Theorem

Assume that f is continuous and 0 outside of $[-A, A]$, and assume that the differential entropy of f:

$$H(f) = -\int_{-A}^{A} f(x) \log f(x)\, dx \quad \text{is finite}$$

Then the N level uniform quantizer (for the entropy $H(Q_N)$ of)

$$\lim_{N \to \infty} \left(H(Q_N) + \log q_N \right) = H(f)$$

So for N large enough $H(Q_N) \approx -\log q_N + H(f)$

$$= \log N \underbrace{- \log 2A + H(f)}_{\text{constant}}$$

Proof

From the continuity of f:

in each $[y_{N,i-1}; y_{N,i}]$ ~~fil~~ there is a point z_i
(interval)

such that $f(z_i) = \dfrac{1}{q_N} \displaystyle\int_{y_{N,i-1}}^{y_{N,i}} f(x)\, dx$ $\quad (*)$

$$H(Q_N) = -\sum_{i=1}^{N} Pr\{Q(x) = x_i\} \log Pr\{Q(x) = x_i\}$$

$$Q(x) = x_i \iff \begin{array}{l} x \in B_i \\ x \in [y_{N,i-1} ; y_{N,i}] \end{array} \implies Pr\{Q(x) = x_i\} = \int_{B_i} f(x)\,dx$$

$$H(Q_N) = -\sum_{i=1}^{N} \left(\int_{y_{N,i-1}}^{y_{N,i}} f(x)\,dx \right) \log \left(\int_{y_{N,i-1}}^{y_{N,i}} f(x)\,dx \right)$$

$$= -\sum_{i=1}^{N} (q_N f(z_i)) \log (q_N f(z_i)) \qquad \text{from } (*)$$

$$= \underbrace{-\sum q_N f(z_i) \log q_N}_{} - \underbrace{\sum q_N f(z_i) \log f(z_i)}_{\lim_{N\to\infty}}$$

$$-\log q_N \sum q_N f(z_i)$$

$$= -\log q_N \sum_{i=1}^{N} \int_{y_{N,i-1}}^{y_{N,i}} f(x)\,dx$$

$$= -\log q_N \int_{-A}^{A} f(x)\,dx = -\log q_N$$

$$= \lim_{N\to\infty} -q_N \sum_{i=1}^{N} f(z_i) \log f(z_i)$$

$$= -\int_{-A}^{A} f(x) \log f(x)\,dx = H(f)$$

appr the integral $\to$ tends to the integral

$$\implies \lim_{N\to\infty} H(Q_N) = -\log q_N + H(f) \qquad \square$$

Since $D(Q_N) \approx \frac{q_N^2}{12}$

$$\implies \lim_{N\to\infty} \left(H(Q_N) + \log \sqrt{12\, D(Q_N)} \right) = H(f)$$

i.e. $$H(Q_N) \approx H(f) - \log \sqrt{12\, D(Q_N)}$$

Differential entropy of some distributions:

1) uniform distribution on $[0, a]$

$$f(x) = \begin{cases} 1/a & x \in [0, a] \\ 0 & \text{otherwise} \end{cases}$$

$$H(f) = -\int_0^a f(x) \log f(x)\, dx = -\int_0^a \frac{1}{a} \log \frac{1}{a}\, dx$$

$$= -\log \frac{1}{a} = \log a$$

2) normal distr. $N(0, \sigma^2)$

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}}$$

$$H(f) = -\int_{-\infty}^{\infty} f(x) \log f(x)\, dx = -\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}} \cdot$$

$$\cdot \left(\log \frac{1}{\sqrt{2\pi\sigma^2}} + \frac{-x^2}{2\sigma^2} \log e \right) dx = -\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}} \cdot \log \frac{1}{\sqrt{2\pi\sigma^2}}\, dx$$

$$+ \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}} \cdot \frac{x^2}{2\sigma^2} \log e\, dx =$$

$$= -\log \frac{1}{\sqrt{2\pi\sigma^2}} + \frac{1}{2} \log e = \frac{1}{2} \log(2\pi\sigma^2 e)$$

since $\int f(x)\, dx = 1$

and $\int x^2 f(x)\, dx = \sigma^2$

$\left(\ln 2\, H(f) = \frac{1}{2} \ln(2\pi e \sigma^2)\right)$

$\mathbb{D}^2 X = \mathbb{E}X^2 - (\mathbb{E}X)^2$

if this is 0, then $\mathbb{D}^2 X = \mathbb{E}X^2$

3) exponential distribution (λ)

$$f(x) = \lambda e^{-\lambda x} \quad \text{for } x \geq 0$$

$$\int_0^\infty x f(x)\, dx = \frac{1}{\lambda} \quad \text{expected value}$$

easier to calculate $\ln 2\, H(f)$ first

$$\ln 2\, H(f) = -\int_0^\infty f(x) \ln f(x)\, dx = \cancel{-\int f(x)}$$

$$= -\int_0^\infty \lambda e^{-\lambda x} \cdot (\ln \lambda - \lambda x)\, dx$$

$$= \underbrace{-\int_0^\infty \lambda e^{-\lambda x} \ln \lambda\, dx}_{-\ln \lambda} + \underbrace{\int_0^\infty \lambda^2 x e^{-\lambda x}\, dx}_{\lambda \underbrace{\int_0^\infty x \cdot \lambda e^{-\lambda x}\, dx}_{\frac{1}{\lambda}}}$$

$$= -\ln \lambda + 1 = \ln \frac{e}{\lambda}$$

$$\Rightarrow \quad H(f) = \log \frac{e}{\lambda}$$

Both (all three) can be negative if ~~it is~~ the distr. is concentrated around 0.

Lemma

For any density functions f and g

$$-\int f(x) \log f(x)\, dx \le -\int f(x) \log g(x)\, dx$$

Like in case of discrete distr., consequence of the Jensen-inequality

Proof

$$\int_{-\infty}^{\infty} f(x) \log \frac{g(x)}{f(x)}\, dx = \mathbb{E}_f\left(\log \frac{g(X)}{f(X)}\right) \underset{\text{Jensen}}{\le} \log \mathbb{E}\left(\frac{g(X)}{f(X)}\right)$$

$$= \log\left(\int_{-\infty}^{\infty} f(x) \frac{g(x)}{f(x)}\, dx\right) = \log\left(\int_{-\infty}^{\infty} g(x)\, dx\right) = \log 1 = 0 \qquad \square$$

Theorem

Let $\sigma > 0$, $f(x)$ density function such that

$$\int x f(x)\,dx = 0, \quad \int x^2 f(x)\,dx = \sigma^2$$

zero mean, σ variance. The Normal distr. has maximal differential entropy:

$$H(f) \le H(\varphi) = \frac{1}{2}\log(2\pi e \sigma^2)$$

where $\varphi \sim N(0, \sigma)$

Proof

$$H(f) = -\int f(x)\log f(x)\,dx \underset{\text{Jensen}}{\le} -\int f(x)\log\varphi(x)\,dx$$

$$\ln 2\, H(f) \le -\int f(x)\ln\left(\frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{x^2}{2\sigma^2}}\right)dx$$

$$= -\int f(x)\ln\frac{1}{\sqrt{2\pi\sigma^2}}\,dx + \frac{1}{2\sigma^2}\underbrace{\int f(x)x^2\,dx}_{\sigma^2}$$

$$= \ln\sqrt{2\pi\sigma^2} + \frac{1}{2} = \frac{1}{2}\ln(2\pi\sigma^2 e) = \ln 2\, H(\varphi).$$

$\square$