Ninth Lecture
November 13, 2023

Solution of the Home work
Let X be a random variable that takes its values on the finite set $\{1,2,3,4\}$ with uniform distribution. (That is $P(X=1)=P(X=2)=P(X=3)=P(X=4)=1/4$.) Calculate the distortion of the following three quantizers:

$$Q_1(1)=1, Q_1(2)=Q_1(3)=Q_1(4)=3;$$

$$Q_2(1)=Q_2(2)=1.5, Q_2(3)=Q_2(4)=3.5;$$

$$Q_3(1)=Q_3(2)=Q_3(3)=2, Q_3(4)=4.$$

It takes an easy calculation to check that $D(Q_1)=D(Q_3)=0.5$, while $D(Q_2)=0.25$. Thus only Q_2 is optimal, although neither of Q_1 and Q_3 can be improved by the Lloyd-Max algorithm.

Uniform quantizer

The simplest quantizer is the uniform quantizer, we investigate it a bit closer. Note that we do not assume now that the distribution we work with is uniform. For simplicity we assume, however, that the density function of our random variable to be quantized is 0 outside the interval $[-A,A]$, and it is continuous within $[-A,A]$. The N-level uniform quantizer is defined by the function

$$Q_N(x) = -A + (2i-1)\frac{A}{N}$$

whenever

$$-A + 2(i-1)\frac{A}{N} < x \leq -A + 2i\frac{A}{N}.$$

(To be precise: for $x=-A$ we also have $Q_N(-A) = -A + \frac{A}{N}$.)
The length of each interval for the elements of which we assign the same value is then $q_N = \frac{2A}{N}$. The following theorem gives the distortion of the uniform quantizer asymptotically (as N goes to infinity) in terms of q_N.

Theorem 1 *If the density function f of the random variable X satisfies the above requirements (continuous in $[-A,A]$ and 0 outside it) then for the distortion of the N-level uniform quantizer Q_N we have*

$$\lim_{N\to\infty} \frac{D(Q_N)}{q_N^2} = \frac{1}{12}.$$

Proof. We will use the following notation. The extreme points of the quantization intervals are

$$y_{N,i} = -A + 2i\frac{A}{N},\ i = 0,1,\ldots,N,$$

while the quantization levels are

$$x_{N,i} = -A + (2i-1)\frac{A}{N},\ i = 1,2,\ldots,N.$$

With this notation the distortion can be written by definition as

$$D(Q_n) = \sum_{i=1}^{N} \int_{y_{N,i-1}}^{y_{N,i}} (x - x_{N,i})^2 f(x)dx.$$

We define the auxiliary density function $f_N(x)$ as

$$f_N(x) := \frac{1}{q_N} \int_{y_{N,i-1}}^{y_{N,i}} f(z)dz \quad \text{if } x \in (y_{N,i-1}, y_{N,i}].$$

First we calculate the distortion $\hat{D}(Q_N)$ of Q_N with respect to this auxiliary density function.

$$\hat{D}(Q_N) = \sum_{i=1}^{N} \int_{y_{N,(i-1)}}^{y_{N,i}} (x - x_{N,i})^2 f_N(x)dx =$$

$$\sum_{i=1}^{N} \frac{1}{q_N} \int_{y_{N,(i-1)}}^{y_{N,i}} f(z)dz \int_{y_{N,(i-1)}}^{y_{N,i}} (x - x_{N,i})^2 dx =$$

$$\sum_{i=1}^{N} \frac{1}{q_N} \int_{y_{N,(i-1)}}^{y_{N,i}} f(z)dz \int_{-\frac{q_N}{2}}^{\frac{q_N}{2}} x^2 dx =$$

$$\frac{q_N^2}{12} \sum_{i=1}^{N} \int_{y_{N,(i-1)}}^{y_{N,i}} f(z)dz = \frac{q_N^2}{12}.$$

To finish the proof we will show that

$$\lim_{N\to\infty} \frac{\hat{D}(Q_N) - D(Q_N)}{\hat{D}(Q_N)} = \lim_{N\to\infty} \frac{\hat{D}(Q_N) - D(Q_N)}{q_N^2/12} = 0,$$

that is clearly enough.

Since f is continuous in the closed interval $[-A, A]$ it is also uniformly continuous. Thus for every $\varepsilon > 0$ there exists N_0 such that if $N \geq N_0$ then $|f(x) - f(x')| < \varepsilon$ whenever $x, x' \in (y_{N,(i-1)}, y_{N,i})$ (since $|y_{N,(i-1)} - y_{N,i}| < q_N$, and $q_N \to 0$ as $N \to \infty$).
So for $N \geq N_0$ we can write

$$\frac{|\hat{D}(Q_N) - D(Q_N)|}{q_N^2/12} =$$

$$\frac{12}{q_N^2} \left| \sum_{i=1}^{N} \int_{y_{N,(i-1)}}^{y_{N,i}} (x - x_{N,i})^2 f(x)dx - \sum_{i=1}^{N} \int_{y_{N,(i-1)}}^{y_{N,i}} (x - x_{N,i})^2 f_N(x)dx \right| \leq$$

$$\frac{12}{q_N^2} \sum_{i=1}^{N} \int_{y_{N,(i-1)}}^{y_{N,i}} (x - x_{N,i})^2 |f(x) - f_N(x)|dx \leq$$

$$\frac{12}{q_N^2} \sum_{i=1}^{N} \int_{-q_N/2}^{q_N/2} z^2 \varepsilon dz = \frac{12}{q_N^2} N \frac{q_N^3}{12} \varepsilon = q_N N \varepsilon = \frac{2A}{N} N \varepsilon = 2A\varepsilon$$

that can be made arbitrarily small by choosing ε small enough. This completes the proof. □