{2}(n-2)!\cdot 2^{\binom{n-1}{2}} + \sum_{i=1}^{n-3}\binom{n}{i}i!\cdot 2^{\binom{i+1}{2}} = \\
&= n!\cdot 2^{\binom{n+1}{2}} + n!\cdot 2^{\binom{n}{2}} + \frac{n!}{2}\cdot 2^{\binom{n-1}{2}} + \sum_{i=1}^{n-3}n(n-1)\cdots(n-i+1)2^{\binom{i+1}{2}} = \\
&= n!\cdot 2^{\binom{n+1}{2}} + O\left(n!\cdot 2^{\binom{n}{2}}\right) + O\left(2^n n!\cdot 2^{\binom{n-2}{2}}\right) = \\
&= n!\cdot 2^{\binom{n+1}{2}} + O\left(n!\cdot 2^{\binom{n}{2}}\right)
\end{aligned}
$$

□

Although for $p_n(2)$ we have a nice closed formula, it is hopeless to get one for $|F_{\mathcal{SL}^2}(n)|$. The case of $\mathcal{SL}^3$ is even more complicated.

Proposition 5.3. *There exist a constant $\alpha > 1$ and a monotone increasing sequence of real numbers $\alpha_n \to \alpha$ such that*

$$p_n(3) = \alpha_n n!(\prod_{i=1}^{n} i!)2^{\binom{n+2}{3}}$$

.

Proof. According to the recurrence formula (6) and formula (7) one can obtain $p_k(3) = p_{k-1}(3)k \sum_{i=0}^{k} \binom{k}{i} i! \cdot 2^{\binom{i+1}{2}}$. To simplify the calculation ε_k be defined by the following:

$$\sum_{i=0}^{k} \binom{k}{i} i! \cdot 2^{\binom{i+1}{2}} = p_k(2)(1+\varepsilon_k). \tag{8}$$

The recurrence formula can be expanded as follows:

$$\begin{aligned} p_n(3) &= p_{n-1}(3)np_n(2)(1+\varepsilon_n) = \\ &= p_{n-2}(3)n(n-1)p_n(2)p_{n-1}(2)(1+\varepsilon_n)(1+\varepsilon_{n-1}) = \cdots = \\ &= n!\left(\prod_{i=2}^{n} p_i(2)(1+\varepsilon_i)\right)p_1(3) = n!\left(\prod_{i=2}^{n} i! \cdot 2^{\binom{i+1}{2}}(1+\varepsilon_i)\right)p_1(3) = \\ &= \frac{3}{2}n!\left(\prod_{i=1}^{n} i!\right)2^{\binom{n+2}{3}}\prod_{i=2}^{n}(1+\varepsilon_i). \end{aligned}$$

From (8) $\varepsilon_k = \sum_{i=0}^{k-1} \frac{1}{(i+1)!}2^{-(i+1)(2k-i)/2} < \sum_{i=0}^{k-1} 2^{-k-i} < 2^{1-k}$ (for $k \geq 2$). Using the inequality $1 + x < e^x$ we obtain

$$\prod_{i=2}^{n}(1+\varepsilon_i) < \prod_{i=2}^{n} e^{\varepsilon_i} < \prod_{i=2}^{n} e^{2^{1-i}} < e.$$

Thus $\alpha_n = \frac{3}{2}\prod_{i=2}^{n}(1+\varepsilon_i) < \frac{3}{2}e$, and the statement holds. □

Note that $\alpha = 1.70506\ldots$

Corollary 5.4. *There exists a sequence of real numbers $\beta_n \to \alpha$ such that $|F_{\mathcal{SL}^3}(n)| = \beta_n n!(\prod_{i=1}^{n} i!)2^{\binom{n+2}{3}}$. In particular,*

$$\log_2 |F_n(3)| = \binom{n+2}{3} + \frac{1}{2\log 2} \cdot n^2 \log n + O(n^2).$$

Proof. By Proposition 5.3 we have $p_k(3) = \alpha_k k!(\prod_{i=1}^{k} i!)2^{\binom{k+2}{3}}$. The first part of the statement holds, since $\frac{|F_{\mathcal{SL}^3}(n)|}{p_n(3)} \to 1$. Indeed,

$$|F_{\mathcal{SL}^3}(n)| = \sum_{i=1}^{n} \binom{n}{i} p_i(3) = \sum_{i=1}^{n} \binom{n}{i} \alpha_i i! \left(\prod_{j=1}^{i} j! \right) 2^{\binom{i+2}{3}} =$$
$$= p_n(3)\left(1 + O(2^{-n(n+1)/2})\right) = p_n(3)(1 + o(1)).$$

For the second part note that the numbers of the form $\prod_{i=1}^{n} i!$ are called superfactorials. From Stirling's formula one can derive the following well-known estimates for the logarithms of factorials and superfactorials.

$$\log_2 \beta_n n! = O(n \log n),$$

$$\log_2 \left(\prod_{i=1}^{n} i! \right) = \frac{1}{2 \log 2} \cdot n^2 \log n + O(n^2). \tag{9}$$

By substituting these to the formula $|F_{\mathcal{SL}^3}(n)| = \beta_n n!(\prod_{i=1}^{n} i!)2^{\binom{n+2}{3}}$, we get

$$\log_2 |F_{\mathcal{SL}^3}(n)| = \binom{n+2}{3} + \frac{1}{2 \log 2} \cdot n^2 \log n + O(n^2).$$

□

Theorem 5.5. *For the p_n sequence of the variety $\mathcal{SL}^t$ the following asymptotic formula holds for $t \geq 3$:*

$$\log_2 p_n(t) = \binom{n+t-1}{t} + \frac{1}{\log 2} \cdot \frac{1}{(t-1)!} \cdot n^{t-1} \log n + O_t(n^{t-1})$$

Proof. Define

$$a_n(t) = \left(\prod_{i_1=1}^{n} \prod_{i_2=1}^{i_1} \cdots \prod_{i_{t-2}=1}^{i_{t-3}} i_{t-2}! \right) 2^{\binom{n+t-1}{t}} \quad \text{and} \quad b_n(t) = e^{n^{t-2} \log n}$$

Now we prove that

$$a_n(t) \leq p_n(t) \leq a_n(t) b_n(t) \qquad \text{for} \quad t \geq 3, n \geq 2, \tag{10}$$

then give an estimate for $\log a_n(t)$.

For the lower bound at first we check the case $n = 2$. Clearly, $a_2(t) = 2^{t+2}$ and $p_2(2) = a_2(2) = 16$, thus $a_2(t) \le p_2(t)$ is true for $t = 2$. By induction on t and the recurrence formula (6)

$$p_2(t) = 2t(p_2(t-1) + 2(t-1) + 1) \ge 2p_2(t-1) \ge 2a_2(t-1) = a_2(t),$$

thus $a_2(t) \le p_2(t)$ holds for every $t \ge 2$.

We prove the inequality $a_n(t) \le p_n(t)$ by induction on t. For $t = 3$ it follows from Proposition 5.3, as $\alpha_n > 1$. Suppose that it is proved for some $t \ge 3$.

The recurrence formula (6) for $p_n(t)$ implies that

$$p_n(t+1) = p_{n-1}(t+1)n\sum_{i=0}^{n}\binom{n}{i}p_i(t) \ge p_{n-1}(t+1)p_n(t) \ge \cdots \ge$$
$$\ge p_n(t)p_{n-1}(t)\cdots p_2(t)p_1(t+1).$$

By the induction hypothesis $a_k(t) \le p_k(t)$ for $2 \le k \le n$ and $p_1(t+1) = t + 1 \ge 2$, and using $\prod_{i=2}^{n} a_i(t) = \frac{1}{2}a_n(t+1)$

$$p_n(t+1) \ge a_n(t)a_{n-1}(t)\cdots a_2(t)\cdot 2 = a_n(t+1).$$

Now we continue with the upper bound of (10). Similarly to the proof of Proposition 5.3 in order to estimate the quotient of the size of the free algebra and $p_n(t)$ define $\eta_k = \frac{|\mathbf{F}_t(k)|}{p_k(t)} = \sum_{i=0}^{k-1}\binom{k}{i}\frac{p_i(t)}{p_k(t)}$. We prove that

$$\prod_{k=2}^{n}\frac{\sum_{i=0}^{k}\binom{k}{i}p_i(t)}{p_k(t)} = \prod_{k=2}^{n}(1+\eta_k) < e. \tag{11}$$

From the recurrence formula (6)

$$\frac{p_{i-1}(t)}{p_i(t)} = \frac{1}{i}\cdot\frac{1}{\sum_{j=0}^{i}\binom{k}{j}p_j(t-1)} \le \frac{1}{i}\cdot\frac{1}{\sum_{j=0}^{i}\binom{k}{j}} = \frac{1}{i2^i}.$$

Then

$$\binom{k}{i}\frac{p_i(t)}{p_k(t)} = \binom{k}{i}\prod_{j=i+1}^{k}\frac{p_{j-1}(t)}{p_j(t)} \le \frac{1}{(k-i)!}\cdot 2^{\binom{i+1}{2}-\binom{k+1}{2}} \le 2^{i-2k+1},$$

so $\eta_k = \sum\limits_{i=0}^{k-1} \binom{k}{i} \frac{p_i(t)}{p_k(t)} < 2^{1-k}$. Using the inequality $1+x<e^x$ we obtain

$$\prod_{i=2}^{n}(1+\eta_i) < \prod_{i=2}^{n} e^{\eta_i} < \prod_{i=2}^{n} e^{2^{1-i}} < e.$$

By proceeding by induction on t we show that $p_n(t) \leq a_n(t)b_n(t)$ if $t \geq 3$, $n \geq 2$ except $t=4$ and $n=2$. For $t=3$ the inequality $p_n(t) \leq a_n(t)b_n(t)$ obviously holds (see Proposition 5.3). By the recurrence formula (6) and the inequality (11)

$$p_n(t+1) = n!(t+1)p_n(t)p_{n-1}(t)\cdots p_2(t) \cdot \prod_{k=2}^{n} \frac{\sum\limits_{i=0}^{k} \binom{k}{i} p_i(t)}{p_k(t)} < \tag{12}$$
$$< n!(t+1)p_n(t)p_{n-1}(t)\cdots p_2(t)e.$$

According to the induction hypothesis we have $p_j(t) \leq a_j(t)b_j(t)$ for any $2 \leq j \leq n$ except the case $t=4$, $j=2$. In this exceptional case, $p_2(4) = 1064$ and $a_2(4)b_2(4) = 1024$, hence $p_2(4) \leq 2a_2(4)b_2(4)$. Applying these estimations we get

$$n!(t+1)p_n(t)p_{n-1}(t)\cdots p_2(t)e < n!(t+1)a_n(t)\cdots a_2(t)b_n(t)\cdots b_2(t)\cdot 2e = \tag{13}$$
$$= a_n(t+1)n!(t+1)e\prod_{i=2}^{n} b_i(t).$$

Hence,

$$p_n(t+1) < a_n(t+1)n!(t+1)e\prod_{i=2}^{n} b_i(t). \tag{14}$$

Now, an estimate for the logarithm of the right-hand side of (14) is going to be given. The function $x^{t-2}\log x$ is increasing, thus for the logarithm of $\prod\limits_{i=2}^{n} b_i(t)$ we get

$$\log\left(\prod_{i=2}^{n} b_i(t)\right) = \sum_{i=2}^{n} i^{t-2}\log i \leq n^{t-2}\log n + \int_2^n x^{t-2}\log x, \tag{15}$$

where

$$\int_2^n x^{t-2}\log x \le \int_2^n x^{t-2}\log x + \frac{1}{t-1}x^{t-2} = \tag{16}$$

$$= \left[\frac{1}{t-1}x^{t-1}\log x\right]_2^n \le \frac{1}{t-1}n^{t-1}\log n.$$

From (15) and (16) we obtain

(17)

$$\log\prod_{i=2}^n b_i(t) \le \left(\frac{1}{n}+\frac{1}{t-1}\right)n^{t-1}\log n = \left(\frac{1}{n}+\frac{1}{t-1}\right)\log b_n(t+1).$$

The following inequality also holds

$$\log(n!(t+1)e) \le n\log n + \log(t+1) + 1 = \tag{18}$$

$$= \left(\frac{1}{n^{t-2}} + \frac{\log(t+1)}{n^{t-1}\log n} + \frac{1}{n^{t-1}\log n}\right)\log b_n(t+1).$$

Taking the logarithm of both sides of (14) and substituting (17) and (18) we obtain that $p_n(t+1) \le a_n(t+1)b_n(t+1)$ if

$$\frac{1}{n}+\frac{1}{t-1}+\frac{1}{n^{t-2}}+\frac{\log(t+1)}{n^{t-1}\log n}+\frac{1}{n^{t-1}\log n} \le 1.$$

This inequality holds, except the cases $(n;t) = (2;3),(2;4),(2;5),(3;3),(4;3)$ (we suppose that $n \ge 2$ and $t \ge 3$). Calculation says that $p_n(t) \le a_n(t)b_n(t)$ holds for the remaining four cases, as well.

Hence, for a fixed t $\log_2 p_n(t) = \log_2 a_n(t) + O(n^{t-2}\log n)$, where

$$\log_2 a_n(t) = \binom{n+t-1}{t} + \log_2\left(\prod_{i_1=1}^{n}\prod_{i_2=1}^{i_1}\cdots\prod_{i_{t-2}=1}^{i_{t-3}} i_{t-2}!\right).$$

Now we show that

(19)

$$\log_2\left(\prod_{i_1=1}^{n}\prod_{i_2=1}^{i_1}\cdots\prod_{i_{t-2}=1}^{i_{t-3}} i_{t-2}!\right) = \frac{1}{\log 2}\cdot\frac{1}{(t-1)!}\cdot n^{t-1}\log n + O_t(n^{t-1}),$$

which proves the statement.

(19) can be proved by induction on t. For $t=3$ this is the estimate for the superfactorials (see (9)). In the induction step it is shown that

$$\sum_{i=1}^n\left(\frac{1}{(t-1)!}\cdot i^{t-1}\log i + O_t(i^{t-1})\right) = \frac{1}{t!}\cdot n^t\log n + O_{t+1}(n^t). \tag{20}$$

From the monotonicity of the function $x^{t-1}\log x$ and estimating the integral on the standard way we get

$$\sum_{i=1}^{n}\frac{1}{(t-1)!}\cdot i^{t-1}\log i=\int_{2}^{n}\frac{1}{(t-1)!}\cdot x^{t-1}\log x+O_{t+1}(n^{t-1}\log n)=$$
$$=\frac{1}{t!}\cdot n^{t}\log n+O_{t+1}(n^{t-1}\log n).$$

As $\sum_{i=1}^{n}O_t(i^{t-1})=O_{t+1}(n^t)$, we obtain (20), hence the statement holds.

□

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