On the generalized Mycielskian of complements of odd cycles

Anna Gujgiczer[1]

Department of Computer Science and Information Theory
Budapest University of Technology and Economics
and
MTA-BME Lendület Arithmetic Combinatorics Research Group, ELKH, Budapest, Hungary
gujgicza@cs.bme.hu

Gábor Simonyi[2]

Alfréd Rényi Institute of Mathematics, Budapest, Hungary
and
Department of Computer Science and Information Theory
Budapest University of Technology and Economics
simonyi@renyi.hu

Gábor Tardos

Alfréd Rényi Institute of Mathematics, Budapest, Hungary
tardos@renyi.hu

Abstract: The main goal of this talk is to popularize a (special case of a) result of Pan and Zhu according to which whether the generalized Mycielski construction applied to the complement $\overline{C_{2k+1}}$ of an odd cycle makes the chromatic number increase or not depends on the residue of $2k+1$ modulo 4. This surprizing phenomenon is explained by the topological properties of the circular complete graphs $K_{p/q}$ and the trivial observation that $\overline{C_{2k+1}}$ is isomorphic to $K_{(2k+1)/2}$.

Keywords: Mycielsky construction, circular coloring, topological method

1 Introduction

The Mycielski construction is one of the best-known constructions that from any graph G creates a graph $M(G)$ with the same clique number and larger chromatic number. Formally, denoting the chromatic number of a graph F by $\chi(F)$ and its clique number by $\omega(F)$, we have

$$\chi(M(G)) = \chi(G) + 1, \quad \text{while} \quad \omega(M(G)) = \omega(G).$$

The generalized Mycielski construction has a further integer parameter r (describing the number of "levels" in the construction) and it creates the graph $M_r(G)$ from a given graph G as follows.

Definition 1 *For a graph G and positive integer r the generalized Mycielskian $M_r(G)$ of G is defined by*

$$V(M_r(G)) = \{(i, v) \colon 0 \le i \le r-1, v \in V(G)\} \cup \{z\};$$

[1]Research is partially supported by the National Research, Development and Innovation Office (NKFIH) grant K–120706 of NKFIH Hungary.

[2]Research is partially supported by the National Research, Development and Innovation Office (NKFIH) grants K–120706, K–132696 and SNN-135643 of NKFIH Hungary.

[3]Research is partially supported by the National Research, Development and Innovation Office (NKFIH) grants K–132696 and SNN-135643 of NKFIH Hungary.

$E(M_r(G)) = \{\{(i,u),(j,v)\}: \{u,v\} \in E(G) \text{ and } (i=j=0 \text{ or } |i-j|=1)\} \cup \{\{z,(r-1,v)\}: v \in V(G)\}.$
The (usual) Mycielskian of graph G is the special case of $r=2$, i.e., $M(G) = M_2(G)$.

It is easy to see that we always have $\chi(M_r(G)) \le \chi(G)+1$, but unlike in the cases $r \le 2$ when this inequality always holds with equality, for $r \ge 3$ we can have $\chi(M_r(G)) = \chi(G)$ as well. For example, for the complementary graph $\overline{C_7}$ of the 7-cycle we have

$$\chi(M_3(\overline{C_7})) = 4 = \chi(\overline{C_7}).$$

This example appears, for example, in Tardif's paper [11] and this is even used in an essential way in his more recent paper [12], where the author gave new counterexamples to the famous Hedetniemi conjecture for much smaller chromatic numbers than it was done before and he used the above phenomenon to improve his construction a little further. (Since then this record was yet further improved first by Wrochna [14] and then by Tardif [13] where he achieved the best possible such value. The latter ones did not use the generalized Mycielski construction.)

On the other hand, it was proved by Stiebitz [10] that if we apply the generalized Mycielski construction iteratively starting with an odd cycle (in fact, one can also start with a single edge and then obtain an odd cycle after the first iteration), then the chromatic number increases at every step. Since [10] is not easily available, this result is given with proof also in [1], cf. also [5].

Recently we observed that for the complementary graph $\overline{C_{2k+1}}$ of an odd cycle of length at least 5 and large enough r whether we have

$$\chi(M_r(\overline{C_{2k+1}})) = k+1 = \chi(\overline{C_{2k+1}})$$

or

$$\chi(M_r(\overline{C_{2k+1}})) = k+2 = \chi(\overline{C_{2k+1}})+1$$

seems to depend on the residue of the length of the complementary odd cycle modulo 4. With some effort we managed to verify that this is indeed the case but later realized that we have just rediscovered a special case of a more general result due to Pan and Zhu [7]. This made us feel that this result is not known well enough, and especially not in the (by our opinion) both very appealing and surprising form of this special case. This is why we would like to popularize it.

2 Complements of odd cycles, circular chromatic number and topology

Although they do not state it in this special form, let us state the above mentioned special case of Pan and Zhu's result formally.

Theorem 2 (Pan and Zhu [7]) *For every even value of $k > 0$ and $r \ge 1$ we have*

$$\chi(M_r(\overline{C_{2k+1}})) = k+2 = \chi(\overline{C_{2k+1}})+1,$$

while for every odd $k > 1$ and large enough r we have

$$\chi(M_r(\overline{C_{2k+1}})) = k+1 = \chi(\overline{C_{2k+1}}).$$

The proof of Stiebitz's result in [1] gives more than just the chromatic number of the iterated generalized Mycielskian of odd cycles. It is based on the topological method to bound the chromatic number from below introduced by Lovász in his seminal paper [4]. The proof in [1] is based on a lemma (Lemma 3.1 in [1], see also as Theorem 5.9.6 in [5]) which implies that if the topological lower bound on the chromatic number provided by Lovász's method is t for a graph G then it is $t+1$ for $M_r(G)$ for every r. This means that whenever this lower bound is tight for G, then we must have $\chi(M_r(G)) \ge \chi(G)+1$ that can hold only with equality (implying that $t+1$ will also be a tight lower bound for $\chi(M_r(G))$).

Graphs for which (a certain version of) the topological lower bound on their chromatic number is t are called *topologically t-chromatic* in [8]. The observation mentioned in the Introduction and the facts mentioned in the previous paragraph suggested that we should be able to prove that the graph $\overline{C_{2k+1}}$ is topologically $(k+1)$-chromatic if and only if k is even. Towards proving this the main observation was a trivial one: $\overline{C_{2k+1}}$ (for $k \geq 2$) is isomorphic to the circular (also called rational, cf. [2]) complete graph $K_{(2k+1)/2}$ that we define next.

Definition 3 *The circular complete graph $K_{p/q}$ is defined for positive integers $p \geq 2q$ as follows.*

$$V(K_{p/q}) = \{0, 1, \ldots, p-1\};$$

$$E(K_{p/q}) = \{\{i, j\} \colon q \leq |i-j| \leq p-q\}.$$

The name circular complete graph refers to the popular coloring parameter called circular chromatic number that can be defined as

$$\chi_c(G) := \inf \left\{ \frac{p}{q} \colon G \to K_{p/q} \right\},$$

where $F \to H$ denotes the existence of a graph homomorphism from F to H (that is an edge-preserving map from $V(F)$ to $V(H)$). It is well-known that

$$\chi(G) - 1 < \chi_c(G) \leq \chi(G)$$

holds for any graph G, in particular, $\chi(K_{p/q}) = \left\lceil \frac{p}{q} \right\rceil$. For more about graph homomorphisms and the circular chromatic number we refer to [2, 15, 16].

In Subsection 3.3.4 of [9] the last two authors already listed the odd-chromatic circular complete graphs $K_{p/q}$ among those graphs G that are topologically $\chi(G)$-chromatic. As also explained there, this follows from the monotonicity of the topological lower bound of the chromatic number for graph homomorphism and the fact that the circular chromatic number of certain odd-chromatic topologically $\chi(G)$-chromatic graphs can be arbitrarily close to the lower bound $\chi(G)-1$. The first example found for such a family was that of generalized Mycielskians of complete graphs by Lam, Lin, Gu and Song [3]. On the other hand, if $\chi(G)$ is even, then topologically $\chi(G)$-chromatic graphs G always have $\chi_c(G) = \chi(G)$ as shown in [8] (and independently for the important special case of Schrijver graphs also by Meunier in [6]). This latter fact implies that even-chromatic circular complete graphs $K_{p/q}$ with non-integral p/q will not have equality between their chromatic number and its topological lower bound. Indeed, if there was equality for such p/q then $\chi_c(K_{p/q}) = \chi(K_{p/q}) = \lceil p/q \rceil > p/q$ would follow, an obvious contradiction.

The proof of the first statement in Theorem 2 already follows from the above: If $k > 0$ is even then $\left\lceil \frac{2k+1}{2} \right\rceil = k+1$ is odd implying that $\overline{C_{2k+1}} \cong K_{(2k+1)/2}$ is odd-chromatic therefore topologically t-chromatic with $t = k+1 = \chi(\overline{C_{2k+1}})$. This implies (by Stiebitz's result) that $\chi(M_r(\overline{C_{2k+1}})) = \chi(\overline{C_{2k+1}}) + 1 = k+2$.

The proof of the second statement can be checked by finding the corresponding coloring that is not too difficult.

Once one realizes the above relations it is quite natural to ask, whether $\chi(M_r(K_{p/q})) = \chi(K_{p/q})$ always holds when r is sufficiently large and $\chi(K_{p/q}) = \left\lceil \frac{p}{q} \right\rceil$ is even (and p/q non-integral). It is not hard to check that the answer is yes. The result of Pan and Zhu [7] also covers this, but it is even more general (since they also consider multicolorings) but we do not state it in its full generality. What we really wanted to emphasize and make better known is the special case we stated as Theorem 2.

3 Acknowledgement

We thank a referee for spotting a small but disturbing mistake in the original version of this extended abstract.

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