Parameterized complexity of candidate nomination for elections based on positional scoring rules

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Abstract

Consider elections where the set of candidates is partitioned into parties, and each party must nominate exactly one candidate. The Possible President problem asks whether some candidate of a given party can become the unique winner of the election for some nominations from other parties. We perform a multivariate computational complexity analysis of Possible President for several classes of elections based on positional scoring rules. We consider the following parameters: the size of the largest party, the number of parties, the number of voters and the number of voter types. We provide a complete computational map of Possible President in the sense that for each choice of the four possible parameters as (i) constant, (ii) parameter, or (iii) unbounded, we classify the computational complexity of the resulting problem as either polynomial-time solvable or NP-complete, and for parameterized versions as either fixed-parameter tractable or W[1]-hard with respect to the parameters considered.

Keywords: computational social choice, candidate nomination, positional scoring rules, parameterized complexity

1 Introduction

There are several situations in the life of a society where various interest groups are engaged in an election with the aim to maximize the chance of the victory of their preferred candidate. Perhaps the most important and famous ones are presidential elections in a country. Here, obviously, political parties prefer to have a person in this position who will best support their political program. Similarly, in elections of academic officials like chairs of departments, deans of faculties or rectors of universities, individual research groups or institutes nominate a candidate so as to have a person in the respective position whom they consider to be able to best represent their interests. We can also think of sports events, like Alpine Ski or Canoe-Kayak World Cup, where countries nominate their participating athletes. Recall that such international circuits of competitions can also be studied as elections with every race representing a voter that awards a certain number of points to each racer on the basis of the position taken in the race. At the end of the season the racer with the most points is the winner.

In some cases, primary elections or internal nomination races are held to choose the candidates that will represent their parties or countries in an upcoming general election or world cup season. However, these take into account only voters belonging to the party or competitions only between athletes of the given country. Therefore, to increase the chances of victory for the party's or the country's candidate, it is more useful to take into account the preferences of all voters.

A formal model of such situations was proposed for the first time by Faliszewski et al. [20]. They assumed that the set of candidates is partitioned into disjoint subsets called *parties* and that the preferences of all voters over all potential candidates are known. Each party nominates a single candidate for the election from its pool of candidates. Faliszewski et al. studied two problems in this setting. Possible President asks whether a given party can nominate one of its candidates in such a way that he or she can become the winner of the election for some nominations from other parties. Necessary President wants to decide whether some nominee of the party will be the winner irrespective of the other nominations.

In this paper we investigate the computational complexity of the Possible President problem in detail using the framework of parameterized complexity. For readers unfamiliar with this framework, we refer to Section 2.2 for the basic concepts of parameterized complexity used in this paper, and to the books [13, 16] for more background.

1.1 The Possible President problem

We start with discussing what is known about the Possible President problem. Faliszewski et al. [20] concentrated on Plurality elections. For this voting rule they proved that Possible President is NP-complete and Necessary President is coNP-complete in case of unrestricted preferences. Refining these hardness results, they showed that Necessary President admits a polynomial-time algorithm for single-peaked profiles. By contrast, Possible President remains NP-complete for single-peaked and even 1D-Euclidean profiles, but admits a polynomial-time algorithm if the elections are restricted to single-peaked profiles where the candidates of any party appear consecutively on the societal axis.

The results of Faliszewski et al. [20] have been extended to other voting rules. Lisowski [29] dealt with tournament solutions and showed that Possible President for the Condorcet rule can be solved in polynomial time but is NP-complete for the Uncovered Set rule. Cechlárová et al. [10] studied the Possible President problem under positional scoring rules, among them ℓ-Approval, ℓ-Veto and Borda; and under Condorcet-consistent rules Copeland, Llull and Maximin. In addition, their paper provides integer programs for the Possible President and the Necessary President problems for all studied voting rules, as well as computational experiments with these integer programs applied to real and synthetic elections.

Misra [31] has initiated the study of the parameterized complexity of Possible President for Plurality. She examined the number t of parties as the parameter, and proved that the problem is W[2]-hard and in XP (i.e., solvable in polynomial time for a constant number of parties), and becomes fixed-parameter tractable (FPT for short) with parameter t when restricted to 1D-Euclidean preference profiles. She also strengthened the results of Faliszewski et al. [20] by proving that Possible President for Plurality is NP-hard even if all parties have size at most two, and the preferences are both single-peaked and single-crossing; hence, the problem is para-NP-hard when parameterized by the size of the largest party even on a very restricted domain. Misra has asked whether Possible President for Plurality is fixed-parameter tractable when parameterized by the number of voters; we will show that this is not the case.

1.2 Our parameters

Instead of focusing on a single parameter and a single voting rule, we examine the Possible President problem for a wide range of voting rules and investigate four possible parameters and their combinations. We perform a detailed multivariate complexity analysis of the Possible President problem considering the following natural parameters:

- s: the size of the largest party;
- t: the number of parties;
- $|V|$: the number of voters;
- τ: the number of voter types, where two voters have the same *type* if their preferences are the same.

The first three parameters are arguably the most natural parameters in connection to the Possible President problem. The parameter "number of voter types" can be interesting in cases when several voters have the same preferences; such a situation might arise as a consequence of assigning weights to voters. These four parameters have a significant effect on the computational complexity of our problem: e.g., it is not hard to see that instances of Possible President with only a constant number of parties can be solved in polynomial time for any efficiently computable voting rule (a simple brute force approach yields a running time of $|I|^{O(t)}$ for an instance I; see Proposition 1); by contrast, restricting the maximum size of a party to two still yields NP-hardness even for Plurality [20]. This motivates us to explore how exactly these parameters influence the computational complexity of the Possible President problem.

We remark that depending on the situation at hand, it is reasonable to assume that some of these parameters have small value: a party might not have too many candidates who are willing to be nominated (resulting in small s), there may only be a few parties (resulting in small t), and it is even possible that there are only a limited number of voters, e.g., when the rector of a university is elected by the members of the academic senate (resulting in small $|V|$).

1.3 Our contribution

We concentrate on positional scoring rules that, for an election over t candidates, are described by a scoring vector $(a_1, a_2, \dots, a_t)$ with $a_1 \geq a_2 \geq \cdots \geq a_t$; the interpretation of such a scoring vector is that each voter assigns a_i points to the candidate they rank at the i^{th} position for each $1 \leq i \leq t$, and the winners of the election are the candidates who obtain the most points.

We consider the following classes of positional scoring rules; see Section 2 for precise definitions and further details:

- *Short scoring rules*, whose scoring vector contains only a constant number of nonzero entries, say ℓ. Short scoring rules include ℓ-Approval, and its special case Plurality.
- *Veto-like scoring rules*, whose scoring vector starts with several entries equal to a constant, say a, and then a constant number of strictly smaller entries. Such scoring rules include ℓ-Veto and its special case Veto.
- (ℓ, ℓ')-*Approval& Veto*: these are scoring rules that are neither short nor Veto-like, but can be interpreted as a combination of ℓ-Approval and ℓ'-Veto, where each voter has ℓ approvals and ℓ' vetoes, possibly with different weights, to distribute among the nominated candidates. More precisely, we consider voting rules based on scoring vectors of the form $(a_1, a_2, \dots, a_\ell, a, a, \dots, a, a'_1, a'_2, \dots, a'_{\ell'})$ for positive integers ℓ, ℓ'.
- *Borda*, a classic scoring rule whose scoring vector is $(t-1, t-2, \dots, 1, 0)$ for an election with t nominated candidates.

For each of the considered scoring rules, we provide a complete computational map in the following sense: for each choice of the four variables $s, t, |V|$, and τ as being (i) constant, (ii) a parameter, or (iii) unbounded, we determine the computational complexity of the Possible President problem as either fixed-parameter tractable or W[1]-hard[1] in the case of parameterized versions, while for versions without parameters we either show NP-hardness or polynomial-time solvability. Hence, our results yield a complete computational description of Possible President for each of the voting rules that fall within the above classes. See Figure 1 for a summary of our results.

Note that the number of voter types is always at most the number of voters, that is, $\tau \leq |V|$. This means that any algorithmic result concerning the parameter τ holds for $|V|$ as well, but the reverse is not necessarily true: an algorithm for, e.g., a constant number of voters does not automatically solve the "weighted" version where only the number of voter types is constant. Similarly, parameterized hardness for parameter $|V|$

[1] In some cases we even show W[2]-hardness which, however, implies W[1]-hardness.

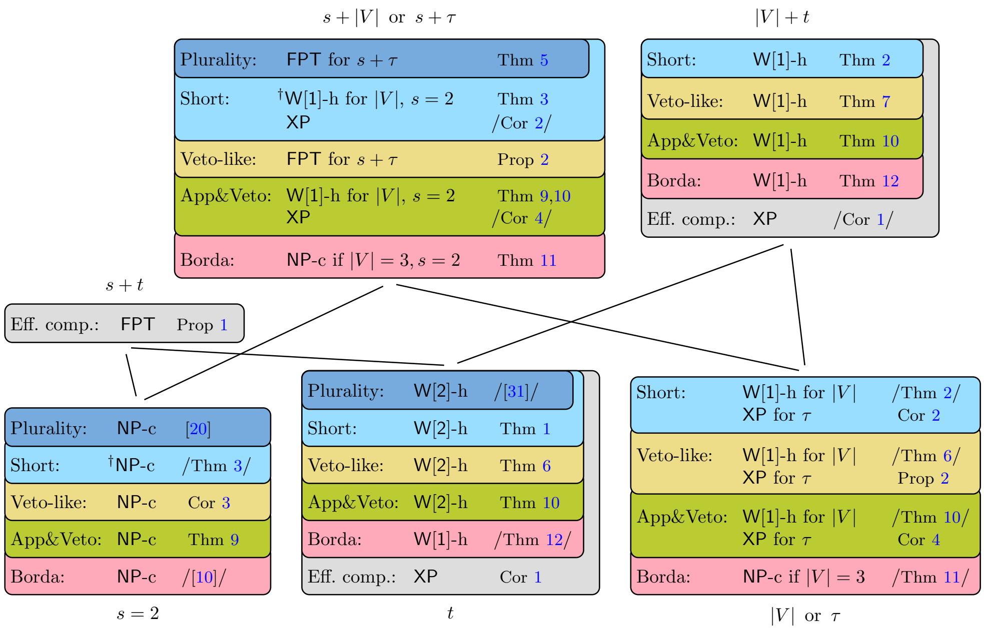

Fig. 1 The complexity landscape of Possible President for positional scoring rules. Results for a given parameterization are grouped together, with the parameter displayed above or under the group. "App&Veto" refers to (ℓ, ℓ')-Approval&Veto with $\ell \geq 1$ and $\ell' \geq 1$, and results marked with the symbol † only hold for short voting rules whose scoring vectors have at least two non-zero positions. "Eff. comp." refers to scoring rules where winner determination is in P. Results that can be inferred from some other result in the figure have references enclosed within slashes $(/\cdots/)$. Note that an algorithmic result for a given class $\mathbb{V}$ of voting rules implies the same result for any subclass of $\mathbb{V}$; this is reflected in the figure as containment between the corresponding rectangles.

implies hardness for parameter τ automatically. Notably, in each case we were able to show the stronger result: all our algorithmic results hold for the parameter τ, and all our intractability results hold for the parameter $|V|$.

Among others, we show the following:

- All efficiently computable voting rules are FPT with respect to $s + t$.
- Possible President for short scoring rules and for (ℓ, ℓ')-Approval&Veto for positive integers ℓ and ℓ' is
  - W[2]-hard when parameterized by t,
  - W[1]-hard with respect to the combined parameter $|V| + t$,
  - W[1]-hard with respect to $|V|$ even if $s = 2$, except for Plurality, but
  - in XP with respect to τ.
- Possible President for Plurality is FPT with respect to $s + \tau$.
- Possible President for Veto-like voting rules is FPT with respect to $s + \tau$.
- Possible President for Borda is
  - NP-hard even if $s = 2$ and $|V| = 3$, and
  - W[1]-hard with respect to t even if $|V| = 6$.

1.4 Related work

We have already discussed all papers that directly deal with the Possible President problem in Section 1.1. We next take a look at research that focuses on different, mostly game-theoretic, aspects of candidate nomination. For a broader view, we also place our work within the literature that concerns elections where, contrary to the classical setup in the theory of voting, some information about the candidates, or voters' preferences over them is missing.

Strategic candidate nomination

Dutta et al. [18] investigated the possibility of strategic behavior of candidates in a setting where each candidate can individually decide to run in the election or to withdraw from candidacy. They found that all non-dictatorial voting rules satisfying unanimity are vulnerable to strategic candidacy. Their concept of *candidate stability* inspired a long line of research, see e.g., [9, 19, 26, 33, 34]; however, the model used in these works does not involve parties and thus differs crucially from our model.

Ding and Lin [15] examined questions about the strategic behavior of parties in a model describing the electoral system in Hong Kong where voters cast their votes over disjoint lists of candidates, and parties can cooperate to form joint lists in order to maximize the number of seats obtained. Under certain assumptions Ding and Lin proved that a pure Nash equilibrium always exists for a two-party election, but is NP-hard to compute. Lin et al. [28] formulated the *two-party election game* that models candidate nomination strategies based on utilities; they examined questions about Nash equilibria and the price of anarchy.

Harrenstein et al. [23] explored the algorithmic properties of Nash equilibria in the Hotelling-Downs model. In this model, the set of voters as well as candidates are arranged along the line. The candidates of each party have predetermined positions on the line, and parties strategize over which candidate to select to attract as many voters as possible under the assumption that each voter votes for the candidate closest to her. Harrenstein et al. show that deciding the existence of a Nash equilibrium is NP-complete for the general case. For two parties, in a discrete version of the model a Nash equilibrium is guaranteed to exist; for the continuous version the problem of checking if a Nash equilibrium exists is linear-time solvable. Deligkas et al. [14] extended this model by assuming that each candidate comes at a different cost and the profit of a party is the number of votes they obtain minus the cost of its nominee. Deligkas et al. examined the parameterized complexity of deciding whether a pure Nash equilibrium exists for this model under several parameters: the number of different positions of the candidates, the discrepancy and the span of the nominees, and the overlap of the parties.

Uncertainty about the candidates

Bartholdi et al. [2] introduced *candidate control* where the chair of an election knows the preferences of voters over the set of candidates, and in order to make a distinguished candidate a winner, can modify the set of candidates that are allowed to run in the election. Chevaleyre et al. [12] considered a different model where voters express their preferences about a set of initial candidates, and afterwards some new candidates arrive; as opposed to candidate control, there is no information about the ranking of new candidates. Chevaleyre et al. studied the problem of identifying those initial candidates who can still be winners once voters' preferences about all candidates are revealed. They showed that computing possible winners can be done in polynomial time for Borda, as well as for ℓ-Approval if $\ell \leq 2$ or if the number or new arrivals is at most two. By contrast, if $\ell = 3$ or if there are at least three new candidates, then the problem becomes NP-complete; they show computational hardness also for the scoring rule defined by the vector $(3, 2, 1, 0, \ldots, 0)$. This NP-completeness result was generalized by Baumeister et al. [4] to scoring vectors of the form $(\alpha_1, \alpha_2, 1, 0, \ldots, 0)$ with $\alpha_1 > \alpha_2 > 1$.

Lou and Boutilier [30] developed the *unavailable candidate* model where only a subset of the candidates is available, but the unavailability of some candidate c is revealed only when c is announced a winner; the task is to construct a policy that gets as close as possible to selecting the "real" winner (in the hypothetical election where only available candidates run), in terms of expectation; see also [1, 8].

Uncertainty about the preferences of voters

Note that in Possible President the uncertainty over the nominated candidates can also be interpreted as uncertainty about the preferences of the voters. Konczak and Lang [25] introduced a model where voters' preferences are *incomplete*, described by a partial ordering, i.e., a transitive and asymmetric but not necessarily total binary relation over the set of candidates. Konczak and Lang were the first to use the notions of *possible winner*[2] and *necessary winner* for those candidates that are the winner of some election or all elections, respectively, obtained via some linear extension of the partial preferences. Note that in this setting, each vote can be extended to a total linear order over the set of candidates independently of other votes, contrasting the Possible President model where deciding on a nominee has consequences for each vote. The paper by Konczak and Lang [25] stimulated a fruitful research area centered around the Possible Winner and Necessary Winner problems, which ask whether a given candidate is a possible or necessary winner. A series of papers [3, 6, 36, 37] established a full dichotomy for the Possible Winner problem by proving that the problem is polynomial-time solvable for Plurality and Veto, and is NP-complete for all remaining pure[3] positional scoring rules. Recently, Chakraborty et al. [11] improved the previously known algorithms for Possible and Necessary winner, proposed integer programs for these problems for all scoring rules, and implemented them on real and generated data.

Betzler et al. [7] investigated the multivariate complexity of Possible Winner. They showed that for all positional scoring rules, the Possible Winner problem is fixed-parameter tractable with respect to the parameter "number of candidates." For ℓ-Approval, they showed that Possible Winner remains NP-complete already for two incomplete votes, if ℓ is part of the input. For Borda, NP-completeness holds even

[2] In some sense we also study the problem of finding *possible winners*, but we keep the name Possible President and the terminology introduced by Faliszewski et al. [20] in our formal definitions and results.

[3] A positional scoring rule is *pure*, if the scoring vector for t candidates is obtained by inserting a single coordinate into the scoring vector for $t-1$ candidates.

if there are only six voters, with only three of them having incomplete preferences. Betzler et al. also derived fixed-parameter tractability results with respect to the parameter "total number of undetermined candidate pairs."

Kenig [24] considered a special case of Possible Winner where the candidate set is partitioned into disjoint subsets, and each voter provides a total linear order over these subsets, while candidates within a subset are incomparable with each other. She gave a polynomial-time algorithm for ℓ-Approval and for the scoring rule based on the vector $(2, 1, \ldots, 1, 0)$, and proved NP-hardness for voting rules whose scoring vector has at least four or, under some additional conditions, three distinct values.

See the survey by Lang [27] for more results on the Possible/Necessary Winner problems.

2 Preliminaries

We use the notation $[i] = \{1, 2, \ldots, i\}$ for each positive integer i.

An election $\mathcal{E} = (C, V, \{\succ_v\}_{v \in V})$ consists of a finite set C of candidates, a finite set V of voters, and the preferences of each voter over the set C of candidates. We assume that the preferences of each voter v are represented by a strict linear order $\succ_v$ over C, where $c \succ_v c'$ means that voter v *prefers* candidate c to candidate c'. We denote the set of all elections over a set C of candidates by $\mathbb{E}_C$. A *voting rule* $f : \mathbb{E}_C \to 2^C$ chooses a set of *winners* of the election.

We shall also assume that a partition $\mathcal{P} = \{P_1, \ldots, P_t\}$ of the set C of candidates is given; each set P_j is interpreted as a *party* that has to decide about whom among its potential candidates to nominate for the election.

Formally, a *reduced election* arises after all parties have nominated a unique candidate, leading to a reduced candidate set $C' \subseteq C$ such that $|C' \cap P_j| = 1$ for each $j \in [t]$. In the reduced election $\mathcal{E}_{C'} = (C', V, \{\succ'_v\}_{v \in V})$ the preference relation $\succ'_v$ of each voter $v \in V$ is the restriction of her original preference relation $\succ_v$ over C to C'.

Now we formulate the problem studied in this paper.

Problem Possible President.
Instance: An election $\mathcal{E}$ with a set of voters V, a set of candidates C with a partition $\mathcal{P}$ into parties, a given party P_j.
Question: Is there a candidate p for party $P_j \in \mathcal{P}$ such that for some nominations of other parties leading to reduced candidate set C', p is the *unique* winner of the reduced election $\mathcal{E}_{C'}$?

Notice that we consider the *unique winner model*, i.e., we aim for a set of nominations that yield $f(\mathcal{E}_{C'}) = \{p\}$ for the candidate p nominated by the designated party in the reduced election $\mathcal{E}_{C'}$ with voting rule f. Also, when the set of candidates in the reduced election $\mathcal{E}_{C'}$ is clear from the context, we shall omit the subscript and write simply $\mathcal{E}$.

2.1 Voting rules

In this paper we shall concentrate on *positional scoring rules*. A positional scoring rule for elections involving t candidates is associated with a *scoring vector* $(a_1, a_2, \ldots, a_t)$

where $a_1 \geq a_2 \geq \cdots \geq a_t$ and at least one inequality is strict. For each candidate c, the rule assigns a_i points to c for each voter that ranks c on the i^{th} position of her preference list. The *winners* of the election are the candidates with the highest *score*, that is, the total number of points obtained. We write $\mathsf{scr}_{\mathcal{E}}(c)$ for the score of some candidate c in an election $\mathcal{E}$. The score of some party P, denoted by $\mathsf{scr}_{\mathcal{E}}(P)$, is the score of its nominated candidate in a reduced election $\mathcal{E}$.

We deal with four classes of positional scoring rules.

Short scoring rules are defined by scoring vectors with only a constant number of non-zero positions, i.e., having the form $(a_1, a_2, \ldots, a_\ell, 0, \ldots, 0)$ for some constant ℓ. A special case of short scoring rules is *ℓ-Approval* for fixed ℓ, because this scoring rule corresponds to the scoring vector with ones in their first ℓ positions and zeros afterwards. For $\ell = 1$ we obtain *Plurality*. In other words, in ℓ-Approval each voter votes for their ℓ most preferred candidates, and in Plurality only for their top candidate.[4]

Veto-like scoring rules have scoring vectors that contain some value a on every position except for the last ℓ positions for some constant ℓ, i.e., they have the form $(a, \ldots, a, a_1, a_2, \ldots, a_\ell)$ for some constant $\ell \geq 1$ where $a > a_1$. In such a scoring rule we shall sometimes say that a candidate c receives a *negative vote* from a voter v if c is ranked in one of the last ℓ positions in v's preference list. Veto-like scoring rules include *ℓ-Veto*, whose scoring vector is $(1, 1, \ldots, 1, 0, 0, \ldots, 0)$ with exactly ℓ zeros; for $\ell = 1$ we obtain *Veto*.

We define a common generalization of short and Veto-like scoring rules through scoring vectors of the form

$$(\underbrace{a_1, a_2, \ldots, a_\ell}_{\ell \text{ positions}}, a, a, \ldots, a, \underbrace{a'_1, a'_2, \ldots, a'_{\ell'-1}, a'_{\ell'}}_{\ell' \text{ positions}})$$

for some constant integers ℓ and ℓ'; we refer to such scoring rules as (ℓ, ℓ')-*Approval& Veto*. This class also generalizes the combination of Plurality and Veto defined by the scoring vector $(2, 1, 1, \ldots, 1, 0)$ that was dealt with by Betzler and Dorn in [6] and by Baumeister and Rothe in [3] when studying the Possible Winner problem for incomplete votes.

The popular *Borda* scoring rule does not fall into any of the previously introduced classes. The Borda scoring vector for elections with t candidates is $(t-1, t-2, \ldots, 1, 0)$. This can also be interpreted in such a way that the number of points some candidate c receives from a voter v is equal to the number of other candidates that are ranked worse than c in the preferences of v.

We will call a voting rule *efficiently computable*, if it is computable in polynomial time. Notice that all voting rules defined by a scoring vector are efficiently computable.

[4]Notice that some authors define approval rules by allowing each voter v_i with her associated preference order to have her own approval count ℓ_i, i.e., such voter approves of her top ℓ_i candidates (Schlotter et al. [35]). Our hardness results clearly hold also in this case. On the other hand, in the model described by Baumeister et al. [5] where each ballot only gives the given voter's approvals or disapprovals for each candidate, Possible President is easy. We simply need to check for each possible nominee p of party P_j whether every other party has a candidate with less approvals than p.

2.2 Parameterized complexity

Parameterized complexity was introduced by Downey and Fellows [17] as a tool to deal with hard problems. Each instance of a parameterized problem Q is a pair (I, k) consisting of an input I and a parameter k, which is usually an integer. A parameterized problem is *fixed-parameter tractable* (FPT) if there is an algorithm which correctly determines for each instance (I, k) of the problem whether (I, k) is a "yes"-instance in running time $f(k) \cdot |I|^{O(1)}$, where f is a computable function.

Downey and Fellows [17] introduced the W-hierarchy in an attempt to capture the exact complexity of various hard parameterized problems, consisting of the following classes: $\mathsf{FPT} \subseteq \mathsf{W}[1] \subseteq \mathsf{W}[2] \subseteq \cdots \subseteq \mathsf{W}[\mathsf{SAT}] \subseteq \mathsf{W}[\mathsf{P}]$. Since these complexity classes are all believed to be distinct, proving that a parameterized problem Q is $\mathsf{W}[t]$-hard for some $t \in \mathbb{N}^+$ provides strong evidence that we cannot expect an FPT algorithm for Q. Such intractability results can be obtained via so-called *parameterized reductions*: given two parameterized problems Q and Q', a parameterized reduction from Q to Q' is an algorithm that runs in FPT time and transforms an instance (I, k) of Q into an equivalent instance (I', k') of Q' such that $k' \leq g(k)$ for some computable function g. If Q is $\mathsf{W}[t]$-hard for some $t \in \mathbb{N}^+$, then such a reduction from Q to Q' implies that Q' is $\mathsf{W}[t]$-hard as well.

If a parameterized problem Q is NP-hard for some fixed constant value of the parameter, then Q is said to be para-NP-hard with respect to this parameter. By contrast, if Q can be solved in polynomial time for all constant values of the parameter, then we say that Q is in the class XP. Clearly, $\mathsf{FPT} \subseteq \mathsf{XP}$. The converse is not true, since an algorithm for Q whose running time is described by a polynomial whose degree depends on the parameter is enough for showing that $Q \in \mathsf{XP}$ but does not imply fixed-parameter tractability.

Parameterized problems appearing in the reductions.

In most of our intractability results, we are going to present reductions from the following two problems. The first one is Hitting Set, which is $\mathsf{W}[2]$-hard when parameterized by the solution size [13]. The second is Multicolored Clique, which is $\mathsf{W}[1]$-hard when parameterized by the size of the desired clique [21].

Problem Hitting Set.
Instance: A set S, a family $\mathcal{F} = \{F_1, F_2, \ldots, F_m\}$ of subsets of S, and an integer parameter k.
Question: Is there a subset $S' \subseteq S$ of size at most k such that for each $i \in [m]$ we have $S' \cap F_i \neq \emptyset$?

Problem Multicolored Clique.
Instance: An undirected graph $G = (U, E)$ and an integer parameter k, with the vertex set U partitioned into k independent sets $U_1, U_2, \ldots, U_k$.
Question: Is there a clique of size k in G?

We now define some notation that we will use in each of our reductions from Multicolored Clique. Note that we may assume w.l.o.g. that there are n vertices in every set U_i for some $n \in \mathbb{N}$; we denote these by $u_i^1, \ldots, u_i^n$. Furthermore, for each pair of distinct indices $i, j \in [k]$ we denote by $E_{\{i,j\}}$ the set of edges in G that run between U_i and U_j; note that $E_{\{i,j\}} = E_{\{j,i\}}$. For a vertex $u \in U \setminus U_i$, we let $E_i(u)$ denote the set of edges incident to u that have their other endpoint in U_i.

2.3 Initial observations

Let us first observe that if the nominated candidates are known, then the winners of the election can be computed in polynomial time for all efficiently computable voting rules. This immediately implies that the POSSIBLE PRESIDENT problem belongs to the class NP for all efficiently computable voting rules. Another consequence is the following observation.

Proposition 1 POSSIBLE PRESIDENT *can be solved in $s^t|I|^{O(1)}$ time on an instance I for any efficiently computable voting rule; recall that s is the size of the largest party, and t is the number of parties in I.*

Proof The statement follows easily by observing that trying all possible nominations for each of the parties yields $O(s^t)$ possibilities. □

Corollary 1 *For any efficiently computable voting rule,* POSSIBLE PRESIDENT *is FPT with parameter $s + t$, and is in* XP *with parameter t.*

3 Short scoring rules

In this section we obtain a handful of results for short scoring rules that describe the complexity landscape of these voting rules in great detail. Recall that a scoring rule is called short if the scoring vector contains only a finite number of nonzero entries. Therefore its special case is ℓ-Approval for any fixed ℓ, in particular Plurality, the most intensively studied voting rule for the candidate nomination problem. Cechlárová et al. [10, Theorem 1] proved that POSSIBLE PRESIDENT for ℓ-Approval is NP-complete for any constant ℓ even in the case when each party has at most two candidates which, in the context of parameterized complexity, means that POSSIBLE PRESIDENT is para-NP-hard for this voting rule when parameterized by s, the size of the largest party. We extend this result by examining a wider range of possible parameters, a larger set of voting rules, and determining not only NP-hardness for the arising problems but also their parameterized complexity.

3.1 Parameterizing by the number of parties

In this section we consider the number of parties as our parameter. Misra [31] proved that POSSIBLE PRESIDENT is W[2]-hard with respect to the parameter t representing the number of parties for Plurality. We strengthen her assertion by showing that this result holds for all short scoring rules.

Theorem 1 *Let $\mathcal{R}$ be a short voting rule based on a positional scoring vector that has the form $(a_1, a_2, \dots, a_\ell, 0, \dots, 0)$ for some constant $\ell \geq 1$ such that $a_\ell > 0$. Then* Possible President *for $\mathcal{R}$ is* NP*-hard, and also* W[2]*-hard with respect to parameter t where t is the number of parties. In particular, the result holds for ℓ-Approval for any fixed $\ell \in \mathbb{N}$.*

Proof We present a parameterized reduction from the W[2]-hard Hitting Set problem [13]. Let $H = (S, \mathcal{F}, k)$ be our instance of Hitting Set with universe $S = \{s_1, s_2, \dots, s_n\}$, set family $\mathcal{F} = \{F_1, F_2, \dots, F_m\} \subseteq 2^S$, and parameter k.

We shall construct an instance of Possible President. We set $P = \{p\}$ as our designated party, and further define parties $P' = \{p'\}$ and $P_i = \{s_1^i, ..., s_n^i\}$ for each $i \in [k]$, where candidate s_r^i represents the i^{th} copy of the element $s_r \in S$. We write $F_j^i = \{s^i : s \in F_j\}$ for the set of i^{th} copies of the elements contained in F_j. Additionally we define a set $D_1 \cup D_2 \cup D_3 \cup D_4$ of dummy candidates with $|D_h| = \ell - 1$ for $h = 1, 2, 3, 4$, and with each dummy having its own single-candidate party. Therefore there are altogether $t = k + 4\ell - 2$ parties.

The set of voters is $V = \{w, v_1, \dots, v_m\} \cup V_0 \cup V_0'$ where $|V_0| = |V_0'| = m + 1$. Thus, there are $3m + 3$ voters. The preference profile is shown below.

$$
\begin{array}{ll}
v_j \text{ for } j \in [m]: & F_j^1 \succ F_j^2 \succ \dots \succ F_j^k \succ D_1 \succ p' \succ p \succ [\dots] \\
v \in V_0: & p \succ D_2 \succ p' \succ [\dots] \\
v \in V_0': & p' \succ D_3 \succ p \succ [\dots] \\
w: & D_4 \succ p \succ [\dots]
\end{array}
$$

Recall that ℓ is a constant, so the number of parties is only a function of k, and thus the presented reduction is a parameterized reduction. It is also a polynomial-time reduction, so by the NP-hardness of Hitting Set, our proof yields not only $W[2]$-hardness for parameter t, but also NP-hardness.

We see that candidate p earns a_1 points from each voter in V_0, a_ℓ points from voter w and, thanks to dummy candidates, no point elsewhere. Therefore her score is

$$\mathsf{scr}_{\mathcal{E}}(p) = (m + 1)a_1 + a_\ell.$$

Candidate p' receives $(m + 1)a_1$ points from the voters in V_0'. Each candidate s_r^i receives at most ma_1 points from voters $v_1, \dots, v_m$. Dummy candidates (present only if $\ell \geq 2$) from D_1, D_2, D_3, and D_4 receive at most ma_1, $(m + 1)a_2$, $(m + 1)a_2$, and a_1 points, respectively. Each of these values is less than $\mathsf{scr}_{\mathcal{E}}(p)$.

Now assume that there is a hitting set $S' = \{s_{(1)}, \dots, s_{(k)}\}$, where $s_{(i)}$ denotes the i^{th} element in S' for some fixed order. Let each party P_i nominate the candidate corresponding to the i^{th} copy of element $s_{(i)} \in S'$, i.e., the candidate $s_{(i)}^i$. Since $S' \cap F_j \neq \emptyset$ for each $j \in [m]$, this ensures that at least one candidate in the set $F_j^1 \cup F_j^2 \cup \dots \cup F_j^k$ is nominated. Therefore, candidate p' receives no more points and so p is the unique winner.

If we now assume that p is the unique winner, then p' cannot receive any additional vote from voters $v_1, \dots, v_m$, since otherwise she would obtain at least a_ℓ additional points which would imply $\mathsf{scr}_{\mathcal{E}}(p') \geq \mathsf{scr}_{\mathcal{E}}(p)$. Consequently, for each $j \in [m]$ at least one candidate in the set $F_j^1 \cup F_j^2 \cup \dots \cup F_j^k$ has to be nominated by some party. As each party P_i nominates exactly one candidate $s_{\sigma(i)}^i$ for some $\sigma(i)$, there are exactly k nominated candidates, and the set $S' = \{s_{\sigma(1)}, s_{\sigma(2)}, \dots, s_{\sigma(k)}\}$ is a hitting set for H of cardinality at most k. $\square$

3.2 Parameterizing by the number of voters

Let us now look at parameterizations where we take $|V|$, the number of voters as parameter. In fact, we present two results which show that even if we combine $|V|$ with either of the possible parameters s and t, we still obtain intractability. Namely, in Theorem 2 we show that Possible President for short scoring rules is W[1]-hard for parameter $|V| + t$, where t is the number of parties, while in Theorem 3 we prove that it is W[1]-hard with parameter $|V|$ even if the maximum size of a party is $s = 2$.

Theorem 2 *Let $\mathcal{R}$ be a short voting rule based on a positional scoring vector that has the form $(a_1, a_2, \ldots, a_\ell, 0, \ldots, 0)$ for some constant $\ell \geq 1$ and $a_\ell > 0$. Then* Possible President *for $\mathcal{R}$ is* W[1]*-hard with respect to the combined parameter $|V|+t$ where V is the set of voters and t the number of parties. In particular, the result holds for ℓ-Approval for any fixed $\ell \in \mathbb{N}$.*

Proof We give a parameterized reduction from the W[1]-hard problem Multicolored Clique [21], with input graph $G = (U, E)$ and parameter k; we are going to use all notation introduced in Section 2.2. In particular, recall that $U = U_1 \cup \cdots \cup U_k$ and $E = \bigcup_{i,j \in [k], i<j} E_{\{i,j\}}$.

We construct an instance of Possible President in which the set of voters is defined as $V = A \cup \{a'\} \cup (\bigcup_{i \in [k]} B_i) \cup F \cup F'$ where $|A| = |B_i| = 4$ for each $i \in [k]$, and

$$F = \{f_{i,j} : i, j \in [k], i \neq j\}, \qquad F' = \{f'_{i,j} : i, j \in [k], i \neq j\}.$$

We set $P = \{p\}$ as our designated party, and we add U_i as a party for each $i \in [k]$, as well as $E_{\{i,j\}}$ for each pair (i, j) with $1 \leq i < j \leq k$. We also add a set D_v of $\ell - 1$ dummy candidates, partitioned into single-element parties, for each voter $v \in V$. We let C denote the set of all candidates.

The preference lists are as follows:

$$\begin{aligned}
a \in A: \quad & p \succ D_a \succ [\ldots] \\
a': \quad & D_{a'} \succ p \succ [\ldots] \\
b_i \in B_i: \quad & U_i \succ D_{b_i} \succ [\ldots] \\
f_{i,j} \in F: \quad & D_{f_{i,j}} \succ E_j(u_i^1) \succ u_i^1 \succ E_j(u_i^2) \succ u_i^2 \succ \cdots \succ E_j(u_i^n) \succ u_i^n \succ [\ldots] \\
f'_{i,j} \in F': \quad & D_{f'_{i,j}} \succ E_j(u_i^n) \succ u_i^n \succ E_j(u_i^{n-1}) \succ u_i^{n-1} \succ \cdots \succ E_j(u_i^1) \succ u_i^1 \succ [\ldots]
\end{aligned}$$

Note that the number of parties is $(\ell - 1)|V| + \binom{k}{2} + k + 1$, while the number of voters is $|V| = 4\binom{k}{2} + 4k + 5$, so since ℓ is a constant fixed for $\mathcal{R}$, the number of parties is $O(k^2)$ and the presented reduction is a parameterized one.

We can observe the following directly from the preferences:

- Candidate p, nominated by P, obtains a score of $4a_1 + a_\ell$; recall that ℓ is the last position earning some points in $\mathcal{R}$.
- Since each dummy appears at a score-earning position in exactly one voter's preference list, and each dummy must be nominated as the unique candidate in its party, we get that every dummy obtains a score of at most a_1.
- A party $E_{\{i,j\}}$ for some $i, j \in [k]$, $i \neq j$ may receive points only from voters $f_{i,j}, f'_{i,j}, f_{j,i}$ and $f'_{j,i}$, a score of at most $4a_\ell < 4a_1 + a_\ell$ in total.

Hence, P gains more points than each party $E_{\{i,j\}}$, as well as each dummy, regardless of the nominations.

From the four voters in B_i, party U_i obtains a score of $4a_1$ irrespective of the nominations. Hence, p wins the election if and only if it receives more points than each of the sets U_i for $i \in [k]$. Moreover, P gets more points than some party U_i, $i \in [k]$, in the election resulting from a set of nominations if and only if for each $i,j \in [k]$, $j \neq i$, voters $f_{i,j}$ and $f'_{i,j}$ all prefer the candidate nominated by party $E_{\{i,j\}}$ to the one nominated by U_i (as otherwise U_i obtains an additional score of a_ℓ). The following observation captures the key property of our construction.

Claim 1 The following two statements are equivalent:

- Parties U_i, $i \in [k]$, and parties $E_{\{i,j\}}$, $i,j \in [k]$, $i \neq j$ can nominate candidates in a way such that for each $i,j \in [k]$, $i \neq j$ both $f_{i,j}$ and $f'_{i,j}$ prefer the candidate nominated by $E_{\{i,j\}}$ to the candidate nominated by U_i.
- There exists a clique of size k in G containing a vertex from each set U_i.

Proof Suppose the first statement holds. Let $e_{\{i,j\}}$ denote the candidate nominated by $E_{\{i,j\}}$, and let $u_i^{\sigma(i)}$ denote the candidate nominated by U_i. Since $f_{i,j}$ prefers $e_{\{i,j\}}$ to $u_i^{\sigma(i)}$, we know that $e_{\{i,j\}}$ is incident to some vertex in the set $\{u_i^1, u_i^2, \dots, u_i^{\sigma(i)}\}$. Similarly, since $f'_{i,j}$ also prefers $e_{\{i,j\}}$ to $u_i^{\sigma(i)}$, we know that $e_{\{i,j\}}$ is incident to some vertex in $\{u_i^n, u_i^{n-1}, \dots, u_i^{\sigma(i)}\}$. Therefore, $e_{\{i,j\}}$ is incident to $u_i^{\sigma(i)}$. By symmetry, $e_{\{i,j\}}$ must be incident to $u_j^{\sigma(j)}$ as well. Hence, $u_i^{\sigma(i)}$ and $u_j^{\sigma(j)}$ are adjacent in G. As this holds for each distinct indices i and j in $[k]$, we get that vertices $u_i^{\sigma(i)}$ for $i \in [k]$ form a clique in G.

For the other direction, suppose that K is a clique in G with a vertex in each set U_i. Then it is easy to verify that nominating the edges and vertices of K we obtain a nomination for each of the parties U_i, $i \in [k]$, and $E_{\{i,j\}}$, $i,j \in [k]$, $i \neq j$ that satisfies the conditions of the first statement. $\lhd$

By Claim 1 and our observations preceding it, there is a set of nominations where p is the unique winner of the reduced election if and only if G contains a clique as desired. $\square$

Theorem 3 *Let $\mathcal{R}$ be a short voting rule based on a scoring vector of the form $(a_1, a_2, \dots, a_\ell, 0, \dots, 0)$ for some constant $\ell \geq 2$ where $a_\ell > 0$. Then* Possible President *for $\mathcal{R}$ is* W[1]*-hard with respect to the parameter $|V|$, the number of voters, even if each party has size at most* 2.

Proof We present a parameterized reduction from the W[1]-hard problem Multicolored Clique [21], with input graph $G = (U, E)$ and parameter k; we are going to use all notation introduced in Section 2.2. In particular, recall that $U = U_1 \cup \dots \cup U_k$ and $E = \bigcup_{i,j \in [k], i<j} E_{\{i,j\}}$. We will assume $k > 2$.

We construct an instance of Possible President as follows. We define the set of voters as

$$V = V_0 \cup V'_0 \cup \{w\} \cup Z \cup Z' \cup \left(\bigcup_{i \in [k]} A_i \right) \cup S_U \cup S'_U \cup S_E \cup S'_E \cup F \cup F'$$

where $|V_0| = |V_0'| = |Z'| = 2$, $|Z| = |A_i| = \binom{k}{2} + k$ for each $i \in [k]$, and

$$\begin{aligned} S_U &= \{s_1, \dots, s_k\}, & F &= \{f_{i,j} : i, j \in [k], i \neq j\}, \\ S_U' &= \{s_1', \dots, s_k'\}, & F' &= \{f_{i,j}' : i, j \in [k], i \neq j\}. \\ S_E &= \{s_{\{i,j\}} : 1 \leq i < j \leq k\}, \\ S_E' &= \{s_{\{i,j\}}' : 1 \leq i < j \leq k\}, \end{aligned}$$

This implies

$$\begin{aligned} |V| &= |V_0| + |V_0'| + 1 + |Z| + |Z'| + \sum_{i \in [k]} |A_i| + |S_U| + |S_U'| + |S_E| + |S_E'| + |F| + |F'| \\ &= 7 + \binom{k}{2} + k + k\left(\binom{k}{2} + k\right) + 2k + 2\binom{k}{2} + 4\binom{k}{2} = \binom{k}{2}(k+7) + k^2 + 3k + 7. \end{aligned}$$

We set $P = \{p\}$ as our distinguished party, and for each edge $e \in E$ and vertex $u \in U$, we add a party $\{e, \overline{e}\}$ and a party $\{u, \overline{u}\}$. Thus, each vertex or edge x corresponds to a single party with two potential candidates: let us call candidate x the *main* candidate and $\overline{x}$ the *minor* candidate. Furthermore, we add a party $P' = \{p'\}$ and a party $Q = \{q\}$, and a set D_v of dummies for each voter $v \in V$, each dummy d being the only candidate in its party $P_d = \{d\}$. We set $|D_v| = \ell - 1$ for each voter $v \in V \setminus (S_U \cup S_U' \cup S_E \cup S_E')$ and $|D_v| = \ell - 2$ for each voter $v \in S_U \cup S_U' \cup S_E \cup S_E'$. We let C denote the set of all candidates we have defined. Recall that we treat vertices and edges of G as candidates in C, so in the preferences below, each set U_i should be interpreted as a set containing n main candidates; similarly for each set $E_{\{i,j\}}$. Note that minor candidates appear in the preference lists only at positions indicated by $[\dots]$.

The preference lists are now as follows:

$$\begin{aligned}
&v \in V_0: && p \succ D_v \succ [\dots] \\
&v' \in V_0': && p' \succ D_{v'} \succ [\dots] \\
&w: && D_w \succ p \succ [\dots] \\
&z \in Z: && q \succ D_z \succ [\dots] \\
&z' \in Z': && D_{z'} \succ q \succ [\dots] \\
&a \in A_i: && D_a \succ U_i \succ p' \succ [\dots] \\
&s_i \in S_U: && U_i \succ D_{s_i} \succ p \succ p' \succ [\dots] \\
&s_i' \in S_U': && U_i \succ D_{s_i'} \succ p' \succ [\dots] \succ p \\
&s_{\{i,j\}} \in S_E: && E_{\{i,j\}} \succ D_{s_{\{i,j\}}} \succ p \succ p' \succ [\dots] \\
&s_{\{i,j\}}' \in S_E': && E_{\{i,j\}} \succ D_{s_{\{i,j\}}'} \succ p' \succ [\dots] \succ p \\
&f_{i,j} \in F: && D_{f_{i,j}} \succ E_j(u_i^1) \succ u_i^1 \succ E_j(u_i^2) \succ u_i^2 \succ \dots \succ E_j(u_i^n) \succ u_i^n \succ p' \succ [\dots] \\
&f_{i,j}' \in F': && D_{f_{i,j}'} \succ E_j(u_i^n) \succ u_i^n \succ E_j(u_i^{n-1}) \succ u_i^{n-1} \succ \dots \succ E_j(u_i^1) \succ u_i^1
\end{aligned}$$

It is clear that the presented reduction is a parameterized reduction due to our bound on $|V|$; note also that each party contains at most two candidates. It remains to prove the correctness of our reduction.

Suppose that there is a nomination for each party such that p is the unique winner of the resulting election $\mathcal{E}$. Let $C_{\mathcal{E}}$ denote the set of nominated candidates. Given a subset $V' \subseteq V$ voters, let us denote the score obtained by a candidate $c \in C$ from voters in V' in the reduced election $\mathcal{E}$ as $\mathsf{scr}_{\mathcal{E}|V'}(c)$.

First note that from the voters in $W := V_0 \cup V_0' \cup \{w\} \cup Z \cup Z'$, only candidates p, p', q and some dummy candidates obtain non-zero points, irrespective of the nominations. Namely,

$$\mathsf{scr}_{\mathcal{E}|W}(p) = 2a_1 + a_\ell, \tag{1}$$

$$\mathsf{scr}_{\mathcal{E}|W}(p') = 2a_1, \tag{2}$$

$$\mathsf{scr}_{\mathcal{E}|W}(q) = 2a_1 + \left(\binom{k}{2} + k\right) a_\ell. \tag{3}$$

Consider now the voters s_i and s_i' for some $i \in [k]$. Then we have the following.

- If $U_i \cap C_\mathcal{E} \neq \emptyset$, then p and p' obtain the same points from these voters: indeed, if $|U_i \cap C_\mathcal{E}| > 1$, then p and p' each receive zero points from both s_i and s_i', and if $|U_i \cap C_\mathcal{E}| = 1$, then p and p' each receive a_ℓ points from both s_i and s_i'.
  Thus, $\mathsf{scr}_{\mathcal{E}|\{s_i,s_i'\}}(p') = \mathsf{scr}_{\mathcal{E}|\{s_i,s_i'\}}(p) \leq a_\ell$.
- If $U_i \cap C_\mathcal{E} = \emptyset$, then p obtains $a_{\ell-1}$ points from s_i and nothing from s_i', while p' obtains $a_{\ell-1} + a_\ell$ points in total from these two voters.
  Thus, $\mathsf{scr}_{\mathcal{E}|\{s_i,s_i'\}}(p') = \mathsf{scr}_{\mathcal{E}|\{s_i,s_i'\}}(p) + a_\ell$.

Considering the voters $s_{\{i,j\}}$ and $s'_{\{i,j\}}$ for some pair of distinct indices $i, j \in [k]$, we similarly obtain that

- if $E_{\{i,j\}} \cap C_\mathcal{E} \neq \emptyset$, then $\mathsf{scr}_{\mathcal{E}|\{s_{\{i,j\}},s'_{\{i,j\}}\}}(p') = \mathsf{scr}_{\mathcal{E}|\{s_{\{i,j\}},s'_{\{i,j\}}\}}(p) \leq a_\ell$,
- if $E_{\{i,j\}} \cap C_\mathcal{E} = \emptyset$, then $\mathsf{scr}_{\mathcal{E}|\{s_{\{i,j\}},s'_{\{i,j\}}\}}(p') = \mathsf{scr}_{\mathcal{E}|\{s_{\{i,j\}},s'_{\{i,j\}}\}}(p) + a_\ell$.

Taking into account that p does not obtain any points from the voters in $\bigcup_{i \in [k]} A_i \cup F \cup F'$, by (1) and (2) it follows that $\mathsf{scr}_\mathcal{E}(p') \geq \mathsf{scr}_\mathcal{E}(p)$ holds unless

(i) $U_i \cap C_\mathcal{E} \neq \emptyset$ for each $i \in [k]$, and

(ii) $E_{\{i,j\}} \cap C_\mathcal{E} \neq \emptyset$ for each indices i, j with $1 \leq i < j \leq k$.

Since p is the unique winner in $\mathcal{E}$, it follows that (i) and (ii) must hold. Summing up the points received by p we get

$$\begin{aligned}\mathsf{scr}_\mathcal{E}(p) &= \mathsf{scr}_{\mathcal{E}|W}(p) + \mathsf{scr}_{\mathcal{E}|S_U \cup S_U' \cup S_E \cup S_E'}(p) \\ &= 2a_1 + a_\ell + \Bigg(\underbrace{\big|\{i : |U_i \cap C_\mathcal{E}| = 1\}\big| + \big|\{\{i,j\} : |E_{\{i,j\}} \cap C_\mathcal{E}| = 1\}\big|}_{\eta_\mathcal{E}}\Bigg) a_\ell.\end{aligned}$$

Using the notation $\eta_\mathcal{E}$ as defined above, it is clear that $\eta_\mathcal{E} \leq \binom{k}{2} + k$. Recall now that $\mathsf{scr}_\mathcal{E}(q) \geq 2a_1 + (\binom{k}{2} + k)a_\ell$ by (3), which implies that only $\eta_\mathcal{E} = \binom{k}{2} + k$ is possible, as otherwise $\mathsf{scr}_\mathcal{E}(q) \geq \mathsf{scr}_\mathcal{E}(p)$. We can conclude that

$$\mathsf{scr}_\mathcal{E}(p) = 2a_1 + \left(\binom{k}{2} + k + 1\right) a_\ell. \tag{4}$$

Furthermore, by $\eta_\mathcal{E} = \binom{k}{2} + k$ we have that for each $i \in [k]$ there exists a unique candidate $u_i^{\sigma(i)}$ in $U_i \cap C_\mathcal{E}$, and for each indices $i, j \in [k]$ with $1 \leq i < j \leq k$ there exists a unique candidate $e_{\{i,j\}}$ in $E_{\{i,j\}} \cap C_\mathcal{E}$. We show that the vertex set $U^\star = \{u_i^{\sigma(i)} : i \in [k]\}$ induces a clique in G.

To this end, consider the score of $u_i^{\sigma(i)}$ and observe that

$$\mathsf{scr}_{\mathcal{E}|A_i \cup S_U \cup S_U'}(u_i^{\sigma(i)}) = 2a_1 + \left(\binom{k}{2} + k\right) a_\ell. \tag{5}$$

Consider now voters $f_{i,j}$ and $f'_{i,j}$ for some $j \in [k] \setminus \{i\}$. Candidate $u_i^{\sigma(i)}$ receives either a_ℓ or 0 points from each of these two voters; by Equations (4) and (5) we know that it must obtain zero points from both of them, as otherwise p is not a unique winner in $\mathcal{E}$. Hence, both $f_{i,j}$ and $f'_{i,j}$ must prefer the candidate $e_{\{i,j\}}$ to $u_i^{\sigma(i)}$. By Claim 1 this is only possible for each pair of distinct indices $i, j \in [k]$ if G contains a clique of size k.

For the reverse direction, assume now that there exist indices $\sigma(i)$ for each $i \in [k]$ such that the vertices $U^\star = \{u_i^{\sigma(i)} : i \in [k]\}$ form a clique in G. For each vertex u in G, let the party $\{u, \overline{u}\}$ nominate its main candidate u if $u \in U^\star$, and its minor candidate $\overline{u}$ otherwise. For each edge e in G, let the party $\{e, \overline{e}\}$ nominate its main candidate e if e connects two vertices of $U^\star$, and its minor candidate $\overline{e}$ otherwise. Let $e_{\{i,j\}}$ denote the edge connecting vertices $u_i^{\sigma(i)}$ and $u_j^{\sigma(j)}$ for any $i, j \in [k]$ with $i \neq j$. We claim that p is the unique winner of the resulting election $\mathcal{E}$.

Note that the set $C_\mathcal{E}$ of nominated candidates satisfies conditions (i) and (ii), since we nominate exactly one candidate from each set U_i and from each set $E_{\{i,j\}}$; in other words, $\eta_\mathcal{E} = \binom{k}{2} + k$. This means that Equation (4) holds. Summing up the score obtained by p' and by q, it is straightforward to verify that $\mathsf{scr}_\mathcal{E}(p) = \mathsf{scr}_\mathcal{E}(p') + a_\ell = \mathsf{scr}_\mathcal{E}(q) + a_\ell$. Thus, both parties P' and Q obtain fewer points than p in $\mathcal{E}$. It is also clear that each dummy obtains at most a_1 points, which is less than $\mathsf{scr}_\mathcal{E}(p)$.

Consider the remaining candidates in $C_\mathcal{E}$, i.e., candidates in $\{e, \overline{e} : e \in E\} \cup \{u, \overline{u} : u \in U\}$. Among these, only candidates corresponding to the vertices and edges of $G[U^\star]$, the subgraph of G induced by $U^\star$, receive a non-zero total score. Consider some index $i \in [k]$. Then $u_i^{\sigma(i)}$ obtains a score of $2a_1 + \left(\binom{k}{2} + k\right)a_\ell$ from voters in $A_i \cup S_U \cup S'_U$. Observe now that for any pair of distinct indices $i, j \in [k]$, both $f_{i,j}$ and $f'_{i,j}$ prefer candidate $e_{\{i,j\}}$ to $u_i^{\sigma(i)}$, and hence $u_i^{\sigma(i)}$ receives zero points from these voters. Hence, we obtain that

$$\mathsf{scr}_\mathcal{E}(u_i^{\sigma(i)}) = \mathsf{scr}_{\mathcal{E}|\bigcup_{i\in[k]} A_i \cup S_U \cup S'_U}(u_i^{\sigma(i)}) = 2a_1 + \left(\binom{k}{2} + k\right) a_\ell = \mathsf{scr}_\mathcal{E}(p) - a_\ell.$$

Finally, let us consider the candidate $e_{\{i,j\}}$ for distinct indices $i, j \in [k]$. Note that $e_{\{i,j\}}$ receives non-zero points only from voters in $S_E \cup S'_E \cup F \cup F'$. Namely, it obtains $2a_1$ points from voters in $S_U \cup S'_U$, and it obtains points from four more voters, namely from $f_{i,j}, f'_{i,j}, f_{j,i}$, and $f'_{j,i}$. Therefore, we get $\mathsf{scr}_\mathcal{E}(e_{\{i,j\}}) = 2a_1 + 4a_\ell < \mathsf{scr}_\mathcal{E}(p)$. Hence, p is indeed a unique winner in $\mathcal{E}$, finishing our proof. □

In view of Theorems 2 and 3, we may ask whether Possible President for short scoring rules becomes tractable if the number of voters is constant. As we will see in Theorem 4, the answer is affirmative. In fact, the algorithm behind Theorem 4 works not only for a constant number of voters, but also when there are only a constant number of voter *types*, i.e., there are only a constant number of different preference lists. We present Theorem 4 as well as another algorithmic result for this parameter in Section 3.3.

3.3 Parameterizing by the number of voter types

We say that two voters have the same *type*, if they have the same preferences over the candidates. In this section we present our algorithmic results for the parameter τ, denoting the number of voter types. Note that $\tau \leq |V|$, so an algorithmic result with respect to parameter τ immediately holds for parameter $|V|$ as well.

Theorem 4 shows that if there are only a constant number of voter types, then Possible President for short scoring rules can be solved in polynomial time. The degree of the polynomial in the presented simple algorithm depends on τ; hence, this is an XP algorithm with respect to parameter τ. We postpone the proof of Theorem 4, since we will present it in a slightly more general setting in Section 5 as Theorem 8.

Theorem 4 *Let $\mathcal{R}$ be a short voting rule based on a positional scoring vector of the form $(a_1, a_2, \ldots, a_\ell, 0, \ldots, 0)$ for some ℓ. Then* Possible President *for $\mathcal{R}$ can be solved in $O(\tau|C|^{\ell\tau+1})$ time where C is the candidate set and τ is the number of voter types.*

Corollary 2 *Let $\mathcal{R}$ be a short voting rule based on a positional scoring vector of the form $(a_1, a_2, \ldots, a_\ell, 0, \ldots, 0)$ for some ℓ. Then* Possible President *for $\mathcal{R}$ is in* XP *with respect to the parameter τ, denoting the number of voter types.*

Interestingly, combining the parameter τ with s, the maximum size of a party, yields tractability in the case of Plurality voting: in Theorem 5 we provide an FPT algorithm with parameter $\tau + s$ that solves Possible President for Plurality. Our algorithm is based on the technique of color-coding [32]. Note that such an FPT algorithm cannot exist for short scoring rules other than Plurality (unless $\mathsf{FPT} = \mathsf{W}[1]$), since Theorem 3 implies W[1]-hardness of these problems with parameter $|V|$, and hence, with parameter τ even if $s = 2$.[5]

When we combine the two parameters τ and s, we get the following positive result for Plurality.

Theorem 5 Possible President *for Plurality is FPT with respect to the parameter $s + \tau$ where s is the size of the largest party and τ is the number of voter types.*

Proof We propose a fixed-parameter tractable algorithm for Possible President with parameter $s + \tau$. Let $(C, V, \{\succ_v\}_{v \in V})$ be the given election and $\mathcal{P}$ the set of parties, with P being our designated party. Assume that there exists a nomination for each party that leads to an election $\mathcal{E}$ where P is the winning party. We propose an algorithm to find such nominations as follows.

Step 1: guessing the distribution of votes. We start with guessing some information about the nominations that enable P to win in the election $\mathcal{E}$. Namely, we guess the candidate $p \in P$ nominated by P in $\mathcal{E}$, and we also guess the partition $(V_0, V_1, V_2, \ldots, V_r)$ of V *induced by* $\mathcal{E}$, that is, where

- p obtains all votes from voters in V_0 in $\mathcal{E}$;
- for each $i \in [r]$ there exists a candidate $c_i \in C \setminus \{p\}$ that obtains all votes from voters in V_i (and no votes from other voters) in $\mathcal{E}$.

[5] In fact, our result for Plurality obviously holds for all voting rules defined by a scoring vector $(a_1, 0, 0, \ldots, 0)$ for some $a_1 > 0$, since such voting rules are equivalent with Plurality.

As we do not know the nominated candidates c_i participating in $\mathcal{E}$, of course we cannot check whether the above conditions hold. Instead, we only ensure that our guess $(p, V_0, V_1, \ldots, V_r)$ is *valid*, meaning that $p \in P$, voters of the same type belong to the same partition in $(V_0, V_1, \ldots, V_r)$, and additionally, that $|V_0| > |V_i|$ holds for all $i \in [r]$.

Step 2: coloring the candidates. Next, we randomly color each candidate in $C \setminus P$ with colors in $[r] \cup \{\varnothing\}$, coloring each candidate independently and uniformly. A coloring is *correct*, if the following holds: for each $i \in [r]$, candidate c_i has color i, and all other candidates in the party containing c_i have color $\varnothing$. Thus, a random coloring is correct with probability at least $\left(\frac{1}{r+1}\right)^{rs}$.

Step 3: deleting candidates. First, we delete any candidate that some voter in V_0 prefers to p. Then, we iteratively delete candidates using the following two deletion rules in a way that we first exhaustively apply Rule 1, and only apply Rule 2 when Rule 1 is no longer applicable.

- **Rule 1:** If c is the most-preferred candidate by some voter in V_i but c does not have color i, then delete c.
- **Rule 2:** If c and c' are distinct candidates, both of them most-preferred candidates by some voters in V_i, but they belong to the same party, then delete both c and c'.

Step 4: constructing the nominations. Finally, when neither Rule 1, nor Rule 2 is applicable, then for each voter in V we nominate the most-preferred candidate among the remaining ones; note that since neither Rule 1 nor Rule 2 is applicable, at most one candidate is thus nominated from each party. Then, for all remaining parties we nominate an arbitrary candidate. If this is not possible because there is a party with no candidates left, then we reject the current set of guesses (made in Step 1); if no guesses in Step 1 yield feasible nominations, then we return "No" for the instance.

Correctness. First we show that whenever the algorithm returns a nomination for each party for some valid guess $(p, V_0, V_1, \ldots, V_r)$, then p wins in the resulting election. Clearly, p wins all votes from voters in V_0, as we deleted all candidates that some voter in V_0 prefers to p. Similarly, since for each voter in V_i we nominate its most-preferred candidate (among the remaining ones), and this candidate has color i (as Rule 1 is not applicable), we obtain that no nominated candidate might win a vote from different sets V_i and V_j. Using that $|V_0| > |V_i|$ for each $i \in [r]$, we get that p indeed wins the resulting election.

Next, we show that in a "yes"-instance, the above algorithm returns with probability at least $(r+1)^{-rs}$ a set of nominations that result in an election $\mathcal{E}$ that P wins. Assume that P nominates some candidate p in $\mathcal{E}$, and $(V_0, V_1, \ldots, V_r)$ is the partition induced by $\mathcal{E}$. For each $i \in [r]$, let c_i denote the candidate that in $\mathcal{E}$ obtains the votes from each voter in V_i, and let P_i be the party that contains c_i. Then $P_1, \ldots, P_r$ are r distinct parties (by the definition of the partition induced by $\mathcal{E}$). Note that $(p, V_0, V_1, \ldots, V_r)$ is a valid guess, and thus our algorithm examines it at one point. With probability at least $(r+1)^{-rs}$ it then produces a correct coloring of C. We claim that in this case, the algorithm will output nominations for each party, which (by the previous paragraph) means that it produces a correct output.

To see this claim, it suffices to show that the algorithm does not reject the current set of guesses, i.e., every party has at least one remaining candidates after the application of Rules 1 and 2. First, note that the algorithm does not delete any of the candidates c_i, $i \in [r]$, using Rule 1, because each c_i has color i, and for each $i \in [r]$, all voters in V_i prefer c_i to each of the candidates in $\{c_j : j \in [r], j \neq i\}$. Second, observe that no candidate c_i, $i \in [r]$, can be deleted by Rule 2 either, because for this to happen, both c_i and some candidate $c' \in P_i$ would have to be most-preferred candidates by some voter in a set V_j; however, as c' has color $\varnothing$, Rule 1 is then applicable, meaning that Rule 2 is not applicable. This proves that the set $C^\star$ of candidates that remain by the end of Step 3 (for the guess $(p, V_0, V_1, \dots, V_r)$) contains c_i for each $i \in [r]$. Consider any party P' other than $P, P_1, \dots, P_r$, and let $c' \in P'$ be the candidate nominated by P' in $\mathcal{E}$. To prove our claim, it suffices to observe that $c' \in C^\star$, since for each $i \in [r]$, c' must be less preferred than c_i by all voters in V_i, and thus can never get deleted by Rule 1 or 2.

Hence, we can conclude that the above randomized algorithm produces a correct output with probability at least $(r+1)^{-rs}$ whenever some feasible nominations enable P to win, and always answers "No" whenever this is not possible. Note that there are $s\tau^\tau$ valid guesses that the algorithm needs to explore, and exploring each one can be done in linear time.

Derandomization. To derandomize the algorithm we can use standard techniques based on universal sets and perfect hash families as defined by Naor et al. [32]. Namely, we set $C' = C \setminus P$, and construct a $(|C'|, s\tau)$-universal set $\mathcal{U}$ over C'; that is, $\mathcal{U}$ contains subsets of C' such that for each $A \subseteq C'$ of size at most $s\tau$, the set $\{A \cap U : U \in \mathcal{U}\}$ contains all $2^{|A|}$ subsets of A. Next, we construct a $(|C'|, \tau)$-perfect family $\mathcal{F}$ of hash functions over C', and take the family Π_τ of all permutations over $[\tau]$. Then $\mathcal{F}' = \{\pi \circ f : f \in \mathcal{F}, \pi \in \Pi_\tau\}$ is a family of hash functions such that for each tuple $(a_1, a_2, \dots, a_\tau)$ of distinct candidates, there exists a function $f' \in \mathcal{F}'$ such that $f'(a_i) = i$ for each $i \in [\tau]$.

For a guess $(p, V_0, V_1, \dots, V_r)$, instead of randomly coloring the candidates in $C' = C \setminus P$, we first try all $U \in \mathcal{U}$, and color all candidates in $C' \cap U$ with $\varnothing$. Then we color the remaining candidates by trying every function $f' \in \mathcal{F}'$, and setting $f'(c)$ as the color of c for each $c \in C' \setminus U$. Let c_i and P_i be defined for some "yes"-instance as before. Then the construction of $\mathcal{U}$ guarantees that there exists a set $U^\star \in \mathcal{U}$ with $\bigcup_{i \in [r]}(P_i \setminus \{c_i\}) \subseteq U$ and $\{c_i : i \in [r]\} \subseteq C' \setminus U$; recall that $r \le \tau$ and $|P_i| \le s$ for each $i \in [r]$, so $|\bigcup_{i \in [r]} P_i| \le s\tau$. Additionally, the construction of $\mathcal{F}'$ guarantees that there is a function $f^\star \in \mathcal{F}'$ such that $f^\star(c_i) = i$ for each $i \in [r]$. Hence, when considering $U^\star$ and $f^\star$, the algorithm will obtain a correct coloring and produce a correct output.

Thanks to the work by Naor et al. [32], it is possible to construct the set family $\mathcal{U}$ such that $|\mathcal{U}| \le 2^{s\tau}(s\tau)^{O(\log(s\tau))} \log |C'|$ in $2^{s\tau}(st)^{O(\log(s\tau))}|C'| \log |C'|$ time. We can also construct $\mathcal{F}'$ such that $|\mathcal{F}'| \le |\Pi_\tau| e^\tau \tau^{O(\log \tau)} \log |C'| = \tau^{\tau + O(\log \tau)} \log |C'|$ in $\tau^{\tau+O(\log \tau)}|C'| \log |C'|$ time. This means that the running time after derandomization is indeed fixed-parameter tractable with respect to $s + \tau$. □

Unfortunately, the ideas used in Theorem 5 do not seem to work for ℓ-Approval when $\ell \ge 2$, since in such elections a new phenomenon arises: in a given vote v it might happen that we need to nominate *exactly* one candidate c from a large set of candidates, all ranked before the designated candidate p in v, in order to ensure that p obtains a point alongside c, and at the same time c prevents a dangerous other candidate (with a high score) following p in v from obtaining a point. We cannot nominate such "prevention candidates" independently from each other, but have to coordinate their choice so that none of them obtains more points than p. As Theorem 3 shows, this phenomenon is a manifestation of a real complexity barrier, since for elections based on short scoring rules other than Plurality (including ℓ-Approval for $\ell \ge 2$) the Possible President problem is W[1]-hard with respect to the number of voters, even if each party has at most two candidates; by $\tau \le |V|$ this trivially implies W[1]-hardness with parameter τ as well under the same assumption.

4 Veto-like rules

In this section we explore Veto-like voting rules where in each vote, almost all candidates receive the highest possible score, say a, with only a constant number of candidates who receive less than a points in the given vote. Clearly, such voting rules generalize Veto and ℓ-Veto for any constant $\ell \geq 1$. Cechlárová et al. [10, Theorem 2] proved that Possible President for ℓ-Veto is NP-hard even for $s = 2$, that is, when the size of each party is at most 2; their results easily imply the following observation.

Corollary 3 ([10]) *Let $\mathcal{R}$ be a Veto-like voting rule based on a scoring vector of the form $(a, \ldots, a, a_1, a_2, \ldots, a_\ell)$ for some constant $\ell \geq 1$ where $a > a_1$. Then* Possible President *is* NP*-hard for $\mathcal{R}$, even if $s = 2$, i.e., each party has size at most two.*

Proof We can reduce Possible President for Veto with $s = 2$ to our problem by adding $\ell - 1$ dummy candidates, each forming its own singleton party, and appending them (in any order) to the end of each vote. □

Corollary 3 means that Possible President is para-NP-hard with respect to s, the maximum size of a party. Hence, we investigate how the number of parties t and the number of voters $|V|$ (or voter types τ) affects the computational complexity of Possible President in Veto-like elections.

As we will see, if we consider the parameters t and $|V|$, then Veto-like rules behave quite similarly to short scoring rules. Theorem 6 showing W[2]-hardness for parameter t is an analog of Theorem 1, and is based on similar techniques. Similarly, Theorem 7 showing W[1]-hardness for the combined parameter $|V| + t$ is an analog of Theorem 2, and is proved by a similar reduction.

Theorem 6 *Let $\mathcal{R}$ be a Veto-like voting rule based on a scoring vector of the form $(a, \ldots, a, a_1, a_2, \ldots, a_\ell)$ for some constant $\ell \geq 1$ where $a > a_1$. Then* Possible President *is* W[2]*-hard for $\mathcal{R}$ when parameterized by t, the number of parties.*

Proof We again present a parameterized reduction from Hitting Set, using similar ideas as in the proof of Theorem 1. Let $H = (S, \mathcal{F}, k)$ be our instance of Hitting Set with $S = \{s_1, \ldots, s_n\}$ and $\mathcal{F} = \{F_1, \ldots, F_m\}$. Again, $P = \{p\}$ is our designated party and we define parties $P' = \{p'\}$ and $P_i = \{s_1^i, \ldots, s_n^i\}$ for each $i \in [k]$. We define $F_j^i = \{s^i : s \in F_j\}$ for the set of i^{th} copies of the elements contained in F_j. Further, we have a set D of $\ell - 1$ dummy candidates, each one constituting its single-element party. The number of parties is therefore $t = k + \ell + 1$.

We now define the set of voters as $V = \bigcup_{i \in [k]} W_i \cup \bigcup_{j \in [m]} V_j \cup \{v_0, v_0', v_0''\}$ where $|W_i| = 2$ for each $i \in [k]$ and $|V_j| = 2$ for each $j \in [m]$. The preferences of voters are

$$
\begin{array}{ll}
v_0 : & [\ldots] \succ p \succ D \\
v_0', v_0'' : & [\ldots] \succ p' \succ D \\
v \in V_j : & [\ldots] \succ p \succ F_j^1 \succ F_j^2 \succ \cdots \succ F_j^k \succ D \\
w \in W_i : & [\ldots] \succ s_1^i \succ s_2^i \succ \cdots \succ s_n^i \succ D
\end{array}
$$

As for Veto, let us call a vote where a given party (or candidate) receives less than a points a *negative vote* for that party (or candidate). It is easy to see that p receives one negative vote from voter v_0, candidate p' two negative votes from voters v_0' and v_0'', each party P_i for some $i \in [k]$ receives two negative votes from voters in W_i, and each dummy candidate gets a negative vote from each voter.

If the Hitting Set instance H is a "yes"-instance, then using the same nomination strategy as in the proof of Theorem 1, where each P_i nominates the i^{th} copy of the i^{th} element in the hitting set, candidate p receives no additional negative votes and so becomes the winner of the election.

Conversely, if p is the unique winner of the reduced election, then she must not get a negative vote from voters in the sets V_j, as in this case she would obtain at least two additional negative votes and would thus loose the election against p'. This implies that the set of elements of S with at least one of their copies nominated forms a hitting set for H of size at most k. □

Theorem 7 *Let $\mathcal{R}$ be a Veto-like voting rule based on a scoring vector of the form $(a, \ldots, a, a_1, a_2, \ldots, a_\ell)$ for some constant $\ell \geq 1$ where $a > a_1$. Then* Possible President *for $\mathcal{R}$ is* W[1]*-hard with respect to the combined parameter $|V| + t$ where $|V|$ is the set of voters and t the number of parties.*

Proof We present a parameterized reduction from the W[1]-hard problem Multicolored Clique [21], with input graph $G = (U, E)$ and parameter k; we are going to use all notation introduced in Section 2.2. Recall that $U = U_1 \cup \cdots \cup U_k$ and $E = \bigcup_{i,j \in [k], i<j} E_{\{i,j\}}$.

We construct an instance of Possible President as follows. We define the set of voters as $V = V_0 \cup \bigcup_{1 \leq i < j \leq k} A_{\{i,j\}} \cup F \cup F'$ where $|V_0| = 2k-3$, $|A_{\{i,j\}}| = 2k-2$ for each distinct pair of indices i and j, and

$$F = \{f_{i,j} : i, j \in [k], i \neq j\}, \qquad F' = \{f'_{i,j} : i, j \in [k], i \neq j\}.$$

This implies $|V| = \binom{k}{2}(2k+2) + 2k - 3$.

We set $P = \{p\}$ as our distinguished party, and we add U_i for each $i \in [k]$ as well as $E_{\{i,j\}}$ for each pair (i, j) with $1 \leq i < j \leq k$ as a party. We shall also have a set D of $\ell - 1$ dummy candidates, each one with its own party. Thus, the number of parties is $t = \binom{k}{2} + k + \ell$. The preference lists are as follows:

$$\begin{aligned}
v \in V_0 :&\quad [\ldots] \succ p \succ D \\
a \in A_{\{i,j\}} :&\quad [\ldots] \succ E_{\{i,j\}} \succ D \\
f_{i,j} \in F :&\quad [\ldots] \succ E_j(u_i^1) \succ u_i^1 \succ E_j(u_i^2) \succ u_i^2 \succ \cdots \succ E_j(u_i^n) \succ u_i^n \succ D \\
f'_{i,j} \in F' :&\quad [\ldots] \succ E_j(u_i^n) \succ u_i^n \succ E_j(u_i^{n-1}) \succ u_i^{n-1} \succ \cdots \succ E_j(u_i^1) \succ u_i^1 \succ D
\end{aligned}$$

It is clear that the presented reduction is a parameterized reduction due to our bounds on $|V|$ and on t. It remains to prove its correctness.

Suppose that p is the unique winner of the reduced election $\mathcal{E}$ resulting from some nominations. Again, let us call a vote where a given party (or candidate) receives less than a points a *negative vote* for that party (or candidate). Observe that p gets $2k-3$ negative votes from the voters in V_0. This means that P can only be a unique winner in $\mathcal{E}$ if all other parties get at least $2k-2$ negative votes. In particular, party U_i has to receive a negative vote from each of the $2k-2$ voters in $\{f_{i,j}, f'_{i,j} : j \in [k], j \neq i\}$. Notice that the preferences of $f_{i,j}$ and $f'_{i,j}$ over the candidates in parties U_i for $i \in [k]$ and $E_{\{i,j\}}$ for $i, j \in [k], i \neq j$ are defined exactly the same as in the proof of Theorem 2. Thus, we can apply Claim 1. It follows that parties U_i for $i \in [k]$ can receive $2k-2$ negative votes only if there is a clique of size k in G.

Conversely, it is straightforward to verify that nominating the vertices and edges of a clique in G that contains a vertex from each of the sets U_i yields an election $\mathcal{E}$ where p receives exactly $2k-3$ negative votes while all other parties receive at least $2k-2$ negative votes. □

Similarly as in the case of short scoring rules, the combination of parameters s and τ renders tractability. Notice that the FPT algorithm is much simpler and more general than the one based on color-coding presented in Theorem 5.

Proposition 2 *Let $\mathcal{R}$ be a Veto-like voting rule based on a scoring vector of the form $(a, \ldots, a, a_1, a_2, \ldots, a_\ell)$ for some constant $\ell \geq 1$ where $a > a_1$. Then* Possible President *for $\mathcal{R}$ is FPT with respect to the parameter $s + \tau$, and is in* XP *with parameter τ, where s is the size of the largest party and τ is the number of voter types.*

Proof First realize that voters of each type allocate less than a points to exactly ℓ parties. Hence, if the number of different voter types is τ, then at most $\ell\tau$ parties receive less than $|V| \cdot a$ points in the election. Therefore, if $t \geq \ell\tau + 2$, then there are at least two parties receiving the maximum score, in which case party P cannot be the unique winner and the instance I is a "no"-instance.

If $t \leq \ell\tau + 1$, then we can simply apply Proposition 1 to solve the problem on instance I in $s^t|I|^{O(1)} = s^{\ell\tau}|I|^{O(1)}$ time. □

5 Combining Approval and Veto

Let us now consider the complexity of Possible President for scoring rules that, roughly speaking, combine Approval and Veto. We have seen that both Plurality and Veto-like rules are tractable when the maximum size of the parties and the number of voter types is small, as Possible President is FPT with parameter $s + \tau$ for these voting rules. As we will see in Theorem 9, fixed-parameter tractability with respect to $s + \tau$ no longer holds for the combination of Plurality and Veto-like rules that allow each voter to distribute one positive vote and a few negative votes over the candidates. Nevertheless, if there are only a fixed number of voter types, i.e., τ is a constant, then we can still obtain a polynomial-time algorithm: the algorithm we present in Theorem 8 provides a proof also for Theorem 4, and can be seen as a generalization of Proposition 2 as well.

Theorem 8 *Let $\mathcal{R}$ be a voting rule based on a positional scoring vector of the form $(a_1, a_2, \ldots, a_\ell, a, a, \ldots, a, a'_1, a'_2, \ldots, a'_{\ell'-1}, a'_{\ell'})$ for some integers $\ell, \ell' \in \mathbb{N}$ where $a_\ell > a$ (unless $\ell = 0$) and $a > a'_1$ (unless $\ell' = 0$). Then* Possible President *for $\mathcal{R}$ can be solved in $O(\tau|C|^{(\ell+\ell')\tau+1})$ time where C is the candidate set and τ is the number of voter types.*

Proof Let us be given an instance of Possible President with $\mathcal{R}$ with candidate set C and voter set V partitioned into voter types $V_1, V_2, \ldots, V_\tau$. We propose the following algorithm.

We guess (by trying all possibilities) the set $C^\star$ of all candidates that obtain either more than a or less than a points from at least one voter in the election $\mathcal{E}$ where P wins; note that we must have $|C^\star| \leq (\ell+\ell')\tau$. Clearly, every candidate in $C \setminus C^\star$ obtains a total score of $a|V|$ in $\mathcal{E}$. Let $\mathcal{E}_{C^\star}$ denote the election restricted to candidates in $C^\star$. A guess $C^\star$ is *valid*, if

- each party has at most one candidate in $C^\star$;
- P contains a candidate $c(P)$ in $C^\star$, and the score of $c(P)$ in $\mathcal{E}_{C^\star}$ is at least $a|V|+1$ and exceeds the score of each candidate in $C^\star \setminus \{c(P)\}$ in $\mathcal{E}_{C^\star}$;
- for each party P' that has no candidate in $C^\star$ there exists a candidate $c(P')$ *enclosed* by $C^\star$ for every voter

where $C^\star$ *encloses* some candidate c' for some voter v if v prefers at least ℓ candidates of $C^\star$ to c', and prefers c' to at least ℓ' candidates of $C^\star$. It is straightforward to verify that a guess $C^\star$ is valid if and only if P wins the election where each party P' nominates its candidate in $C^\star$ if there is such a candidate, and nominates $c(P')$ otherwise.

We can compute in linear time the score obtained by each candidate in $C^\star$ in the election $\mathcal{E}_{C^\star}$. Also, for each party P' we can check whether any of its candidates fulfills the last condition of validity. Thus, we can decide for each guess $C^\star$ whether it is valid in $O(\tau|C|)$ time. Since there are at most $|C|^{(\ell+\ell')\tau}$ guesses, it takes $O(\tau|C|^{(\ell+\ell')\tau+1})$ time to check the validity of all possible guesses. □

Corollary 4 *Let $\mathcal{R}$ be a voting rule based on a positional scoring vector of the form $(a_1, a_2, \ldots, a_\ell, a, a, \ldots, a, a'_1, a'_2, \ldots, a'_{\ell'-1}, a'_{\ell'})$ for some integers $\ell, \ell' \in \mathbb{N}$ where $a_\ell > a$ (unless $\ell = 0$) and $a > a'_1$ (unless $\ell' = 0$). Then* Possible President *for $\mathcal{R}$ can be solved in polynomial time for every constant value of τ, denoting the number of voter types, i.e., it is in* XP *with parameter τ.*

Next we show that the XP algorithm with parameter τ presented in Corollary 4 is optimal in the sense that we cannot expect to find an FPT algorithm with parameter τ for Possible President for scoring rules that combine Plurality with Veto-like voting rules, since the problem for such voting rules is W[1]-hard with parameter τ, even if we fix $s = 2$. Hence, Possible President for such voting rules is significantly harder than it is for Plurality or for Veto-like voting rules, since Possible President admits an FPT algorithm with parameter $s + \tau$ both for Plurality (Theorem 5) and for Veto-like rules (Proposition 2).

Theorem 9 *Let $\mathcal{R}$ be a voting rule based on a positional scoring vector of the form $(a_1, a, a, \ldots, a, a'_1, a'_2, \ldots, a'_{\ell'-1}, a'_{\ell'})$ for some constant integer $\ell' \geq 1$ where $a_1 > a$ and $a > a'_1$ (unless $\ell' = 0$). Then* Possible President *for $\mathcal{R}$ is* W[1]*-hard with respect to the parameter $|V|$, the number of voters, even if each party has size at most* 2.

Proof We present a parameterized reduction from the W[1]-hard problem Multicolored Clique [21]; let $G = (U, E)$ with $U = U_1 \cup \cdots \cup U_k$ be our instance with parameter k. We are going to use all notation introduced in Section 2.2; recall $E = \bigcup_{i,j \in [k], i<j} E_{\{i,j\}}$.

We construct an instance of Possible President as follows. Set the values of α and α' such that $a_1 = a + \alpha$ and $a'_1 = a - \alpha'$; note that these values are constant, since $\mathcal{R}$ is a fixed scoring rule, and they are positive. We set β as the smallest integer larger than α/α', that is, $\beta = \lfloor \alpha/\alpha' \rfloor + 1$; then β is a constant as well. Next, we set γ as an integer that satisfies

$$(2k-3)\beta\frac{\alpha'}{\alpha} + 2 \leq \gamma < (2k-2)\beta\frac{\alpha'}{\alpha} + 2. \tag{6}$$

Since the difference between the lower and upper bound on γ above is $\beta\frac{\alpha'}{\alpha} > \alpha/\alpha' \cdot \frac{\alpha'}{\alpha} = 1$, we can choose such an integer γ. Note also that $\gamma = O(k)$.

We can now define the set of voters as $V = V_0 \cup \{v'\} \cup \bigcup_{i \in [k]} A_i \cup B \cup F \cup F'$ where

$$B = \{b_{\{i,j\}} : i,j \in [k], i < j\},$$
$$F = \bigcup\{F_{i,j} : i,j \in [k], i \neq j\},$$
$$F' = \bigcup\{F'_{i,j} : i,j \in [k], i \neq j\},$$

and we set $|V_0| = 2$, $|A_i| = \gamma$ for each $i \in [k]$, and $|F_{i,j}| = |F'_{i,j}| = \beta$ for each $i,j \in [k]$ with $i \neq j$.

We set $P = \{p\}$ as our designated party, and we add a party $\{u, \overline{u}\}$ for each vertex $u \in U$, as well as a party $\{e, \overline{e}\}$ for each edge $e \in E$. Thus, each vertex or edge x corresponds to a single party with two potential candidates: let us call candidate x the *main* candidate and $\overline{x}$ the *minor* candidate. We also define a party $P' = \{p'\}$. Additionally, we add a set $D \cup \{d\}$ of dummy candidates with $|D| = \ell' - 1$, and for each voter $f \in F \cup F'$ we add an additional dummy candidate d_f. Each dummy candidate is contained as a singleton in its own party. Observe that in the preference lists below we explicitly list only the main candidates, while minor candidates only appear at positions indicated by $[\dots]$. The preferences are as follows.

$$
\begin{array}{ll}
v \in V_0: & p \succ [\dots] \succ d \succ D \\
v': & p' \succ [\dots] \succ d \succ D \\
a \in A_i: & u_i^1 \succ u_i^2 \succ \dots \succ u_i^n \succ p' \succ [\dots] \succ d \succ D \\
b_{\{i,j\}}: & E_{\{i,j\}} \succ p' \succ [\dots] \succ d \succ D \\
f \in F_{i,j}: & d_f \succ p \succ p' \succ [\dots] \\
 & \quad \succ E_j(u_i^1) \succ u_i^1 \succ E_j(u_i^2) \succ u_i^2 \succ \dots \succ E_j(u_i^n) \succ u_i^n \succ D \\
f' \in F'_{i,j}: & d_{f'} \succ p \succ p' \succ [\dots] \\
 & \quad \succ E_j(u_i^n) \succ u_i^n \succ E_j(u_i^{n-1}) \succ u_i^{n-1} \succ \dots \succ E_j(u_i^1) \succ u_i^1 \succ D
\end{array}
$$

Observe that there are $|V| = \binom{k}{2}(1+4\beta) + k\gamma + 3 = O(k^2)$ voters (recall that β is a constant and γ is linear in k), and the maximum size of each party is $s = 2$; hence, this is a parameterized reduction. Clearly, it is a polynomial-time reduction as well.

To see its correctness, let $\mathcal{E}$ be the reduced election resulting from some nominations. In this election, p obtains $a_1 = a + \alpha$ points from both votes in V_0, while p' receives $a + \alpha$ points from v'. Note also that neither p nor p' can get less than a points from any of the voters, as both are always followed by at least ℓ' dummy candidates: this is clear for a voter from $V \setminus (F \cup F')$, and observe that for some voter $f \in F \cup F'$, both p and p' are followed by all dummy candidates except for d_f plus all nominees from parties corresponding to vertices and edges of G. Therefore,

$$\mathsf{scr}_{\mathcal{E}}(p) = a|V| + 2\alpha$$

and $\mathsf{scr}_{\mathcal{E}}(p') \geq a|V| + \alpha$.

Suppose now that p is the unique winner in $\mathcal{E}$, and let $C_{\mathcal{E}}$ denote the nominated candidates in $\mathcal{E}$. Note that p' cannot receive $a+\alpha$ points from any of the voters in $\bigcup_{i \in [k]} A_i \cup B$, as that would increase her score to at least $a|V| + 2\alpha$. This means that $U_i \cap C_{\mathcal{E}} \neq \emptyset$ for each $i \in [k]$, and $E_{\{i,j\}} \cap C_{\mathcal{E}} \neq \emptyset$ for each distinct pair of indices $i, j \in [k]$. For each $i \in [k]$, let $\sigma(i)$ denote the smallest index h for which the party $\{u_i^h, \overline{u}_i^h\}$ nominates its main candidate u_i^h in $\mathcal{E}$. Let us fix also some candidate $e_{\{i,j\}} \in E_{\{i,j\}} \cap C_{\mathcal{E}}$ for each pair of indices $i, j \in [k]$ with $i < j$.

Clearly, $u_i^{\sigma(i)}$ for some $i \in [k]$ receives $a + \alpha$ points from each voter in A_i, at least $a - \alpha'$ points from each voter in $F_i = \bigcup\{F_{i,j} \cup F'_{i,j} : j \in [k], j \neq i\}$, and exactly a points from each remaining voter. We have $|F_i| = (2k-2)\beta$. Note that if $u_i^{\sigma(i)}$ obtains a points from some voter in F_i, then it obtains a points from β voters in F_i, and this would yield

$$\begin{aligned} \mathsf{scr}_{\mathcal{E}}(u_i^{\sigma(i)}) &\geq a|V| + |A_i|\alpha - (|F_i| - \beta)\alpha' = a|V| + \gamma\alpha - (2k-3)\beta\alpha' \\ &= \mathsf{scr}_{\mathcal{E}}(p) + (\gamma - 2)\alpha - (2k-3)\beta\alpha' \geq \mathsf{scr}_{\mathcal{E}}(p), \end{aligned}$$

due to lower bound on γ in Inequality (6). Thus, $u_i^{\sigma(i)}$ must obtain less than a points from each voter in F_i. Observe that this is only possible if all voters in $F_{i,j} \cup F'_{i,j} \cup F_{j,i} \cup F'_{j,i}$ prefer $e_{\{i,j\}}$ to $u_i^{\sigma(i)}$. By the same arguments as used in the proof of Claim 1, this yields that G has a clique of size k.

For the reverse direction, assume that there exists a clique of size K in G. Consider the election $\mathcal{E}$ that results from nominating the main candidate x for some vertex or edge x if x is contained in K (as an edge or a vertex), and nominating the minor candidate $\overline{x}$ otherwise. Recall that $\mathsf{scr}_{\mathcal{E}}(p) = a|V| + 2\alpha$, and observe that $\mathsf{scr}_{\mathcal{E}}(p') = a|V| + \alpha$ is immediate. It is also straightforward to verify that every dummy in $D \cup \{d\}$ gets a score less than $a|V|$ in $\mathcal{E}$, while the dummy candidate d_f for some voter $f \in F \cup F'$ gets a score of $a|V| + \alpha < \mathsf{scr}_{\mathcal{E}}(p)$, since it obtains $a_1 = a + \alpha$ points in the vote by f, and a points in every other vote.

For each $i \in [k]$, let $u_i^{\sigma(i)}$ denote the unique vertex of K in U_i. Observe that for each distinct pairs of indices $i, j \in [k]$, each voter in $F_{i,j} \cup F'_{i,j}$ puts the single nominated candidate in U_i, i.e., the main candidate $u_i^{\sigma(i)}$, at the position earning $a'_1 = a - \alpha'$ points in her vote, because all such voters prefer the main candidate corresponding to the edge $u_i^{\sigma(i)} u_j^{\sigma(j)}$ in K to the candidate $u_i^{\sigma(i)}$. As a consequence, we get

$$\begin{aligned} \mathsf{scr}_{\mathcal{E}}(u_i^{\sigma(i)}) &= a|V| + |A_i|\alpha - |F_i|\alpha' = a|V| + \gamma\alpha - (2k-2)\beta\alpha' \\ &= \mathsf{scr}_{\mathcal{E}}(P) + (\gamma - 2)\alpha - (2k-2)\beta\alpha' < \mathsf{scr}_{\mathcal{E}}(P), \end{aligned}$$

due to upper bound on γ in Inequality (6).

For each $u \in U$ that is not contained in K, the nominated minor candidate $\overline{u}$ receives a score of $|V|$ in total, as do all candidates $\overline{e}$ where $e \in E$ is not an edge of K. Finally, we also know that for an edge e of K, candidate e receives a score at most $a|V| + \alpha < \mathsf{scr}_{\mathcal{E}}(p)$, since candidate e is preferred to p' by only one voter in B, and no voters in $\bigcup_{i \in [k]} A_i \cup F \cup F'$. Hence, p is the unique winner in $\mathcal{E}$, proving the correctness of our reduction. $\square$

Next we show that Possible President for scoring rules that fall within the category of (ℓ, ℓ')-Approval&Veto for some integers $\ell \geq 1$ and $\ell' \geq 0$, generalizing both short scoring rules as well as Veto-like scoring rules, "inherit" the hardness results obtained in Section 3 for short scoring rules. The proof gives a simple reduction that adds a few dummy agents whose sole purpose is to receive all "negative" votes, so that the competition between the remaining candidates essentially becomes an election based on a short voting rule.

Theorem 10 *Let $\mathcal{R}$ be a voting rule based on a positional scoring vector of the form $(a_1, a_2, \ldots, a_\ell, a, a, \ldots, a, a'_1, a'_2, \ldots, a'_{\ell'-1}, a'_{\ell'})$ for some constant integers $\ell \geq 1$ and $\ell' \geq 0$ where $a_\ell > a$ and $a > a'_1$ (unless $\ell' = 0$). Then* Possible President *for $\mathcal{R}$ is*

- NP*-hard even if each party has size at most two;*
- W[2]*-hard with respect to the parameter t, denoting the number of parties;*
- W[1]*-hard with respect to the combined parameter $|V| + t$ where V is the set of voters.*
- W[1]*-hard with respect to the parameter $|V|$, the number of voters, even if each party has size at most 2, assuming that $\ell \geq 2$.*[6]

Proof Consider the positional scoring vector $\vec{a}_0 = (a_1 - a, a_2 - a, \ldots, a_\ell - a, 0, 0, \ldots, 0)$, and let $\mathcal{R}_0$ be the voting rule based on $\vec{a}_0$; note that since $a_1 \geq a_2 \geq \cdots \geq a_\ell > a$, we know that the vector $\vec{a}_0$ is indeed a valid scoring vector. By Theorems 1, 2, and 3 and the result by Cechlárová et al. [10] (necessary for the NP-hardness of the case $\ell = 1$ and $s = 2$) we know that Possible President for $\mathcal{R}_0$ is NP-hard even if $s = 2$, W[2]-hard for parameter t, and W[1]-hard with respect to the combined parameter $|V| + t$, and also, in case $\ell \geq 2$ holds, with respect to parameter τ even if $s = 2$. We present a reduction from this problem to Possible President for $\mathcal{R}$ as follows.

Given an instance I_0 of Possible President for $\mathcal{R}_0$ with voter set V, we create a set D of ℓ' additional dummy candidates, with a unique party $\{d\}$ for each dummy $d \in D$. We append these dummy candidates in some arbitrary order to the preference list of each voter in V. We claim that the resulting instance I of Possible President for $\mathcal{R}$ is equivalent with I_0.

First, it is clear that each dummy candidate obtains a score less than $a|V|$ in I, while each non-dummy candidate obtains a score at least $a|V|$. Second, observe that for each reduced $\mathcal{R}_0$-election $\mathcal{E}_0$ resulting from some nominations in I_0, there is a corresponding reduced $\mathcal{R}$-election $\mathcal{E}$ in I where each non-dummy party nominates the same candidate as in $\mathcal{E}_0$ and all dummies are nominated; clearly, this is a bijection. Moreover, we get that $\mathsf{scr}_{\mathcal{E}}(c) = \mathsf{scr}_{\mathcal{E}_0}(c) + a|V|$ for each candidate c present in I_0. Hence, some candidate p is the unique winner in $\mathcal{E}_0$ if and only if it is a unique winner in $\mathcal{E}$, proving the correctness of our reduction.

Notice that the reduction does not change the number of voters, which remains $|V|$, and increases the number of parties by ℓ', a which is a constant; the maximum size of the parties, s, remains the same. Therefore, the presented reduction is not only a polynomial-time reduction, but also a parameterized one with respect to both t and $t + |V|$. □

6 Borda

Borda voting rule seems to be harder than the previously considered voting rules. Theorem 11 is a strengthening of a result proved by Cechlárová et al. [10, Theorem 3] who showed that Possible President for Borda is NP-complete even if each party has size at most 2. Here we show that it is even harder: it is intractable already for three voters, even if all parties have size at most 2. If we allow larger parties but only a few of them, computational intractability persists: in Theorem 12 we prove that Possible President for Borda is W[1]-hard when parameterized by the number of parties, even for a constant number of voters.

[6] See Theorem 9 for the case when $\ell = 1$.

Theorem 11 Possible President *for Borda is* NP*-hard even if each party has size at most* 2*, and there are only three voters.*

Proof We present a reduction from the following variant of 3-SAT, known to be NP-complete [22, Problem LO1]: the input is a propositional formula φ over variables $x_1, \ldots, x_n$ in conjunctive normal form as $\varphi = \bigwedge_{j\in[m]} c_j$ where each clause c_j contains at most 3 literals, and each variable occurs in at most 3 clauses; the task is to decide whether φ admits a satisfying truth assignment.

We construct an instance of Possible President for Borda as follows. For each variable x_i, we create a party $X_i = \{x_i, \overline{x}_i\}$ as well as three additional parties $Y_i^h = \{y_i^h, \overline{y}_i^h\}$ for $h = 1, 2, 3$. For each clause c_j, we create a party C_j containing the single candidate c_j. Additionally, we set $P = \{p\}$ as our designated party, and we also add a set D of dummy candidates, each $d \in D$ belonging to a distinct party $P_d = \{d\}$. To distinguish between three types of dummies, we partition D into sets D_1, D_2, and D_3; we set $|D_1| = m - 1$, $|D_2| = 7n-4$, and $|D_3| = 3m-m^\star+1$ where $m^\star$ denotes the total number of literals summed over all clauses in φ. We will denote dummies in D_1, D_2, and D_3 using $\bullet, \circ$, and $\diamond$ symbols, respectively. We write $\{\bullet\bullet\cdots\bullet\}_r$ for a set of r distinct dummies from D_1, and we use the analogous notation for sets of dummies from D_2 or from D_3. Observe that the number of parties is

$$t = 4n + m + 1 + |D| = 11n + 5m - m^\star - 3. \tag{7}$$

We set $\{a, b, c\}$ as the set of voters. To define their preferences, we will use the following shorthand for each $i \in [n]$ and $j \in [m]$:

$$\begin{aligned} &Z_i \ \text{ for } \ \overline{x}_i \succ y_i^1 \succ y_i^2 \succ y_i^3 \succ x_i \succ \overline{y}_i^1 \succ \overline{y}_i^2 \succ \overline{y}_i^3, \\ &\overline{Z}_i \ \text{ for } \ x_i \succ \overline{y}_i^1 \succ \overline{y}_i^2 \succ \overline{y}_i^3 \succ \overline{x}_i \succ y_i^1 \succ y_i^2 \succ y_i^3, \text{ and} \\ &S_j \ \text{ for } \ L_j \succ c_j \succ \overline{L}_j \succ \{\diamond\diamond\cdots\diamond\}_{3-|L_j|} \end{aligned}$$

where L_j and $\overline{L}_j$ are constructed as follows:

- if x_i is a (positive) literal in clause c_j, and this literal is the q^{th} occurrence of variable x_i in φ (as either a positive or a negative literal), then we put y_i^q into L_j, and we put $\overline{y}_i^q$ into $\overline{L}_j$;
- if $\overline{x}_i$ is a (negative) literal in clause c_j, and this literal is the q^{th} occurrence of variable x_i in φ, then we put $\overline{y}_i^q$ into L_j, and we put y_i^q into $\overline{L}_j$.

Observe that $|L_j| = |\overline{L}_j|$. As the clause c_j may have less than three literals, the dummies involved ensure that S_j will contain exactly four nominated candidates, irrespective of the nominations.

Observe also that the lists S_j for all $j \in [m]$ together contain exactly $3m-m^\star = |D_3|-1$ dummies. We also let $Y^\star$ denote the union of those candidates y that are contained in some party Y_i^h, $i \in [n]$ and $h \in [3]$, but do not appear in any of the sets S_j, $j \in [m]$; such candidates may exist, because a variable may occur only once or twice in φ.

We are now ready to define the preferences of the voters a, b, and c:

$$\begin{aligned} &a: c_m \succ \bullet \succ c_{m-1} \succ \bullet \succ \cdots \succ c_2 \succ \bullet \succ c_1 \succ p \succ Z_1 \succ Z_2 \succ \cdots \succ Z_n \succ D_3 \succ D_2 \\ &b: c_m \succ \bullet \succ c_{m-1} \succ \bullet \succ \cdots \succ c_2 \succ \bullet \succ c_1 \succ p \succ \overline{Z}_1 \succ \overline{Z}_2 \succ \cdots \succ \overline{Z}_n \succ D_3 \succ D_2 \\ &c: X_n \succ \{\circ\circ\cdots\circ\}_7 \succ X_{n-1} \succ \{\circ\circ\cdots\circ\}_7 \succ \cdots \succ X_2 \succ \{\circ\circ\cdots\circ\}_7 \succ X_1 \\ &\qquad \succ \{\circ\circ\circ\}_3 \succ p \succ \diamond \succ S_1 \succ S_2 \succ \cdots \succ S_m \succ Y^\star \succ D_1 \end{aligned}$$

It is easy to verify that our choice of the sizes $|D_1|, |D_2|$, and $|D_3|$ ensure that each dummy appears exactly once in each preference list.

In the constructed instance of Possible President, each party has size at most 2 and there are exactly three voters. It remains to prove that the reduction is correct.

Let us first compute the score obtained by p in an election $\mathcal{E}$ resulting from some nominations, according to the Borda rule. Clearly, p obtains $t-2m$ points from a, and also from b, while it obtains $t-8(n-1)-5$ points from c; hence, we have

$$\mathsf{scr}_{\mathcal{E}}(P) = 3t - 4m - 8n + 3. \tag{8}$$

Let us now consider some party X_i for $i \in [n]$. Observe that both Z_i and $\overline{Z}_i$ contain all candidates from the four parties X_i, Y_i^1, Y_i^2, and Y_i^3. If X_i nominates x_i, then it obtains $t-2m-4(i-1)-1$ points from b, since it is the first candidate among those in $\overline{Z}_i$. By contrast, it obtains at most $t-2m-4(i-1)-1$ but at least $t-2m-4(i-1)-4$ points from a, depending on the number of candidates among y_i^1, y_i^2, and y_i^3 that participate in $\mathcal{E}$. Analogously, if X_i nominates $\overline{x}_i$, then it obtains $t-2m-4(i-1)-1$ points from a, and obtains at most as many but at least $t-2m-4(i-1)-4$ points from b. Hence, in either case X_i obtains at least $2t-4m-8i+3$ points from a and b together. Since X_i obtains $t-8(n-i)-1$ points from the vote of c, we get

$$\mathsf{scr}_{\mathcal{E}}(X_i) \geq 3t - 4m - 8n + 2. \tag{9}$$

Third, consider the party C_j representing clause c_j for some $j \in [m]$. Note that c_j obtains $t-2(m-j)-1$ points from each of a and b. Recall also that the union of all candidates in S_j comprises the candidate set of four parties. Hence, c_j obtains at most

$$t - (8(n-1) + 4(j-1) + 6) - 1 = t - 8n - 4j + 5$$

points from c; note that C_j obtains less points than that from c exactly if at least one of the candidates in L_j, corresponding to the literals in c_j, participates in the election $\mathcal{E}$. This yields

$$\mathsf{scr}_{\mathcal{E}}(C_j) \leq 3t - 4m - 8n + 3; \tag{10}$$

moreover, the inequality is strict only if at least one of the candidates in L_j is nominated in $\mathcal{E}$.

Suppose now that P wins in election $\mathcal{E}$. We define a truth assignment α that sets variable x_i to `true` if and only if X_i nominates x_i, for each $i \in [n]$. We are going to show that α satisfies φ.

First observe that if P is a (unique) winner in $\mathcal{E}$, then Inequality (9) must hold with equality for each $i \in [n]$, as otherwise $\mathsf{scr}_{\mathcal{E}}(X_i) \geq \mathsf{scr}_{\mathcal{E}}(P)$. Hence, if X_i nominates x_i in $\mathcal{E}$, then X_i must obtain exactly $t-2m-4i$ points from a, which means that each of the candidates y_i^1, y_i^2, and y_i^3 must participate in $\mathcal{E}$, i.e., they must be nominated. Similarly, if X_i nominates $\overline{x}_i$ in $\mathcal{E}$, then X_i must obtain exactly $t-2m-4i$ points from b, which means that each of the candidates $\overline{y}_i^1, \overline{y}_i^2$, and $\overline{y}_i^3$ must participate in $\mathcal{E}$. This means that $\alpha(x_i) =$ `true` exactly if all candidates in $Y_i := \{y_i^1, y_i^2, y_i^3\}$ are nominated in $\mathcal{E}$, and $\alpha(x_i) =$ `false` exactly if all candidates $\overline{Y}_i := \{\overline{y}_i^1, \overline{y}_i^2, \overline{y}_i^3\}$ are nominated in $\mathcal{E}$.

Consider now clause c_j for some $j \in [m]$. Clearly, if P is the unique winner in $\mathcal{E}$, then Inequality (10) must hold with strict inequality; hence, at least one of the candidates in L_j must be nominated in $\mathcal{E}$. By the definition of these candidates, this means that there exists a variable x_i appearing in c_j such that

- if x_i appears as a (positive) literal in c_j, then some candidate of Y_i, preferred to c_j by voter c, must be nominated in $\mathcal{E}$;
- if $\overline{x}_i$ appears as a (negative) literal in c_j, then some candidate of $\overline{Y}_i$, preferred to c_j by voter c, must be nominated in $\mathcal{E}$.

By the facts of the previous paragraph, this implies that at least one of the literals in c_j is set to $\texttt{true}$ by α in φ. Thus, α is indeed a satisfying truth assignment, as required.

For the reverse direction, let α be a truth assignment on variables $x_1, \dots, x_n$ that satisfies φ. Consider the election $\mathcal{E}$ where for each $i \in [n]$,

- X_i nominates the literal, x_i or $\overline{x}_i$, that has value $\texttt{true}$ according to α,
- if $\alpha(x_i) = \texttt{true}$, then each party Y_i^h, $h \in [3]$, nominates y_i^h, while if $\alpha(x_i) = \texttt{false}$, then each party Y_i^h, $h \in [3]$, nominates $\overline{y}_i^h$;
- all remaining parties nominate the single candidate they contain.

It is straightforward to check that in this case, Inequality (9) will hold with equality (by the nominations for parties Y_i^h, $i \in [n]$ and $h \in [3]$), and Inequality (10) will hold with strict inequality (due to α being a satisfying truth assignment). Furthermore, observe that each of the three voters prefers p to both candidates of a party Y_i^h for some $i \in [n]$ and $h \in [3]$. Thus, it suffices to check that each dummy has score less than $\mathsf{scr}_{\mathcal{E}}(P)$ in $\mathcal{E}$.

First, consider a dummy $d_1 \in D_1$. From the votes by a and b it obtains at most $2t - 4$ points, and from the vote of c it obtains at most $|D_1| - 1 = m - 2$ points. Hence, its total score is less than $\mathsf{scr}_{\mathcal{E}}(P)$ due to the value of t (see (7)), the value of $\mathsf{scr}_{\mathcal{E}}(P)$ (see (8)), and the fact $m^\star \leq 3n$:

$$\begin{aligned} \mathsf{scr}_{\mathcal{E}}(P_{d_1}) &\leq 2t + m - 6 = 3t - 11n - 4m + m^\star - 3 \\ &= \mathsf{scr}_{\mathcal{E}}(P) - (3n - m^\star) - 6 < \mathsf{scr}_{\mathcal{E}}(P). \end{aligned}$$

Second, consider a dummy $d_2 \in D_2$. From the votes by a and b, it obtains at most $2|D_2| - 2 = 14n - 10$ points in total, and from the vote of c it obtains at most $t - 2$ points. Hence, its total score is less than $\mathsf{scr}_{\mathcal{E}}(P)$ due to the value of t, the value of $\mathsf{scr}_{\mathcal{E}}(P)$, and the fact $m^\star \leq 3m$:

$$\begin{aligned} \mathsf{scr}_{\mathcal{E}}(P_{d_2}) &\leq t + 14n - 12 = 3t - 8n - 10m + 2m^\star - 6 \\ &= \mathsf{scr}_{\mathcal{E}}(P) - 2(3m - m^\star) - 9 < \mathsf{scr}_{\mathcal{E}}(P). \end{aligned}$$

Third, observe that all voters prefer p to every dummy of D_3. Therefore, we obtain that P's score exceeds the score of each dummy, proving that P is indeed a unique winner in the election $\mathcal{E}$. □

Theorem 12 Possible President *is* $\mathsf{W}[1]$*-hard for Borda with respect to the parameter t, denoting the number of parties, even if there are only six voters.*

Proof Again, we are going to construct a reduction from the $\mathsf{W}[1]$-hard Multicolored Clique. Let $G = (U, E)$ with $U = U_1 \cup \dots \cup U_k$ be our input with parameter k. We will assume that $k \geq 4$ and that k is even; note that if necessary, we can increase the parameter k to $k+1$ by simply adding a set U_{k+1} of n newly introduced vertices, each of them connected to every vertex in U. Besides the notation from Section 2.2, we define the edge sets $E_{<i}(u) = \bigcup_{j \in [i-1]} E_j(u)$ and $E_{>i}(u) = \bigcup_{j \in [k] \setminus [i]} E_j(u)$ for some $u \in U_i$ and $j \in [k] \setminus \{i\}$.

Again, we let $P = \{p\}$ be our designated party, and we add U_i as a party for each $i \in [k]$, as well as $E_{\{i,j\}}$ for each pair (i, j) with $1 \leq i < j \leq k$. We also add a set D of dummy candidates, where each dummy candidate $d \in D$ belongs to a distinct party $P_d = \{d\}$. We set $|D| = \binom{k}{2} - 1$, hence the number of parties is $t = |D| + \binom{k}{2} + k + 1 = k^2$. We define the set of voters as $\{a_1, a_2, \widetilde{a}_1, \widetilde{a}_2, b, \widetilde{b}\}$.

To define the preferences, we fix an arbitrary ordering $e_1, \dots, e_{|E|}$ over the set E, and let $E^{\rightarrow}$ serve as an abbreviation for $e_1 \succ \cdots \succ e_{|E|}$, and let $E^{\leftarrow}$ stand for the reversed preferences $e_{|E|} \succ \cdots \succ e_1$.

We further define $k-4$ pairwise disjoint subsets $D_2, D_4, \dots, D_{k-4}$ and $\widetilde{D}_2, \widetilde{D}_4, \dots, \widetilde{D}_{k-4}$ of D, with $|D_r| = |\widetilde{D}_r| = r$ for each $r \in \{2, 4, \dots, k-4\}$. In addition, we will use the notation $D_b = D_2 \cup D_4 \cup \cdots \cup D_{k-4}$, and $D_{\widetilde{b}} = \widetilde{D}_2 \cup \widetilde{D}_4 \cup \cdots \cup \widetilde{D}_{k-4}$, and we distinguish two dummies d_1 and d_2 with $d_1, d_2 \in D \setminus (D_b \cup D_{\widetilde{b}})$.

For some index i, we will use the shorthand

$S_{<i}$ for $E_{<i}(u_i^1) \succ u_i^1 \succ E_{<i}(u_i^2) \succ u_i^2 \succ \cdots \succ E_{<i}(u_i^n) \succ u_i^n$,
$\widetilde{S}_{<i}$ for $E_{<i}(u_i^n) \succ u_i^n \succ E_{<i}(u_i^{n-1}) \succ u_i^{n-1} \succ \cdots \succ E_{<i}(u_i^1) \succ u_i^1$,
$S_{>i}$ for $E_{>i}(u_i^1) \succ u_i^1 \succ E_{>i}(u_i^2) \succ u_i^2 \succ \cdots \succ E_{>i}(u_i^n) \succ u_i^n$, and
$\widetilde{S}_{>i}$ for $E_{>i}(u_i^n) \succ u_i^n \succ E_{>i}(u_i^{n-1}) \succ u_i^{n-1} \succ \cdots \succ E_{>i}(u_i^1) \succ u_i^1$.

Observe that $S_{<1} = U_1$ and $S_{>k} = U_k$. Moreover, the set of candidates in $S_{<i}$, which equals the set of candidates in $\widetilde{S}_{<i}$, is exactly $U_i \cup \left(\bigcup_{j \in [i-1]} E_{\{i,j\}}\right)$; the analogous statement holds for the sets $S_{>i}$ and $\widetilde{S}_{>i}$. Using this, we define the preferences of the voters as

$$a_1 : p \succ U_1 \succ S_{<2} \succ S_{<3} \succ \cdots \succ S_{<k} \succ D$$
$$\widetilde{a}_1 : U_1 \succ \widetilde{S}_{<2} \succ \widetilde{S}_{<3} \succ \cdots \succ \widetilde{S}_{<k} \succ d_1 \succ p \succ D \setminus \{d_1\}$$
$$a_2 : p \succ U_k \succ S_{>k-1} \succ \cdots \succ S_{>2} \succ S_{>1} \succ D$$
$$\widetilde{a}_2 : U_k \succ \widetilde{S}_{>k-1} \succ \cdots \succ \widetilde{S}_{>2} \succ \widetilde{S}_{>1} \succ d_2 \succ p \succ D \setminus \{d_2\}$$
$$\begin{aligned} b :\ & p \succ U_1 \succ U_k \succ D_{k-4} \succ U_2 \succ U_{k-1} \succ D_{k-6} \succ \cdots \succ U_{k/2-2} \succ U_{k/2+3} \\ & \succ D_2 \succ U_{k/2-1} \succ U_{k/2+2} \succ U_{k/2} \succ U_{k/2+1} \succ D \setminus D_b \succ E^{\rightarrow} \end{aligned}$$
$$\begin{aligned} \widetilde{b} :\ & p \succ U_k \succ U_1 \succ \widetilde{D}_{k-4} \succ U_{k-1} \succ U_2 \succ \widetilde{D}_{k-6} \succ \cdots \succ U_{k/2+3} \succ U_{k/2-2} \\ & \succ \widetilde{D}_2 \succ U_{k/2+2} \succ U_{k/2-1} \succ U_{k/2+1} \succ U_{k/2} \succ D \setminus D_{\widetilde{b}} \succ E^{\leftarrow} \end{aligned}$$

Note that $|V| = 6$. Since $t = k^2$, the presented reduction is a parameterized reduction; we now prove its correctness.

Recall that in an election $\mathcal{E}$ induced by some nominations, we use the positional scoring rule $(t-1, t-2, \dots, 1, 0)$ corresponding to Borda voting. Observe that p obtains $t-1$ votes from each voter other than $\widetilde{a}_1$ and $\widetilde{a}_2$, and obtains $t - \binom{k}{2} - k - 2$ points from each of $\widetilde{a}_1$ and $\widetilde{a}_2$, since both of them rank all candidates in $U_1 \cup \cdots \cup U_k \cup E$ and exactly one dummy before p. Therefore, we know

$$\mathsf{scr}_{\mathcal{E}}(P) = 6t - k^2 - k - 8. \tag{11}$$

Note also that for each dummy $d \in D$, either all voters prefer p to d, or five of them prefer p to d while the remaining voter ranks d right above p (resulting in a score difference of 1 from this vote). Thus

$$\mathsf{scr}_{\mathcal{E}}(P_d) < \mathsf{scr}_{\mathcal{E}}(P)$$

for each dummy $d \in D$. Consider now some candidate $e \in E_{\{i,j\}}$ for some indices $i, j \in [k]$ with $i \neq j$. Note that both a_1 and a_2 prefer at least two candidates to e irrespective of the nominations in $\mathcal{E}$, while $\widetilde{a}_1$ and $\widetilde{a}_2$ prefer at least one candidate to e in $\mathcal{E}$; so e obtains at most $2(t-2) + 2(t-3)$ points from these four voters. By contrast voters b and $\widetilde{b}$ together contribute $\binom{k}{2} - 1$ points to the score of e, due to the reversed ordering of the candidates in E by b and by $\widetilde{b}$. Hence the total score of $E_{\{i,j\}}$ can be upper-bounded as

$$\mathsf{scr}_{\mathcal{E}}(E_{\{i,j\}}) \leq 2(t-2) + 2(t-3) + \binom{k}{2} - 1 = 4t + \frac{k^2 - k}{2} - 11 < \mathsf{scr}_{\mathcal{E}}(P)$$

where the last line follows by $t = k^2$ and Equality (11).

This implies that P wins the election $\mathcal{E}$ obtained as a result of some nominations if and only if p's score exceeds the score of each of the parties $U_1, \dots, U_k$ in $\mathcal{E}$. For each $i \in [k]$, let $u_i^{\sigma(i)}$ denote the candidate nominated by U_i in $\mathcal{E}$, and for each $j \in [k] \setminus \{i\}$ let $e_{\{i,j\}}$ denote the candidate nominated by $E_{\{i,j\}}$ in $\mathcal{E}$.

Consider therefore the party U_i for some $i \in [k]$. Let us compute for each voter the score it contributes to $\mathsf{scr}_{\mathcal{E}}(U_i)$.

- Voter a_1: First, note that given some index $i' \in [k]$ with $i' \neq i$, voter a_1 prefers all candidates of $U_{i'} \cup \left(\bigcup_{j \in [i'-1]} E_{\{i',j\}}\right)$ to $u_i^{\sigma(i)}$ if $i' < i$, and prefers $u_i^{\sigma(i)}$ to all of these candidates if $i' > i$. Second, a_1 prefers the candidate $e_{\{i,j\}}$ to $u_i^{\sigma(i)}$ in $\mathcal{E}$ if and only if $j < i$ and $e_{\{i,j\}}$ is incident to some vertex u_i^r with $r \in \{1, \dots, \sigma(i)\}$. Third, a_1 prefers p to $u_i^{\sigma(i)}$. Hence, $u_i^{\sigma(i)}$ obtains at least $t - 2 - \sum_{i' \in [i-1]} i' - (i-1) = t - \binom{i+1}{2} - 1$ points from voter a_1, and equality holds if and only if for each $j < i$, edge $e_{\{i,j\}}$ is incident to some vertex u_i^r with $r \in \{1, \dots, \sigma(i)\}$.
- Voter $\widetilde{a}_1$: Similarly to a_1, for each $i' \in [k]$ with $i' \neq i$, voter $\widetilde{a}_1$ prefers all candidates of $U_{i'} \cup \left(\bigcup_{j \in [i'-1]} E_{\{i',j\}}\right)$ to $u_i^{\sigma(i)}$ if $i' < i$, and prefers $u_i^{\sigma(i)}$ to all of these candidates if $i' > i$. Also, $\widetilde{a}_1$ prefers $e_{\{i,j\}}$ to $u_i^{\sigma(i)}$ in $\mathcal{E}$ if and only if $j < i$ and $e_{\{i,j\}}$ is incident to some vertex u_i^r with $r \in \{\sigma(i), \dots, n\}$. Taking into account that $\widetilde{a}_1$ prefers $u_i^{\sigma(i)}$ to p, unlike voter a_1, we get that $u_i^{\sigma(i)}$ obtains at least $t - \binom{i+1}{2}$ points from voter $\widetilde{a}_1$, and equality holds if and only if for each $j < i$, edge $e_{\{i,j\}}$ is incident to some vertex u_i^r with $r \in \{\sigma(i), \dots, n\}$.

Hence, a_1 and $\widetilde{a}_1$ together contribute at least $2t - 2\binom{i+1}{2} - 1$ points to the score of U_i in total, and equality holds exactly if $e_{\{i,j\}}$ is incident to $u_i^{\sigma(i)}$ for each $j \in [i-1]$.

- Voters a_2 and $\widetilde{a}_2$: with a symmetric argument we obtain that together they contribute at least $2t - 3 - 2\sum_{i' \in [k] \setminus [i]}(k - i' + 1) - 2(k - i) = 2t - 2\binom{k-i+2}{2} - 1$ points, and equality holds exactly if $e_{\{i,j\}}$ is incident to $u_i^{\sigma(i)}$ for each $j \in [k] \setminus [i]$.
- Voters b and $\widetilde{b}$: we can see that b and $\widetilde{b}$ together contribute the same score to U_i and to U_{k-i+1}, since in these two votes they appear in different order, and otherwise are preceded by the same number of dummies and the same set of candidates from $U_1 \cup \dots \cup U_k$ in $\mathcal{E}$. Hence, for simplicity we now suppose $i \leq k/2$. Then U_i is preceded in each of these votes by $(k-4)+(k-6)+\dots+(k-2i) = (k-i-2)(i-1)$ dummies, and by the nominees of $U_1, \dots, U_{i-1}$ as well as the nominees of $U_k, U_{k-1}, \dots, U_{k-i+2}$. Thus, b and $\widetilde{b}$ contribute with a total of $2t-5-2(k-i-2)(i-1)-4(i-1) = 2t-5-2(k-i)(i-1)$ points to the score of U_i, and therefore also to the score of U_{k-i+1}, in $\mathcal{E}$.

Summarizing the above observations, we get for each $i \in [k]$ that

$$\begin{aligned} \mathsf{scr}_{\mathcal{E}}(U_i) &\geq 2t - 2\binom{i+1}{2} - 1 + 2t - 2\binom{k-i+2}{2} - 1 + 2t - 5 - 2(k-i)(i-1) \qquad (12) \\ &= 6t - k^2 - k - 9. \end{aligned}$$

By Equality (11) this means

$$\mathsf{scr}_{\mathcal{E}}(U_i) \geq \mathsf{scr}_{\mathcal{E}}(P) - 1 \qquad (13)$$

Furthermore, equality holds for (13) if and only if it holds for (12), which in turn happens if and only $u_i^{\sigma(i)}$ is incident to $e_{\{i,j\}}$, the candidate nominated by $E_{\{i,j\}}$, for each $j \in [k] \setminus \{i\}$.

Observe that P uniquely wins the election $\mathcal{E}$ if and only if Inequality (13) holds with equality for each $i \in [k]$, which in turn holds if and only if the set of k vertices nominated by parties $U_1, \ldots, U_k$ are each adjacent to $k-1$ edges among the $\binom{k}{2}$ edges nominated as candidates by parties $E_{\{i,j\}}$ for $i, j \in [k]$ with $i \neq j$. Thus, we obtain that P can win under some nominations if and only if there exists a clique in G containing a vertex from each set U_i. Therefore, the presented reduction is correct. □

7 Conclusions and future research

In this paper we provided a detailed multivariate complexity analysis of the Possible President problem in the framework of candidate nomination by parties for several classes of positional scoring rules. This research could be extended in several possible directions.

- Is it possible to obtain a complete map of the computational complexity of all scoring rules?
- For scoring rules and parameterizations where Possible President is fixed-parameter tractable, can we obtain a polynomial kernel?
- If the elections are restricted to voters with single-peaked profiles, Faliszewski et al. [20] proved that Possible President for Plurality remains NP-complete. Misra [31] strengthened this result for profiles that are both single-peaked and single-crossing, even when each party has at most two candidates. We are not aware of any results for other positional scoring rules for these and similar special cases. What can be said about parameterized complexity in these cases?
- In this paper we have not dealt with voting rules that cannot be derived from a scoring vector. Results of Cechlárová et al. [10] imply para-NP-hardness for Condorcet-consistent voting rules (Copeland, Llull, Maximin) when parameterized by the size of the largest party s. What is the parameterized complexity of Possible President for these voting rules when parameterized by the number of parties or the number of voters?
- We have also not considered the related Necessary President problem in the candidate nomination framework. Recall that for Plurality Faliszewski et al. [20] showed that Necessary President is coNP-complete, even when the size of the largest party is two, and Cechlárová et al. [10] added the same results for ℓ-Approval, ℓ-Veto and Plurality with run-off. As far as we know, the parameterized complexity of this problem has not been considered yet.

Declarations

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Author Contribution

All authors contributed extensively to the theoretical analysis presented in this paper, and took part in preparing the manuscript. All authors read and approved the final manuscript.

Competing Interest

All authors certify that they have no affiliations with or involvement in any organization or entity with any financial interest or non-financial interest in the subject matter or materials discussed in this manuscript.

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