

Winding number and circular 4-coloring of signed graphs

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Abstract

Concerning the recent notion of circular chromatic number of signed graphs, for each given integer k we introduce two signed bipartite graphs, each on $2k^2 - k + 1$ vertices, having shortest negative cycle of length $2k$, and the circular chromatic number 4.

Each of the construction can be viewed as a bipartite analogue of the generalized Mycielski graphs on odd cycles, $M_\ell(C_{2k+1})$. In the course of proving our result, we also obtain a simple proof of the fact that $M_\ell(C_{2k+1})$ and some similar quadrangulations of the projective plane have circular chromatic number 4. These proofs have the advantage that they illuminate, in an elementary manner, the strong relation between algebraic topology and graph coloring problems.

1 Introduction

The problem of building graphs of high girth and high chromatic number is one of the basic questions of graph coloring and its study has led to many further developments. In particular, the original proof of Erdős for the existence of such graphs has led to the development of probabilistic methods in graph theory. Since then several constructive methods were presented, but none are easy to grasp. With a weaker condition of high odd girth instead of high girth, there are several natural classes of graphs. In particular, in the family of the Kneser graphs one can find examples of high odd girth and high chromatic number. The proof of the lower bound for the chromatic number of the Kneser graphs, by L. Lovász [14], was the birthplace of the connection between algebraic topology and graph coloring. Further developing this method, Stiebitz introduced a generalization of the Mycielski construction in [25] to build small graphs of high odd girth and high chromatic number. Generalized Mycielski on odd cycles have been studied independently by many authors and several results on their chromatic number [23, 27, 28], circular chromatic number [1, 3, 7, 12, 24] and on various other related parameters [13, 26] are proved.

In this work, building on the ideas from several works in the literature, we first present a relatively short proof that the generalized Mycielski graphs on odd cycles have circular chromatic number 4. The proof has the advantage of capturing the connection between algebraic topology and graph coloring with elementary

techniques. We then present three similar classes of signed graphs of high negative girth and circular chromatic number 4. The graphs are built similarly to the generalized Mycielski on odd cycles when viewed as a quadrangulation of the projective plane, the main difference being that the subgraph induced by the outer layer induces a Möbius ladder.

In Section 2, we give the necessary notation and the terminology. In Section 3, we provide a historical account of what is known. In Section 4, we discuss three families of signed graphs and in the Section 5, we prove that their circular chromatic number is 4. Finally, we conclude our paper with the Section 6.

2 Notation

We consider simple graphs unless clearly stated otherwise. A *signed (simple) graph* (G, σ) is a graph G together with the assignment σ of signs to the edges. We denote by $(G, -)$ the signed graph G with all edges negative (and $(G, +)$ accordingly). If G is bipartite, then (G, σ) is called a *signed bipartite graph* (in some literature, this term is used to refer to a balanced signed graph, that is a signed graph with no negative cycle). The sign of a structure in (G, σ) (such as a cycle, a closed walk, a path) is the product of the signs of edges in the said structure counting multiplicity.

Given an integer n , $n \geq 3$, we denote by C_n the cycle (graph) on n vertices. That is a 2-regular connected graph on n vertices. Furthermore, we view C_n as a plane graph, that is, the graph together with a planar embedding. For topological use of C_n , one may identify it with the regular polygon on n vertices. Vertices of C_n are normally labeled as v_1, v_2, \dots, v_n . The *exact square* of C_n , denoted $C_n^{\#2}$, is the graph on the same set of vertices where two vertices are adjacent if they are at a distance (exactly) 2 in C_n . Observe that for odd values of n , $C_n^{\#2}$ is also a cycle of length n . For even values of n , $C_n^{\#2}$ consists of two connected components, each isomorphic to a cycle of length $\frac{n}{2}$. They are induced on sets of vertices with odd and even indices and will be denoted, respectively, by $C_n^{\#2o}$ and $C_n^{\#2e}$.

Given a positive real number, we denote by O_r the (geometric) circle of circumference r . That would be a circle of radius $\frac{r}{2\pi}$. The *antipodal* of a point x on O_r is the unique point \bar{x} on O_r which is collinear with x and the center of the circle.

Given a real number r , $r \geq 2$, a *circular r -coloring* of a signed graph (G, σ) is a mapping ψ of the vertices of G to the points of O_r in such a way that when xy is a negative edge, then the distance of $\psi(x)$ from $\psi(y)$ on O_r is at least 1 and if xy is a positive edge, then the distance of $\psi(x)$ from $\overline{\psi(y)}$ is at least 1, equivalently, the distance between $\psi(x)$ and $\psi(y)$ is at most $\frac{r}{2} - 1$. The *circular chromatic number* of (G, σ) , denoted $\chi_c(G, \sigma)$, is the infimum of r such that (G, σ) admits a circular r -coloring. When restricted to signed graphs where all edges are negative, we have the classic notion of circular coloring of graphs. This extension to signed graphs is first presented in [20] noting that a different but similar parameter under a similar name has been introduced in [10]. However, compared to [20], the role of positive and negative edges are exchanged for better suitability with literature on structural theory on signed graphs, especially in regard to the minor theory of signed graphs.

Among basic results, the following should be noted for the purpose of this work. The infimum in the definition is always attained for finite graphs, even allowing multi-edges and positive loops, but a negative loop cannot be colored with a finite r . For the class of signed bipartite (multi)graphs, we have the trivial upper bound of $\chi_c(G, \sigma) \leq 4$, to see this, map the vertices of one part of G to the north pole of O_4 and the vertices of the other part to the east point. Even with such a strong upper bound the problem of determining the exact value of the circular chromatic number of a given signed bipartite graph is of high importance and, in general, quite a difficult problem. In particular, as it is pointed out in [20], using some basic graph operations, namely indicators, one can transform a graph G into a signed bipartite graph $F(G)$ such that

the circular chromatic number of $F(G)$ determines the circular chromatic number of G . A basic example of this sort is the construction $S(G)$, which is obtained from a given graph G by replacing each edge uv of G with a negative 4-cycle $ux_{uv}vy_{uv}$ where x_{uv} and y_{uv} are new and distinct vertices. It is then shown in [20] that $\chi_c(S(G)) = 4 - \frac{4}{\chi_c(G)+1}$. Further connections with some well-known study and theorems, such as the four-color theorem, is discussed in [11] and [19].

Motivated by these observations and in connection with some other studies, some of which are mentioned in the last section, the question of constructing signed bipartite graphs of high negative girth but circular chromatic number 4 is of high interest. In this work, we present two bipartite analogues of the generalized Mycielski graph on odd cycles as examples of signed bipartite graphs.

The proofs also lead to an elementary understanding of the relation between coloring problems of graphs and basic notions of algebraic topology, namely the *winding number*. Recall that given a closed curve γ on the plane, the winding number of γ , defined rather intuitively, is the number of times γ is winded around the origin in the clockwise direction, noting that: if the origin is not in the part bounded by γ , then the winding number is 0 and that winding in anticlockwise direction is presented by a negative number. Here the closed curves we work with are mappings to O_r with the center of O_r being the center of the plane. They can be thought of as continuous mappings of $[0, 1]$ to O_r with the condition that the two endpoints, i.e., 0 and 1 are mapped to the same point.

3 A historical note

In 1955 Mycielski introduced the construction [17] that is now known as the Mycielski construction. His goal of the construction was to build triangle-free graphs of high chromatic number. In this construction, given a graph G one adds a vertex v' for each vertex v of G , which is joined to all neighbors of v in G and then adds a vertex u which is joined to all vertices v' . It is not difficult to prove that the resulting graph has chromatic number $\chi(G) + 1$.

Generalization of the construction, where one adds several layers of copy vertices before adding a universal vertex to the last layer, was first considered independently in Habilitation thesis of M. Stiebitz [25] and Ph.D. thesis of N. Van Ngoc [21]. (The former is written in German, but its result can also be found in [6, 15] and the latter is in Hungarian.) Stiebitz applied methods of algebraic topology to prove that if one starts with K_2 and iteratively builds a generalized Mycielski, at each step the chromatic number would increase by 1. This does not hold for every graph, though. For example, the chromatic number of the complement of C_7 is 4, and any generalized Mycielski of it, except the original one, is also of chromatic number 4. It has been shown recently in [16] that the result of Stiebitz is equivalent to the Borsuk-Ulam theorem.

First English publications of the fact that the generalized Mycielski based on an odd cycle has chromatic number 4 appeared independently in [23, 27, 28]. The proof of Payan [23] is about the special case of $M_k(C_{2k+1})$ as they appear as subgraphs of nonbipartite Cayley graphs on binary groups, but it works the same for any $M_\ell(C_{2k+1})$. This proof has strongly motivated the work presented here. The proof of [27] is presented quite differently, but the hidden idea behind the proof is the same. The result of [28] is more general. It is shown that if G is not bipartite but admits an embedding on the projective plane where all facial cycles are 4-cycles, then $\chi(G) = 4$. That such structures are necessary for 4-chromatic triangle-free projective planar graphs was conjectured in [28] and proved in [5]. The well-known fact that $M_\ell(C_{2k+1})$ quadrangulate the projective plane is evident from our presentation of these graphs in the next section.

The circular chromatic number of Mycielski constructions was first studied in [1]. That of the generalized Mycielski is studied in [3, 7, 12, 24] among others. In particular, that $\chi_c(M_\ell(C_{2k+1})) = 4$ follows, independently, from the general results of [3] and of [24]. In the latter, it is shown that if the lower bound of $2k$

for the chromatic number is proved using topological connectivity, then the same lower bound works for the circular chromatic number as well.

4 The construction

The main body of the construction we will work with is an almost quadrangulation of the cylinder which we define here. Given positive integers ℓ and k , $C_{\ell \times (2k+1)}$ is the graph whose vertex set is $V = \{v_{i,j} \mid 1 \leq i \leq \ell, 1 \leq j \leq 2k+1\}$ with the edge set $E = \{v_{i,j}v_{i+1,j-1}, v_{i,j}v_{i+1,j+1} \mid 1 \leq i \leq \ell-1, 1 \leq j \leq 2k+1\}$. Here, and in the rest of this work, the addition on the indices is taken modular the maximum value of the said index, which is $(\text{mod } 2k+1)$ in this case. We note that, as a graph $C_{\ell \times (2k+1)}$ is isomorphic to the categorical product $P_\ell \times C_{2k+1}$, but the standard labeling of this product does not fit well with our purpose. A general picture of this graph is depicted in Figure 1 where the dashed circles are only presenting the layers, but they will play a key role.

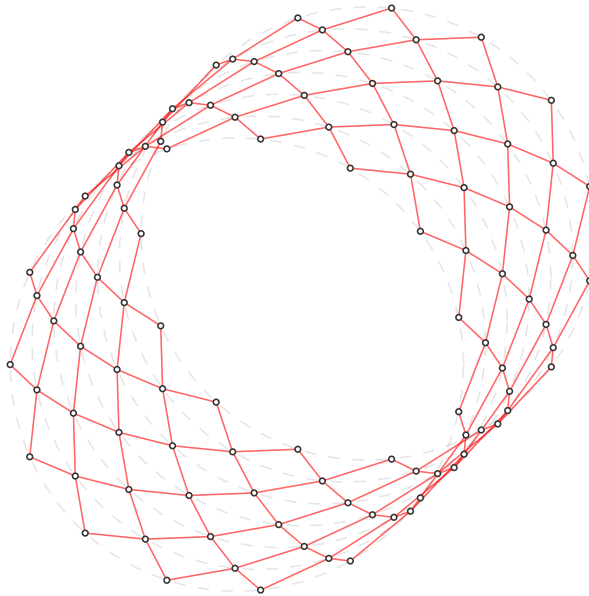


Figure 1: $C_{\ell \times (2k+1)}$ with layers highlighted.

4.1 $M_\ell(C_{2k+1})$

Given positive integers ℓ and k , the generalized Mycielski graph of the odd cycle C_{2k+1} , $M_\ell(C_{2k+1})$ is built from $C_{\ell \times (2k+1)}$ by the following two steps:

- Connect $v_{1,j}$ to $v_{1,j+k}$ (Figure 2, right).
- Add a new vertex u and connect it to all vertices $v_{\ell,j}$, $j = 1, \dots, 2k+1$ (Figure 2, left).

Observe that the added edges in the first item form an isomorphic copy of C_{2k+1} . One can easily observe that starting with this cycle, the classic definition of a generalized Mycielski graph results in the same graph. The graph $M_1(C_3)$ is K_4 . The graph $M_2(C_5)$ is the well-known Gröszsch graph. To show that $M_2(C_5)$ is the smallest 4-chromatic triangle-free graph is proposed as an exercise in [8]. Furthermore, Chvátal showed in [2] that $M_2(C_5)$ is the only 4-chromatic triangle-free graph on 11 vertices.

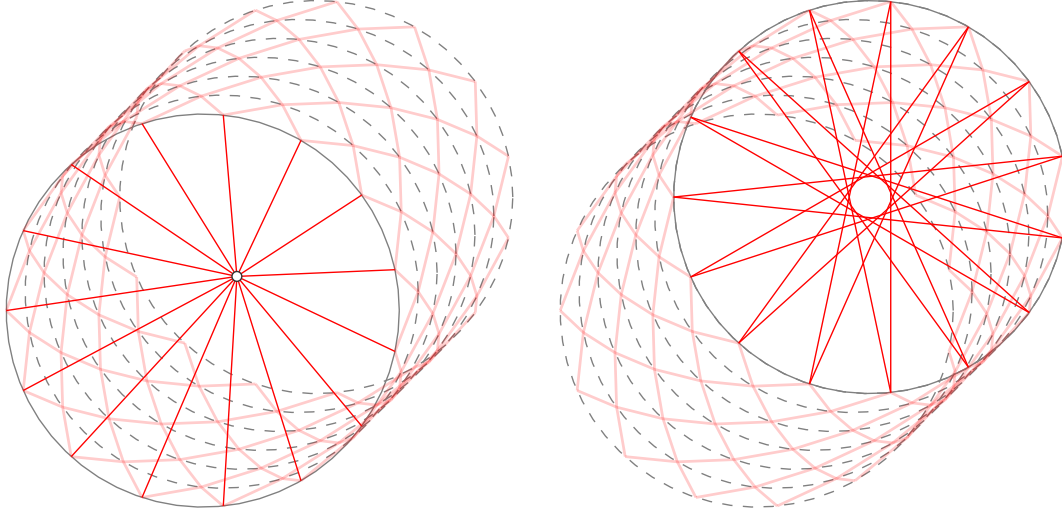


Figure 2: Constructions on bottom and top layers.

The following is a key property of $M_\ell(C_{2k+1})$.

Proposition 1. *The shortest odd cycle of $M_\ell(C_{2k+1})$ is the minimum of $2k + 1$ and $2l + 1$.*

Since this is a folklore fact, we do not provide a proof but we note that the main idea to verify it is also presented in the next proposition.

4.2 $\widehat{BQ}(\ell, 2k - 1)$

Next, given integers ℓ and k satisfying $\ell, k \geq 2$, we define the signed bipartite graph $\widehat{BQ}(\ell, 2k - 1)$ also from $C_{\ell \times (2k-1)}$ as follows.

- Edges of $C_{\ell \times (2k-1)}$ are all negative.
- Connect $v_{1,j}$ to $v_{2,j+k}$ by a positive edge (Figure 3, right).
- Add a new vertex u and connect it to each of the vertices $v_{\ell,j}$, $j = 1, \dots, 2k - 1$, with a negative edge (Figure 3, left).

We view this construction as one of the bipartite analogues of the generalized Mycielski. The second item of the construction, which is presented in Figure 3 (right) is the main difference with the previously known constructions: While in construction of $M_\ell(C_{2k+1})$ we add some edges between vertices of the first layer, in this new construction we add some connection between vertices of the first layer and the second layer. Therefore this operation preserves the bipartition. The underlying graph of the induced subgraph on the first two layers is isomorphic to what is known as the Möbius ladder with $2k - 1$ steps. We will refer to it as such.

The case of $\widehat{BQ}(2, 3)$ is $(K_{3,4}, M)$ depicted in Figure 4. It is the signed bipartite graph where each one of the edges of a maximum matching of $K_{3,4}$ is assigned a positive sign and all the other edges are assigned a negative sign.

The fact that the underlying graph of $\widehat{BQ}(\ell, 2k - 1)$ is bipartite is easily observed. The parity of the levels gives a natural bipartition of the graph. We show that based on the choice of k and l this signed bipartite graph does not have a short negative cycle.

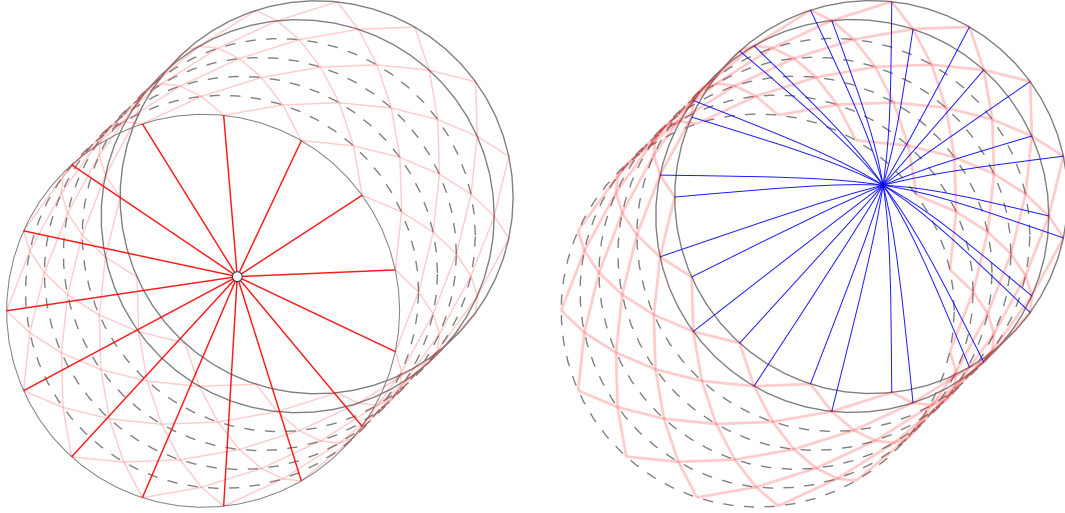


Figure 3: Construction of $\widehat{BQ}(\ell, 2k - 1)$.

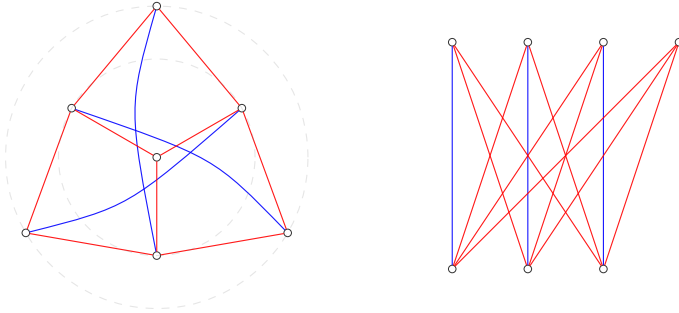


Figure 4: $\widehat{BQ}(2, 3)$, presented two different ways.

Proposition 2. *Given integers l and k where $l, k \geq 2$, the shortest negative cycle of $\widehat{BQ}(\ell, 2k - 1)$ is of length $\min\{2l, 2k\}$.*

Proof. We first present two natural choices for a negative cycle, one of length $2k$ and another of length $2l$. The first is a negative cycle on the first two layers. Take a positive edge and connect its two ends with one of the two paths using only the negative edges that connect the two layers. This would result in a negative cycle of length $2k$. The second negative cycle we consider is by taking a positive edge and connecting each of its ends to the vertex u by a shortest path (all edges negative). One of these paths will be of length l and the other would be of length $l - 1$. Together with the first chosen edge itself then, they form a negative cycle of length $2l$.

It remains to show that the shortest of these two types of cycles gives us the negative girth. To that end, we will first show that a shortest negative cycle can only use one positive edge of $\widehat{BQ}(\ell, 2k - 1)$. Towards a contradiction, let C be a negative cycle with more than two positive edges. We aim to present a negative cycle C' whose length is at most $|C| - 2$. We take two positive edges of C that come consecutively on the cyclic order. Assume xy and $x'y'$ are these two edges and that x' is followed by y in the cyclic order of C (that is to say, there is no positive edge in the $x' - y$ path in C). We remove the two positive edges xy and $x'y'$ and the $x'y$ path connecting them in C , but then we add a xy' copy of this path (which also has no positive edge). The result is a closed walk whose sign is the same as that of C , and whose length is $|C| - 2$. But then this closed walk must contain a negative cycle, whose length then is also at most $|C| - 2$, a contradiction.

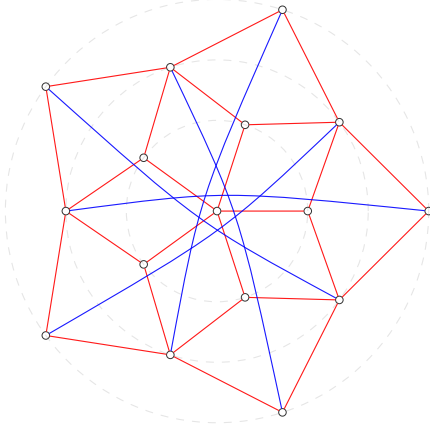


Figure 5: $\widehat{BQ}(3, 5)$.

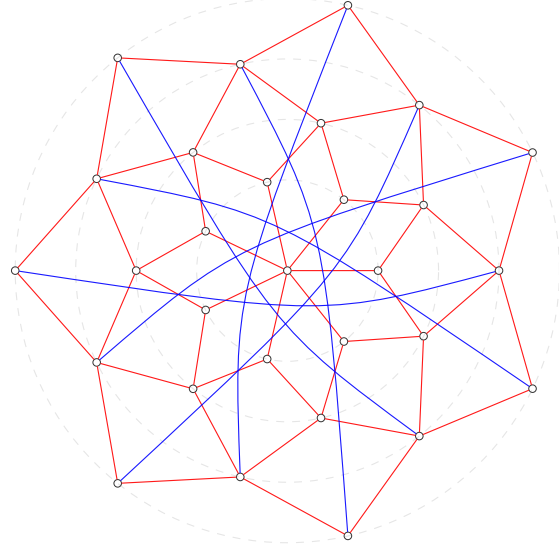


Figure 6: $\widehat{BQ}(4, 7)$.

Finally, if C is a cycle that uses exactly one negative edge, say xy , then the $x - y$ path $P_{xy} = C - xy$ either passes through u in which case we have at least $2l$ edges in C , or the natural image of P_{xy} to the cycle in between the first and second layers also connects x to y . But the shortest such path is of length $2k - 1$, thus P_{xy} is of length at least $2k - 1$, and the negative cycle is of length at least $2k$. \square

4.3 $\widehat{BQ}(\ell, 2k)$

The third family of (signed) graphs we consider in this work are built quite similar to the previous construction. More precisely, given integers ℓ and k satisfying $\ell, k \geq 2$, we define the (signed) graph $\widehat{BQ}(\ell, 2k)$ from $C_{\ell \times (2k)}$ as follows.

- Edges of $C_{\ell \times 2k}$ are all negative.
- Connect $v_{1,j}$ to $v_{1,j+k}$ and $v_{2,j}$ to $v_{2,j+k}$ by negative edges (Figure 7, right).
- Add a new vertex u and connect it to each of the vertices $v_{\ell,j}$, $j = 1, \dots, 2k$, with a negative edge (Figure 7, left).

As an example, the (signed) graphs $\widehat{BQ}(3, 4)$ and $\widehat{BQ}(4, 6)$ are depicted in Figures 8 and 9 respectively.

Proposition 3. *Given integers l and k , where $l, k \geq 2$, the shortest negative cycle of $\widehat{BQ}(l, 2k)$ is of length $\min\{2l - 1, 2k + 1\}$.*

Proof. A cycle of $\widehat{BQ}(l, 2k)$ which does not contain any step of the Möbius ladder induced by the first two layers is even. That is to say any odd cycle has at least one step of this Möbius ladder. A step of the Möbius ladder together with one of the two paths that are connecting the end vertices of this step through the first two layers form an odd cycle of length $2k + 1$. Also, there is another natural choice for an odd cycle constituted by this step and the shortest path which contains the universal vertex u connecting the end vertices of this step. This cycle is of length $2l - 1$. In a similar way as in the proof of Proposition 2 one may conclude that one of these two odd cycles of $\widehat{BQ}(l, 2k)$ is the shortest. \square

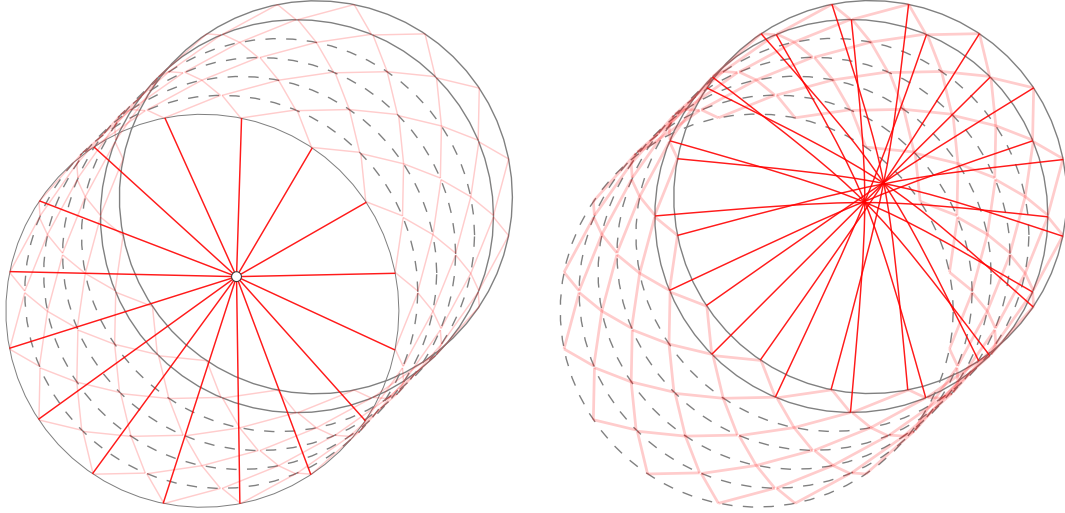


Figure 7: Construction of $\widehat{BQ}(\ell, 2k)$.

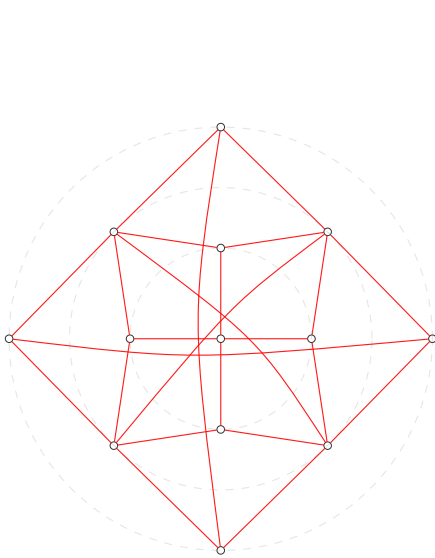


Figure 8: $\widehat{BQ}(3, 4)$.

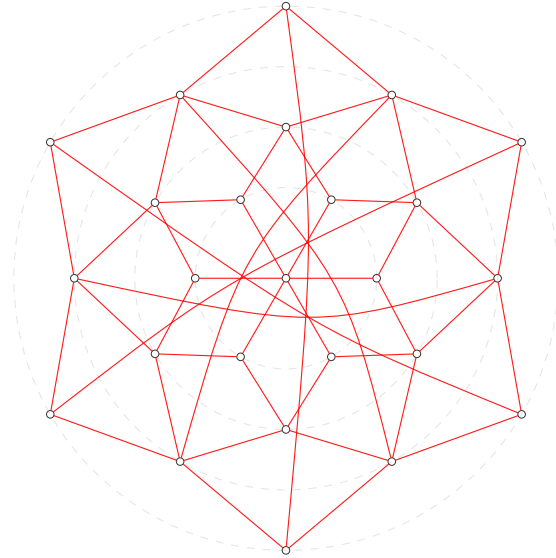


Figure 9: $\widehat{BQ}(4, 6)$.

The two constructions $\widehat{BQ}(\ell, 2k - 1)$ and $\widehat{BQ}(\ell, 2k)$ can be defined uniformly as follows. Starting with an i -star ($i = 2k - 1$ or $i = 2k$) on the projective plane, we complete it to quadrangulation of the planar part except for the vertices on the outer layer which are at distance ℓ or $\ell - 1$ from the center of the star, and assign a negative sign to everything so that all facial 4-cycles are positive. We then complete the outer layer to a Möbius ladder, choosing signs for the crossing edges so that all faces are positive 4-cycles but the non-contractible cycles are negative.

We view this class of signed graphs as *Basic Quadrangulations* of the projective plane and thus use the notation $\widehat{BQ}(\ell, i)$.

4.4 $BM(\ell, 2k)$

The last construction we present here, $BM(\ell, 2k)$, is built from $C_{\ell, 2k}$ as follows. Taking all the edges of this graph as negative edges, on the last layer of the cylinder, as in the other cases, we add a (universal) vertex

which is joined to all vertices of this layer with negative edges. On the first layer we add a set $\{u_1, \dots, u_k\}$ of vertices, then join each $u_i, i = 1, \dots, k$, to $v_{1,i}$ and $v_{1,i+1}$ with negative edges and to $v_{1,i+k}$ and $v_{1,i+k+1}$ with positive edges. See Figure 10 for a depiction.

We leave it to the reader to check the following.

Proposition 4. *Given integers $l, k \geq 2$, the shortest negative cycle of the signed bipartite graph $BM(l, 2k)$ is of length $\min\{2l + 2, 2k\}$.*

In particular, $BM(k - 1, 2k)$ has $2k^2 - k + 1$ vertices and its shortest negative cycles is of length $2k$.

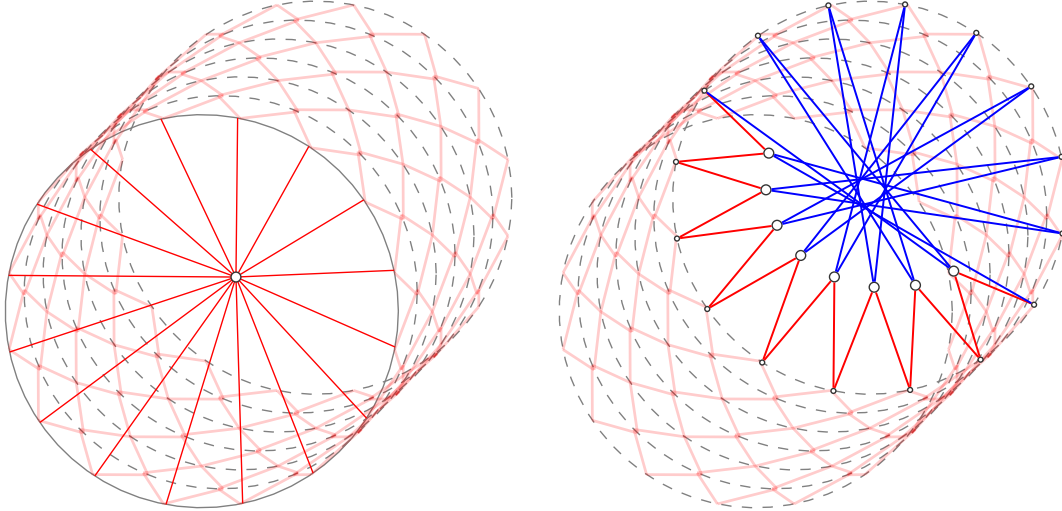


Figure 10: $BM(l, 2k)$

5 Winding number and coloring

Given a simple closed curve γ on the plane, and a continuous mapping φ of γ to O_r , we define the *winding number* of the pair (γ, φ) to be the winding number of the curve $\varphi(\gamma)$ with center of O_r considered as the center of the plane. Intuitively speaking, (γ, φ) tells us how many times the curve γ is wrapped around O_r in the clockwise direction noting that a negative number reflects an anticlockwise mapping. This value then will be denoted by $\omega(\gamma, \varphi)$.

A mapping c of the vertices of the cycle C_n to the points of O_r can be extended to a continuous mapping of C_n to O_r with the former being viewed as the closed curve or the polygon. There are 2^n natural ways to do this. For each pair v_i, v_{i+1} of the vertices of C_n , the pair $c(v_i), c(v_{i+1})$ partitions the circle O_r into two parts. The segment of the polygon that represents the edge $v_i v_{i+1}$ can be projected into one of these two parts. We note that c is allowed to map several vertices of C_n to the same point and that even if v_i and v_{i+1} are mapped to the same point, in our view, they partition the circle O_r into two parts: a part of length 0 and a part of length r .

These 2^n extensions are in a one-to-one correspondence with the 2^n possible orientations of C_n : orient the edge $v_i v_{i+1}$ in such a way that the mapping follows the clockwise direction of O_r .

Given a coloring c of the vertices of the cycle C_n , two extensions of c to a mapping of the polygon to O_r are of special importance. The first is the extension corresponding to the directed cycle C_n . Here $v_i v_{i+1}$ is mapped to the part of the circle where $c(v_{i+1})$ follows $c(v_i)$ in the clockwise direction. Let us denote this extension by c^D . A trivial observation here is that the winding number of (C_n, c^D) is never 0.

The other natural extension is to choose the shortest of the two parts of the circle determined by $c(v_i)$ and $c(v_{i+1})$ and project the line $v_i v_{i+1}$ onto it. The orientation corresponding to this extension then depends on whether $c(v_i)$ is the start or the end of this shorter part of the circle with respect to the clockwise orientation. We denote this extension by c^{sh} and observe that this extension may result in winding number 0 for some choices of c (and r).

Given the cycle C_n , a mapping c of its vertices to O_r and an extension φ of c to the polygon, a combinatorial way to compute $\omega(C_n, \varphi)$ is as follows: take an (open) interval I on O_r which does not contain any image of the vertices of C_n . Then in an extension φ of c to a mapping of the polygon to O_r , each edge of C_n either traverses I completely or does not touch any point of it. Now the winding number $\omega(C_n, \varphi)$ is the number of edges that traverse I in the clockwise direction minus the number of edges that traverse it in the anticlockwise direction (and thus independent of the choice of I).

Let c be a mapping of the vertices of a cycle C_n to the circle O_r . Consider the continuous mapping (C_n, c^D) and an (open) interval I of O_r which does not contain any point $c(v_i)$. Color the edges of C_n with two colors, say green and orange, as follows: if the image of an edge e under c^D contains I , then color it green, otherwise, color it orange. We are interested in the pairs of consecutive edges $v_{i-1}v_i$ and $v_i v_{i+1}$, which are colored differently. If in such a pair, the first edge is colored green, then in the next pair of this sort (next in the cyclic order of indices), the first edge must be orange and vice versa. Thus, the total number of such pairs is even, that is regardless of the choices of n and c .

To use this observation, we will work with certain types of mappings c . We say a mapping c of the vertices of C_n to the points on O_r is *far-polar* if the followings hold: for each i the pair of the points $c(v_{i-1})$ and $c(v_{i+1})$ on O_r partitions O_r into two unequal parts and that $c(v_i)$ is on the larger of the two parts. More generally, a mapping ϕ of the vertices of a graph G to the circle O_r is called far-polar if for each vertex x of G there is a diameter D_x which separates $\phi(x)$ from $\phi(y)$ for all neighbors y of x .

In the following, we present how the condition of c being a far-polar mapping provides a connection between c^D extension of c on C_n and c^{sh} extension of the mapping c on $C_n^{\#2}$.

Lemma 5. *Let c be a far-polar mapping of C_n to O_r and let I be an interval of O_r which does not contain any $c(v_i)$. Then in the extension c^{sh} of a mapping of the one or two cycles in $C_n^{\#2}$ to O_r , the number of edges $v_{i-1}v_{i+1}$ that does not cross over I is an even number.*

Proof. Consider three consecutive vertices v_{i-1}, v_i, v_{i+1} of the cycle. If, following the c^D extension of C_n , both edges $v_{i-1}v_i$ and $v_i v_{i+1}$ are colored orange, that is to say, in the extension they do not pass through I , then since c is far-polar, $c(v_i)$ must be on the longer of $c(v_{i-1})c(v_{i+1})$ or $c(v_{i+1})c(v_{i-1})$ and thus I is on the shorter part. Thus in the extension c^{sh} of $C_n^{\#2}$, $c(v_{i-1})c(v_{i+1})$ passes through I . Similarly, if both edges $v_{i-1}v_i$ and $v_i v_{i+1}$ are colored green, then $c(v_{i-1})c(v_{i+1})$ passes through I . On the other hand, if one of the edges is green and the other orange, then together, they must cover more than half of the O_r . Implying that $c(v_{i-1})c(v_{i+1})$ does not pass through I .

Overall the number of edges of $C_n^{\#2}$ that do not pass through I in c^{sh} extension is the number of vertices of C_n incident with both green and orange edges, where the colors are determined by the extension c^D of C_n . The number of such pairs then must be even as it is observed above. \square

We may now observe that, given a real number r , $r < 4$, any circular r -coloring of C_n must be a far-polar coloring. Thus we have the following two consequences depending on the parity of n .

Lemma 6. *Let c be a circular r -coloring of an even cycle C_n . Let c_o (resp. c_e) be its restriction on the vertices with odd (resp. even) indices. Then the winding numbers of $(C^{\#2o}, c_o^{sh})$ and $(C^{\#2e}, c_e^{sh})$ are of the same parity.*

Proof. That is because after choosing a suitable interval I , by Lemma 5, the total number of edges of $C^{\#2}$ that does not cross over I in the extension c^{sh} is even. As the total number of edges is also even (that is n), the number of edges of $C^{\#2}$ that cross over I is also even. However, the winding number of each of $(C^{\#2o}, c_o^{sh})$ and $(C^{\#2e}, c_e^{sh})$, which is the difference of the number of edges crossing I in the clockwise direction and the number of edges crossing it in the anticlockwise direction, has the same parity as the total number of the edges of the cycle in consideration that cross over I (in the c^{sh} extension). This proves our claim as the sum of the two winding numbers is an even number. \square

Using this lemma, we can build a cylinder of many layers, as shown in the example of Figure 1, with the property that in any circular r -coloring c of the red graph ($r < 4$), all of the dashed grey cycles must have winding numbers of the same parity. Observe that in this construction, the zigzag red cycle between two consecutive layers is an even cycle, and its exact square consists of the two grey cycles presenting the two layers. If we then add structures to the two ends in such a way that one force an odd winding number on one of the grey cycles and the other forces an even winding number on another one of them, then the result would be a graph which admits no circular r -coloring for $r < 4$.

A basic method to achieve these conditions is presented next.

Lemma 7. *Given an odd integer n , a positive real number r , and a far-polar mapping c of C_n to O_r , the winding number $\omega(C_n^{\#2}, c^{sh})$ is an odd number.*

Proof. By Lemma 5, the total number of edges of $C_n^{\#2}$ that does not cross over I is even. As n is an odd number, $C_n^{\#2}$ is isomorphic to C_n , and, hence, the number of edges crossing over I is odd. This is the sum of the number of edges crossing over I in the clockwise direction and in the anticlockwise direction. Thus the winding number, which is the difference between these two numbers, is also an odd number. \square

Applying this lemma on circular r -coloring for $r < 4$ we have the following.

Lemma 8. *Given an odd integer n , $n = 2k + 1$, a real number r satisfying $2 + \frac{1}{k} \leq r < 4$, and a circular r -coloring c of C_n , the winding number $\omega(C_n^{\#2}, c^{sh})$ is an odd number.*

Proof. We observe that if $r < 4$ and c is a circular r -coloring of C_n , then it is, in particular, a far-polar mapping of C_n . That is because for three consecutive vertices v_{i-1} , v_i , and v_{i+1} , having partitioned O_r to two parts based on $c(v_{i-1})$ and $c(v_{i+1})$, the part that contains $c(v_i)$ must be of length at least 2. As $r < 4$, this must be the larger part. Then the statement follows from the previous lemma. \square

Observation 9. *Let G be the star $K_{1,n}$ with u being the central vertex and A being the independent set of order n . Let c be a circular r -coloring of G with $r < 4$. Then for any cycle C built on A , the winding number of (C, c^{sh}) is 0.*

This is observed by taking a small interval I sufficiently close to $c(u)$ and noting that first of all, a vertex of C cannot be mapped to $c(u)$; secondly, since $r < 4$, for any pair x and y of vertices in A in the partition of O_r to two parts by $c(x)$ and $c(y)$, the part containing $c(u)$ is of length at least 2 and thus it is the larger of the two, meaning in the shortest extension, $c(x)c(y)$ will never cross over I .

We may now give a new proof of the following theorem.

Theorem 10. *For any positive integers ℓ and k , we have $\chi_c(M_\ell(C_{2k+1})) = 4$.*

Proof. It is enough to observe that $M_\ell(C_{2k+1})$ is obtained from the $\ell \times (2k + 1)$ cylindrical grid of Figure 1 by adding diagonal edges to the bottom layer (that is connecting pairs at a distance k of the grey cycle) and

adding a universal vertex to the top layer (as mentioned in the previous section). As any circular r -coloring with $r < 4$ is also far-polar, any such a coloring would imply an odd winding number for the layers in c^{sh} extension from one end and an even winding number for the layers from the other end. So a proper mapping to O_r where $r < 4$ is impossible. On the other hand, one can easily color $M_\ell(C_{2k+1})$ with 4 colors, which gives the upper bound 4 on the circular chromatic number as well. \square

Next, we show that $\widehat{BQ}(\ell, 2k - 1)$ shares the same property. We will note later that Theorem 10 follows from the next theorem.

Theorem 11. *For given positive integers ℓ and k , satisfying $\ell, k \geq 2$, we have $\chi_c(\widehat{BQ}(\ell, 2k - 1)) = 4$.*

Proof. Towards a contradiction, let c be a circular r -coloring of $\widehat{BQ}(\ell, 2k - 1)$ with $r < 4$. We will have a contradiction if we show that the cycle C' formed on $v_{1,1}v_{1,2} \cdots v_{1,2k-1}$ in this cyclic order has an odd winding number under the mapping c^{sh} (restricted on the vertices of this cycle). We emphasize that edges of C' are not in $\widehat{BQ}(\ell, 2k - 1)$.

To this end we first consider another cycle, C^* , (also not part of our graph) by considering the following sequence of vertices of the first layer of $\widehat{BQ}(\ell, 2k - 1)$: $v_{1,1}v_{1,k+1}v_{1,2}v_{1,k+2} \cdots v_{1,k}$. Note that in this cycle $v_{1,j}$ is followed by $v_{1,j+k}$ where the addition is taken $(\text{mod } 2k - 1)$. We may also note that this is the diagonally drawn cycle on the first layer of Figure 2 (right).

Our claim is that the mapping c , viewed as a mapping of the vertices of C^* to O_r , is a far-polar mapping. Toward proving the claim, we consider $c(v_{1,j})$, $c(v_{1,j+k})$, and $c(v_{1,j+1})$. The first observation is that since $v_{2,j}$ is adjacent to both $v_{1,j}$ and $v_{1,j+1}$ with negative edges, the points $c(v_{1,j})$ and $c(v_{1,j+1})$ of O_r partition O_r in such a way that the part containing $c(v_{2,j})$ is at least 2. As $r < 4$, it follows that $c(v_{2,j})$ is on the larger part of O_r when it is partitioned by $c(v_{1,j})$ and $c(v_{1,j+1})$. It remains to show that $c(v_{1,j+k})$ is also on the same part. If not, that is if $c(v_{1,j+k})$ is on the shorter side of O_r , then one of the arcs $c(v_{1,j+k})c(v_{2,j})$ and $c(v_{2,j})c(v_{1,j+k})$ contains the shorter side of $c(v_{1,j})c(v_{2,j})$ and the other contains the shorter side of $c(v_{1,j+1})c(v_{2,j})$. As each of these shorter arcs is of length at least one, we conclude that the distance of $c(v_{1,j+k})$ and $c(v_{2,j})$ is at least one. However, since c is a circular r -coloring where $r < 4$ and $v_{1,j+k}v_{2,j}$ is a positive edge, they should be at a distance at most $\frac{r}{2} - 1 < 1$, a contradiction.

Finally, observing that C' is the exact square of C^* , and by Lemma 7, we conclude that the winding number of C' is odd. \square

To prove that $\chi_c(\widehat{BQ}(\ell, 2k)) = 4$ we need a few more lemmas.

Lemma 12. *Let c be a far-polar mapping of C_4 to O_r . Then the winding number $w(C_4, c^D)$ is 2.*

Proof. The points $c(v_1)$ and $c(v_3)$ partition O_r into two unequal parts. Since c is a far-polar coloring, we know that $c(v_2)$ and $c(v_4)$ both should be on the larger of these two parts. Without loss of generality, we may assume that images of the vertices are in the following cyclic order: $c(v_1), c(v_3), c(v_2), c(v_4)$. Let I be an interval of O_r that does not contain any $c(v_i)$. As the winding number is independent of the choice of I we can choose I to be in $c(v_3)c(v_2)$. Then following the orientation of C_4 and clockwise direction of O_r , the arcs $c(v_1)c(v_2)$ and $c(v_3)c(v_4)$ contain the interval I while the arcs $c(v_2)c(v_3)$ and $c(v_4)c(v_1)$ do not intersect it. \square

Lemma 13. *Let c be a far-polar mapping of C_{2k} to O_r and let the edges of C_{2k} be e_1, e_2, \dots, e_{2k} . The number of edges colored green in the extension c^D of c has the same parity as the number of odd (or even) indexed vertices being incident to both green and orange edges.*

Proof. Let J be the set of vertices of C_{2k} incident to both green and orange edges. Let J_o and J_e be the partition of J into odd and even indexed vertices (a natural bipartition of on C_{2k}).

Consider a maximal green path in C_{2k} . Thus the two ends of each such path are in J . Moreover, if the length of the path is even, then both ends of the path belong to the same subset J_o or J_e of J . Thus each even green path contributes 0 to one of J_o or J_e and 2 to the other. If the length of the path is odd, then one of its ends is in J_o and the other is in J_e , thus contributing 1 to each of these two sets. The claim then follows as the odd length green-paths determine the parity of the total number of green edges. \square

Lemma 14. *Let c be a far-polar coloring of the cycle C_{4k} . The number of edges colored green in the extension c^D of c and the winding number $w(C_{4k}^{\#2e}, c^{sh})$ (or similarly $w(C_{4k}^{\#2o}, c^{sh})$) are of the same parity.*

Proof. For each edge $v_{i-1}v_{i+1}$ of $C_{4k}^{\#2e}$ there is an odd indexed vertex v_i of C_{4k} corresponding to it. (And similarly, for each edge $v_{i-1}v_{i+1}$ of $C_{4k}^{\#2o}$ there is an even indexed vertex v_i of C_{4k} .) As we have observed before, an edge $v_{i-1}v_{i+1}$ in $C^{\#2}$ does not cross over the interval I in the c^{sh} extension if and only if the edges incident to v_i are colored differently (i.e. one of $v_{i-1}v_i$ and $v_i v_{i+1}$ is green, the other is orange). So the number of non-crossing edges in $C_{4k}^{\#2e}$ in the c^{sh} extension is just the number of odd indexed vertices being incident to both green and orange edges in C_{4k} . From the previous lemma, we know that this number has the same parity as the total number of green edges in the cycle C_{4k} . As $C_{4k}^{\#2e}$ is a cycle on $2k$ vertices, its total number of edges is an even number, so the total number of edges which cross I should also have the same parity, and so does the difference of the number of edges cross I in the clockwise direction and anticlockwise direction. This completes the proof. \square

We use M_{2k} to denote the Möbius ladder with $2k$ steps. As a graph that is isomorphic to the graph build on C_{4k} by adding an edge between each pair of vertices at a distance $2k$. In the next lemma we show that the Möbius ladder M_{2k} can replace the role of the odd cycle in Lemma 7. It can then be used similarly to build families of graphs with circular chromatic number at least 4.

Lemma 15. *For any far-polar mapping c of M_{2k} to O_r , the winding number $w(C_{4k}^{\#2e}, c^{sh})$ (or similarly $w(C_{4k}^{\#2o}, c^{sh})$) is odd.*

Proof. By Lemma 14, it is enough to prove that the number of green edges of C_{4k} in the extension c^D of c is odd. We use the notation $C_{1,2,3,\dots,t}$ for oriented cycle with vertices v_1, v_2, \dots, v_t and directed edges $\overrightarrow{v_i v_{i+1}}$ for $i \pmod t$. We will view M_{2k} as union of $2k$ 4-cycles, see Figure 11 for reference.

Consider all oriented 4-cycles formed by two consecutive steps of ladder, $C_1 : C_{1,2,(2k+2),(2k+1)}$, $C_2 : C_{2,3,(2k+3),(2k+2)}$, \dots , $C_{2k} : C_{2k,(2k+1),1,4k}$. By Lemma 12 we know that each C_i 's has two green edges in c^D extension. Therefore, in total, the sum of the number of their green edges is an even number as well. To prove our claim, we present a different counting of this number. Consider the oriented $4k$ -cycle, $C_{1,2,3,\dots,4k}$, half of its edges (from $\overrightarrow{v_1 v_2}$ to $\overrightarrow{v_{2k} v_{2k+1}}$) agree in orientation with the one in the corresponding C_i , but the other half is oriented the opposite direction. So if we want to get back the same orientation as in the C_i 's, we should switch $2k$ edges. As changing the orientation of a green edge makes it orange and vice versa, we switch the parity of the number of green edges an even number of times. Now we have to consider the steps of the ladder as well. Except for the edge between v_1 and v_{2k+1} , every other step $v_i v_{2k+i}$ is oriented as $\overrightarrow{v_{2k+i} v_i}$ in C_i and as $\overrightarrow{v_i v_{2k+i}}$ in C_{i-1} (for $1 < i \leq 2k$). So they contribute exactly one green edge (in one of their orientations) to the total sum. The edge between v_1 and v_{2k+1} is oriented as $\overrightarrow{v_{2k+1} v_1}$ in both C_1 and C_{2k} , contributing 0 or 2 to the total sum. Therefore the contribution of the steps is odd in total. So, in summary, starting with the oriented C_{4k} , changing the orientation of an even number of its edges, and then adding the steps of the ladder, we should get back the same number of green edges as we had in total in the C_i 's. Since that is an even number, the oriented C_{4k} must have an odd number of green edges. \square

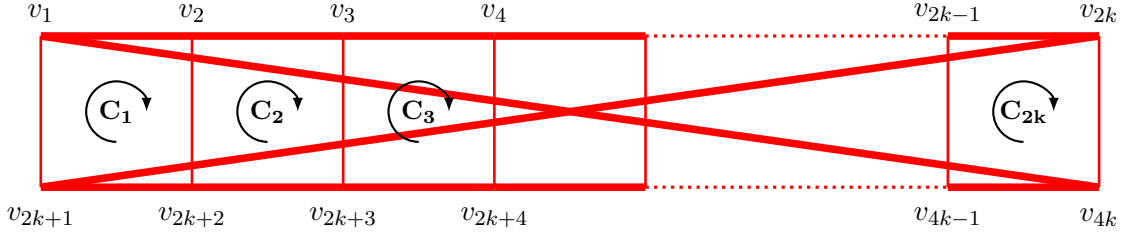


Figure 11: Möbius ladder M_{2k} .

We can now state our theorem for $\widehat{BQ}(\ell, 2k)$.

Theorem 16. *For any positive integers ℓ and k , we have $\chi_c \widehat{BQ}(\ell, 2k) = 4$.*

Proof. As in the previous cases, we can consider $\widehat{BQ}(\ell, 2k)$ as a graph obtained from the $\ell \times 2k$ cylindrical grid by adding a universal vertex on the first layer and completing the last two layers into a Möbius ladder. Any $r < 4$ r -coloring would be a far-polar coloring of this graph, which, by Lemma 15 would imply an odd winding number for each of the last layers in the C^{sh} extension, but by Observation 9 the first layer has the winding number 0, but by Lemma 6 all layers have the same parity of winding number. \square

Finally, we use this to prove that $BM(\ell, 2k)$ also has the same circular chromatic number.

Theorem 17. *For any positive integers ℓ and k , we have $\chi_c(BM(\ell, 2k)) = 4$.*

Proof. As $BM(\ell, 2k)$ is a signed bipartite graph, 4 is an upper bound on its circular chromatic number. To prove that it is also a lower bound, we consider $\widehat{BQ}(\ell + 2, 2k)$ and switch at first $2k$ vertices of the Möbius ladder built on the first two layers. These are vertices labelled $v_{1,1}, v_{2,1}, v_{1,2}, v_{2,2}, \dots, v_{1,k}, v_{2,k}$. At the end all diagonal edges of the Möbius ladder are positive. We consider a homomorphic image of this signed graph by identifying two ends of each diagonal edge of the Möbius ladder. That is $v_{1,1}$ is identified with $v_{1,k+1}$, $v_{2,1}$ is identified with $v_{2,k+1}$ and so on. Then, we can identify each $v_{1,i}$ with $v_{3,i+k}$ as well ($i \in \{1, 2, \dots, k\}$). It can then be verified that the image is the signed graph obtained from $BM(\ell, 2k)$ by adding a positive loop to each of the k vertices u_j and to the next layer. However, a positive loop does not change the circular chromatic number of a signed graph. Thus we have $\chi_c(BM(\ell, 2k)) \geq \chi_c(\widehat{BQ}(\ell + 2, 2k)) = 4$. \square

We note that by identifying the two ends of each positive edge in $\widehat{BQ}(\ell, 2k + 1)$ we get a copy of $M_\ell(C_{2k+1})$ together with some positive loops. Thus, in a similar fashion, one can view Theorem 10 as a corollary of Theorem 11.

6 Concluding remarks

The special subclass of $M_k(C_{2k+1})$, on $2k^2 + k + 1$ vertices, is conjectured in [27] to have the smallest number of vertices among 4-chromatic graphs of odd-girth $2k + 1$. In [4], this is verified to be the case with an added assumption that every pair of odd cycles share a vertex. For the general case, a lower bound of $(k - 1)^2$ for the number of vertices of a 4-critical graph of odd girth $2k + 1$ is given in [9] modifying the method of [22]. A natural bipartite analogue of this question is to find the smallest number of vertices of a signed bipartite graph of negative girth $2k$ whose circular chromatic number is 4. Here we gave two families of such graphs, where the graphs of negative girth $2k$ have $2k^2 - k + 1$ vertices.

The starting point of this work has been a joint work of the second author with Lan Anh Pham and Zhouningxin Wang on the study of C_{-4} -critical signed graphs (for a definition, see [18]). In an unpublished work, they have shown that a C_{-4} -critical signed graph of negative girth at least $2k$ must have at least k^2 vertices. Based on the fact that $\chi_c(C_{-4}) = \frac{8}{3}$, our result in this work implies that $\widehat{BQ}(k, 2k - 1)$ is a signed bipartite graph of negative girth $2k$ which does not map to C_{-4} . Thus $\widehat{BQ}(k, 2k - 1)$ contains a C_{-4} -critical signed graph. As $\widehat{BQ}(k, 2k - 1)$ has $2k^2 - k + 1$ vertices, this implies the smallest number of the vertices of a C_{-4} -critical signed graph of negative girth $2k$ is somewhere between k^2 and $2k^2 - k + 1$.

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References

- [1] Gerard J. Chang, Lingling Huang, and Xuding Zhu. Circular chromatic numbers of Mycielski’s graphs. Discrete Math., 205(1-3):23–37, 1999.
- [2] Vašek Chvátal. The minimality of the Mycielski graph. In Graphs and combinatorics (Proc. Capital Conf., George Washington Univ., Washington, D.C., 1973), Lecture Notes in Math., Vol. 406, pages 243–246. Springer, Berlin, 1974.
- [3] Matt DeVos, Luis Goddyn, Bojan Mohar, Dirk Vertigan, and Xuding Zhu. Coloring-flow duality of embedded graphs. Trans. Amer. Math. Soc., 357(10):3993–4016, 2005.
- [4] Louis Esperet and Matěj Stehlík. The width of quadrangulations of the projective plane. J. Graph Theory, 89(1):76–88, 2018.
- [5] John Gimbel and Carsten Thomassen. Coloring graphs with fixed genus and girth. Trans. Amer. Math. Soc., 349(11):4555–4564, 1997.
- [6] András Gyárfás, Tommy Jensen, and Michael Stiebitz. On graphs with strongly independent color-classes. J. Graph Theory, 46(1):1–14, 2004.
- [7] Hossein Hajiabolhassan and Xuding Zhu. Circular chromatic number and Mycielski construction. J. Graph Theory, 44(2):106–115, 2003.
- [8] Frank Harary. Graph theory. Addison-Wesley Publishing Co., Reading, Mass.-Menlo Park, Calif.-London, 1969.
- [9] Tao Jiang. Small odd cycles in 4-chromatic graphs. J. Graph Theory, 37(2):115–117, 2001.
- [10] Yingli Kang and Eckhard Steffen. Circular coloring of signed graphs. J. Graph Theory, 87(2):135–148, 2018.
- [11] Kardoš, František, Narboni, Jonathan Narboni, Reza Naserasr, and Zhouningxin Wang. Circular $(4 - \epsilon)$ -coloring of signed graphs. Submitted.
- [12] Peter Che Bor Lam, Wensong Lin, Guohua Gu, and Zengmin Song. Circular chromatic number and a generalization of the construction of Mycielski. J. Combin. Theory Ser. B, 89(2):195–205, 2003.

- [13] Wensong Lin, Jianzhuan Wu, Peter Che Bor Lam, and Guohua Gu. Several parameters of generalized mycielskians. Discrete applied mathematics, 154(8):1173–1182, 2006.
- [14] László Lovász. Kneser’s conjecture, chromatic number, and homotopy. J. Combin. Theory Ser. A, 25(3):319–324, 1978.
- [15] Jiří Matoušek. Using the Borsuk-Ulam theorem: lectures on topological methods in combinatorics and geometry. Springer, 2008.
- [16] Tobias Müller and Matěj Stehlík. Generalised Mycielski graphs and the Borsuk-Ulam theorem. Electron. J. Combin., 26(4):Paper No. 4.8, 8, 2019.
- [17] Jan Mycielski. Sur le coloriage des graphes. Colloq. Math., 3:161–162, 1955.
- [18] Reza Naserasr, Lan Anh Pham, and Zhouningxin Wang. Density of C_{-4} -critical signed graphs. J. Combin. Theory Ser. B, 153:81–104, 2022.
- [19] Reza Naserasr and Zhouningxin Wang. Signed bipartite circular cliques and a bipartite analogue of grötzsch’s theorem. Submitted.
- [20] Reza Naserasr, Zhouningxin Wang, and Xuding Zhu. Circular chromatic number of signed graphs. Electron. J. Combin., 28(2):Paper No. 2.44, 40, 2021.
- [21] N. Van Ngoc. On graph colorings (Hungarian). Ph.D. Thesis, Hungarian Academy of Sciences, Budapest. 1987.
- [22] A. Nilli. Short odd cycles in 4-chromatic graphs. J. Graph Theory, 31(2):145–147, 1999.
- [23] Charles Payan. On the chromatic number of cube-like graphs. Discrete Math., 103(3):271–277, 1992.
- [24] Gábor Simonyi and Gábor Tardos. Local chromatic number, Ky Fan’s theorem and circular colorings. Combinatorica, 26(5):587–626, 2006.
- [25] Michael Stiebitz. Beiträge zur Theorie der färbungskritischen Graphen. 1985.
- [26] Claude Tardif. Fractional chromatic numbers of cones over graphs. J. Graph Theory, 38(2):87–94, 2001.
- [27] Nguyen Van Ngoc and Zsolt Tuza. 4-chromatic graphs with large odd girth. volume 138, pages 387–392. 1995. 14th British Combinatorial Conference (Keele, 1993).
- [28] Dale A. Youngs. 4-chromatic projective graphs. J. Graph Theory, 21(2):219–227, 1996.