A NOTE ON SYSTEMS OF RECTANGULAR ISLANDS: THE CONTINUOUS CASE

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ABSTRACT. A real-valued height function f is defined on a closed rectangle R. A rectangle S is an f-island if there exists an open set G containing S such that $f(x) < \inf_S f$ for every $x \in G \setminus S$. The set of all f-islands is called a *system of (rectangular) islands.* The discrete version of this notion was introduced by G. Czédli. A system of islands $\mathcal{H}$ is laminar that is for any two $P, Q \in \mathcal{H}$ either $P \subseteq Q$ or $Q \subseteq P$ or $P \cap Q = \emptyset$. In this paper, we prove that there exists a maximal system of islands of cardinality $\aleph_0$ and the size of a maximal laminar system is either countable or continuum.

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1. INTRODUCTION

The concept of systems of rectangular islands was introduced by G. Czédli [1] with motivations from coding theory coming from a paper of S. Földes and N. M. Singhi [2]. Let R be a closed m by n rectangle with vertices at $(0, 0)$, $(m, 0)$, (m, n) and $(0, n)$, where m and n are positive integers. A real number is written in each cell. Let S be a rectangle (which has vertices with integer coordinates, sides parallel to the coordinate axes, $S \subseteq R$), and let s be the least element (or one of the least elements) of S. If in every cell neighbouring S there is a smaller number than s then S is called a rectangular island.

For the size of maximal systems of rectangular islands, upper- and lower bounds were established in [1] and [5]. These results can be summarized as follows: If $\mathcal{H}$ is a maximal system of rectangular islands on an m by n rectangle then

$$m + n - 1 \leq |\mathcal{H}| \leq \lfloor mn + m + n - 1 \rfloor / 2,$$

where both the upper- and the lower bounds are sharp. Several papers have been published on the subject since, investigating various extensions and generalizations (see G. Pluhár [7], and E. K. Horváth, Z. Németh, and G. Pluhár [3]). In this note, we examine the continuous version by allowing the members of $\mathcal{H}$ to have vertices with non-integer coordinates. Now, we give the definition of the system of islands in the continuous case. A real-valued height function f is defined on R. Let S be a rectangle (which has sides parallel to the coordinate axes) and s be the infimum of f on S. If there exists an open set G containing the (closed) rectangle S such that $f(x) < s$ for every $x \in G \setminus S$, then S is an *f-island*. For a real function f, the set of all f-islands is called a *system of (rectangular) islands.*

One of the most useful properties of a system of rectangular islands is laminarity. A system of subsets $\mathcal{H}$ of a set S is called *laminar* if for any two $P, Q \in \mathcal{H}$ either $P \subseteq Q$ or $Q \subseteq P$ or $P \cap Q = \emptyset$. As in the discrete case, in the continuous case a system of islands is always laminar.

Lemma 1. *If $\mathcal{H}$ is a rectangular island system then $\mathcal{H}$ is laminar.*

Proof. Let f be a height function and $\mathcal{H}$ the system of its islands. Assume indirectly that there exist $S_1, S_2 \in \mathcal{H}$ such that $S_1 \nsubseteq S_2$, $S_2 \nsubseteq S_1$ and $S_1 \cap S_2 \neq \emptyset$. As S_1 and S_2 are f-islands there exist open sets G_1 and G_2 such that $S_1 \subseteq G_1$, $S_2 \subseteq G_2$ and $\inf_{S_i} f = s_i > f(x)$ for every $x \in G_i \setminus S_i$ $(i = 1, 2)$. By the assumption, the set $S_1 \cap (G_2 \setminus S_2)$ is nonempty. Let $y_1 \in S_1 \cap (G_2 \setminus S_2)$. Then $s_2 > f(y_1) \geq \inf_{S_1} f = s_1$. Similarly, there exists $y_2 \in S_2 \cap (G_1 \setminus S_1)$ and $s_1 > f(y_2) \geq \inf_{S_2} f = s_2$. Thus $s_2 > s_1$ and $s_1 > s_2$ is a contradiction. $\square$

Every continuous maximal laminar system of rectangles $\mathcal{H}$ contains infinitely many rectangles. Also, the total number of rectangles in $[0, 1]^2$ is continuum, hence we have

$$\aleph_0 \leq |\mathcal{H}| \leq 2^{\aleph_0}.$$

Our aim is to determine what cardinality a maximal laminar system can have. One can easily find examples where the cardinality of $\mathcal{H}$ is continuum.

Example 1. The height function $f(x, y) = -\max(x, y)$ gives the island system $\mathcal{H} = \{[0, a] \times [0, a] : 0 \leq a \leq 1\}$.

Now, we present a countable maximal laminar system of closed intervals, then we prove that every countable maximal laminar system is a system of rectangular islands for some function f. Hence, there is an analogy in that both in the discrete- and the continuous cases we have a big gap between the size of the largest- and the size of the smallest maximal system of rectangular islands. Later we show that contrary to the discrete case, no other cardinalities can occur, i.e. the cardinality of a maximal laminar system is either countable or continuum.

2. Maximal systems of islands

We consider the one-dimensional case from which higher dimensional examples can easily be obtained. At first, we construct a countable maximal system $\mathcal{H}$ of closed intervals satisfying the laminar property.

Proposition 2. *There exists a maximal system of closed intervals $\mathcal{H}$ on $[0,1]$ such that for any two $A, B \in \mathcal{H}$ either $A \subseteq B$ or $B \subseteq A$ or $A \cap B = \emptyset$, and $\mathcal{H}$ is countable.*

Proof. We define H, a set of closed intervals, as follows. For an interval A with endpoints $a^A < b^A$ let (a_n^A) and (b_n^A) be sequences such that (a_n^A) is strictly decreasing, (b_n^A) is strictly increasing, $a_1^A = (3a^A + b^A)/4$, $b_1^A = (a^A + 3b^A)/4$, $\lim_{n\to\infty} a_n^A = a^A$, and $\lim_{n\to\infty} b_n^A = b^A$. Furthermore, define

$$A^f = \{a_n^A : n = 1, 2, ...\} \cup \{b_n^A : n = 1, 2, ...\},$$

$$A^g = \{(a_{n+1}^A, a_n^A) : n = 1, 2, ...\} \cup \{(a_1^A, b_1^A)\} \cup \{(b_n^A, b_{n+1}^A) : n = 1, 2, ...\},$$

and

$$A^h = \{[a_n^A, b_n^A] : n = 1, 2, ...\}.$$

For a set of intervals I, let

$$I^f = \bigcup_{A\in I} A^f,\ I^g = \bigcup_{A\in I} A^g, \text{ and } I^h = \bigcup_{A\in I} A^h.$$

We define the sets F_n, G_n and H_n recursively by letting

$$F_0 = \{0, 1\},\ G_0 = \{(0, 1)\},\ H_0 = \{[0, 1]\},$$

and for $n > 0$ by letting

$$F_n = G_{n-1}^f,\ G_n = G_{n-1}^g \text{ and } H_n = G_{n-1}^h.$$

Finally, we set

$$H = \bigcup_{n=1}^{\infty} H_n.$$

Now, we show that H is a laminar system. Let $A \in H_m$ and $B \in H_n$, $1 \le m \le n$. By the construction, if $m \le n$ then for every $D \in G_n$ there exists a (unique) $C \in G_m$ such that $D \subseteq C$. Then $A = [a_p^C, b_p^C]$ for some $C \in G_{m-1}$, and $B = [a_q^D, b_q^D]$ for some $D \in G_{n-1}$. We either have $D \subseteq C$ or $D \cap C = \emptyset$, and the latter implies $A \cap B = \emptyset$. Assume $D \subseteq C$. One possibility is that $D = C$ (when $m = n$), in which case we are done because C^h itself has the laminar property. Otherwise, $D \subseteq (a_{r+1}^C, a_r^C)$ for some r or $D \subseteq (a_1^C, b_1^C)$ or $D \subseteq (b_r^C, b_{r+1}^C)$ for some r. The first case can happen with $r < p$ or $r \ge p$. If $r < p$ then $D \subseteq A$ which implies $B \subseteq A$. If $r \ge p$ then D and A are disjoint, and so are B

and A. The case when $D \subseteq (b_r^C, b_{r+1}^C)$ is similar. In the remaining case when $D \subseteq (a_1^C, b_1^C)$, we clearly have $D \subseteq A$, and again $B \subseteq A$ follows.

Note that H is not maximal with respect to the laminar property, however, any maximal laminar system which contains H is countable. For an interval A the union of the intervals in A^h covers A except for the endpoints of A. Similarly, the union of the intervals in A^g covers A except for countably many numbers in A (which are a_n^A, b_n^A, a^A and b^A). It follows that for every n the intervals in G_n cover $[0,1]$ except for countably many numbers in $[0,1]$, and the same is true for H_n. Note also that the lengths of the intervals in G_n are bounded by 2^{-n}, and the lengths of the intervals in H_n are bounded by 2^{-n+1}. We use these observations to show if $H \cup \{[x,y]\}$ has the laminar property then $x, y \in F = \bigcup_{n=0}^{\infty} F_n$. Indeed, if $x \notin F$ then for every n we have that x belongs to the interior of some $A_n \in H_n$. For a sufficiently large n the length of A_n will be smaller than that of $[x,y]$ implying a nontrivial intersection for A_n and $[x,y]$. The case when $y \notin F$ is similar.

By Zorn's lemma, there exists a maximal laminar system $\mathcal{H}$ containing H. Since each H_n consists of countably many intervals, their union H contains countably many intervals, as well. The set F is also countable since each F_n is countable. There are countably many intervals $[x,y]$ with both x and y in F, therefore $\mathcal{H}$ is countable. □

Before proving that the size of a maximal laminar system is either countable or continuum, we show that there exists a countable maximal system of islands.

Proposition 3. *Every maximal laminar system $\mathcal{H}$ of cardinality $\aleph_0$ is a maximal island system.*

Proof. Let $\mathcal{H} = (I_n)_{n=1}^{\infty}$ be a maximal system of closed intervals in $[0,1]$ satisfying the laminar property. We show that these are the islands generated by the following height function:

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{3^n} \chi_{I_n}(x),$$

where $\chi_{I_n}(x)$ is the characteristic function of I_n. First, we prove that each $I_k = [a^{I_k}, b^{I_k}]$ is an f-island. Let us denote the interval $[a^{I_k} - \varepsilon, b^{I_k} + \varepsilon] \cap [0,1]$ by I_k^{ε} (where $\varepsilon > 0$). We can choose $\varepsilon > 0$ such that for all $1 \leq j \leq k$ either $I_j \subseteq I_k$ or $I_k \subseteq I_j$ or I_k^{ε} and I_j are disjoint. If

$x \in I_k$, then $f(x) \geq \sum_{n:I_k \subseteq I_n} \frac{1}{3^n}$. If $x \in I_k^\varepsilon \setminus I_k$, then

$$f(x) \leq \sum_{n:I_k \subsetneq I_n} \frac{1}{3^n} + \sum_{n=k+1}^{\infty} \frac{1}{3^n} < \sum_{n:I_k \subsetneq I_n} \frac{1}{3^n} + \frac{1}{3^k} = \sum_{n:I_k \subseteq I_n} \frac{1}{3^n},$$

hence I_k is an f-island. Therefore, the system of f-islands contains $\mathcal{H}$. As $\mathcal{H}$ is a maximal laminar system we have that it is equal to the island system corresponding to f. □

By slightly modifying the construction in Proposition 2 and with a little more work we can obtain an explicit example—one that itself is a *maximal* continuous island system of countable size.

Higher dimensional examples are obtained by expanding the one-dimensional construction as follows: if $\mathcal{H}$ is a maximal continuous island system on the one dimensional $[0,1]$ interval, then

$$\mathcal{H}^{(n)} = \{A \times [0,1]^{n-1} : A \in \mathcal{H}\}$$

is a maximal continuous island system on the n-dimensional cube $[0,1]^n$. Obviously, $\mathcal{H}^{(n)}$ has the same cardinality as $\mathcal{H}$. It is also fairly obvious that if f is a height function of $\mathcal{H}$, then $\tilde{f}(x,y) = f(x)$ (where $(x,y) \in [0,1] \times [0,1]^{n-1}$) is a height function of $\mathcal{H}^{(n)}$, so $\mathcal{H}^{(n)}$ is an island system. We show that it is a maximal laminar system, hence it is a maximal island system, too. Suppose that for an n-dimensional brick $A \subseteq [0,1]^n$ we have $A \notin \mathcal{H}^{(n)}$. Let the projection of A to the first coordinate be $A_{(1)}$, and the projection of A to the last $n-1$ coordinates be $A_{(2)}$. Then either $A_{(1)} \notin \mathcal{H}$ or $A_{(2)} \neq [0,1]^{n-1}$. If the former holds, then for some $B \in \mathcal{H}$ there must be a nontrivial intersection between $A_{(1)}$ and some $C \in \mathcal{H}$. However, the intersection of A and $C \times [0,1]^{n-1}$ is also nontrivial. Otherwise, we can assume $A_{(1)} \in \mathcal{H}$ and $A_{(2)} \neq [0,1]^{n-1}$. By the maximality of $\mathcal{H}$ we have some $D \in \mathcal{H}$ such that D is a proper subinterval of $A_{(1)}$, and the intersection of A and $D \times [0,1]^{n-1}$ is nontrivial.

The following proposition shows that no cardinality can occur as the size of a maximal laminar system other than $\aleph_0$ or continuum.

Proposition 4. *If $\mathcal{H}$ is a maximal system of closed intervals on $[0,1]$ such that for every $A, B \in \mathcal{H}$ either $A \subseteq B$ or $B \subseteq A$ or $A \cap B = \emptyset$ then the cardinality of $\mathcal{H}$ is $\aleph_0$ or continuum.*

Proof. We have seen that $\aleph_0 \leq |\mathcal{H}| \leq 2^{\aleph_0}$, and there are maximal sets of intervals satisfying the laminar property of sizes both $\aleph_0$ and continuum. We prove that no other cardinality can occur.

Let $\mathcal{C}$ be a maximal chain in $\mathcal{H}$. Let J_i $(i \in S)$ be intervals in $\mathcal{C}$. We show that $J = \bigcap_{i\in S} J_i$ is either a single point or J itself is an interval in $\mathcal{C}$. The chain $\mathcal{C}$ consists of closed intervals, hence if J is not a point, then it is a proper closed interval. We shall prove that in the latter case $\mathcal{H} \cup \{J\}$ satisfies the laminar property and $\mathcal{C} \cup \{J\}$ is a chain. Let $I \in \mathcal{H}$. If $I \cap J \neq \emptyset$ then $I \cap J_i \neq \emptyset$ for all $i \in S$. As $\mathcal{H}$ is a laminar system, for every $i \in S$ either $J_i \subseteq I$ or $I \subseteq J_i$ holds. If there exists an $i \in S$ such that $J_i \subseteq I$, then $J \subseteq I$. If $I \subseteq J_i$ for every $i \in S$, then $I \subseteq \bigcap_{i\in S} J_i = J$. Hence $\mathcal{H} \cup \{J\}$ is a laminar system and consequently $J \in \mathcal{H}$. Assume that $I \in \mathcal{C}$. Then either $I \subseteq J$ or there exists an $i \in S$ such that $J_i \subseteq I$. We have $I \cap J \neq \emptyset$ in both cases, thus $J \subseteq I$ or $I \subseteq J$. Therefore, $\mathcal{C} \cup \{J\}$ is a chain. As $\mathcal{C}$ is maximal, we have $J \in \mathcal{C}$.

We prove that the cardinality of $\mathcal{C}$ is countable or continuum. Let R denote the set of right endpoints of the intervals in $\mathcal{C}$ and let b be a right limit point of R. As $\mathcal{C}$ is a chain, $\bigcap_{b\in J\in\mathcal{C}} J$ is a non-empty closed interval, hence either $b = \bigcap \mathcal{C} = \inf R$ or $\bigcap_{b\in J\in\mathcal{C}} J = [a, b]$ for some $0 \leq a < b$. Hence we get that $[a, b] \in \mathcal{C}$ and $b \in R$ or $b = \inf R$. Let clR be the closure of R and $x \in clR \setminus R$. Then either x is a left but not right limit point of R or $x = \inf R$. For every x that is not a right limit point of R, there exists a $y > x$ rational number such that $(x, y) \cap R = \emptyset$. Clearly, for distinct left (but not right) limit points the corresponding rational number y is different, therefore the cardinality of the set of left but not right limit points is countable.

Hence, it follows that $clR \setminus R$ is countable. It is well-known that the cardinality of a closed set in $\mathbb{R}$ is either countable or continuum, hence $|R| \leq \aleph_0$ or $|R| = 2^{\aleph_0}$. The same holds for L, the set of left endpoints of the elements of $\mathcal{C}$, as well. If the cardinality of L or R is continuum then the cardinality of $\mathcal{C}$ is continuum, too. If $|L| \leq \aleph_0$ and $|R| \leq \aleph_0$ then $|\mathcal{C}| \leq |L| \cdot |R| \leq \aleph_0$. Therefore the cardinality of a maximal chain is countable or continuum.

For showing that $|\mathcal{H}| = \aleph_0$ or $|\mathcal{H}| = 2^{\aleph_0}$ we shall distinguish two cases depending on the length of the maximal chains in $\mathcal{H}$. At first, if the cardinality of a maximal chain is continuum, then clearly $|\mathcal{H}| = 2^{\aleph_0}$. Secondly, suppose that every chain has only countably many elements. For an $r \in [0, 1]$ the laminar property implies that the set $\mathcal{C}_r = \{I : r \in I \in \mathcal{H}\}$ is a chain, hence its cardinality is countable. Every proper interval contains a rational point, thus $\mathcal{H} = \bigcup_{r\in[0,1]\cap\mathbb{Q}} \mathcal{C}_r$, and we have that $\mathcal{H}$ is the union of countably many sets of countable size, hence $|\mathcal{H}|$ is countable, as well. $\square$

3. Further remarks

The statement of Proposition 2 remains true if we replace closed intervals by open ones. Indeed, the entire argument will be valid if we modify the definition of A^h to be

$$A^h = \{(a_n^A, b_n^A) : n = 1, 2, ...\}.$$

If we allow open or half-open intervals, there exist some simpler constructions. For example, the collection of dyadic intervals

$$\left(\frac{k}{2^n}, \frac{k+1}{2^n}\right)$$

with $k = 0, 1, ..., 2^n - 1$, and $n = 0, 1, 2, ...$ form a maximal laminar system. Similarly, half-open intervals as well as mixed intervals (closed or open or half-open) could be used.

In his original paper, G. Czédli [1] introduced island systems for rectangles, but at least two additional shapes were considered by others. G. Pluhár, E. K. Horváth, and Z. Németh [3] investigated triangles (on a triangular grid), and in E. K. Horváth, G. Horváth, Z. Németh, and Cs. Szabó [4] and also in [6], the case of squares was examined. Allowing objects other than rectangles seems even more compelling in the continuous case. However, we do not have an argument to show that in the plane there exists a countable set of, say, circles that would form a maximal island system.

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