Which crossing number is it, anyway?

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Abstract

A *drawing* of a graph G is a mapping which assigns to each vertex a point of the plane and to each edge a simple continuous arc connecting the corresponding two points. The *crossing number* of G is the minimum number of crossing points in any drawing of G. We define two new parameters, as follows. The *pairwise crossing number* (resp. the *odd-crossing number*) of G is the minimum number of pairs of edges that cross (resp. cross an odd number of times) over all drawings of G. We prove that the largest of these numbers (the crossing number) cannot exceed twice the square of the smallest (the odd-crossing number). Our proof is based on the following generalization of an old result of Hanani, which is of independent interest. Let G be a graph and let E_0 be a subset of its edges such that there is a drawing of G, in which every edge belonging to E_0 crosses any other edge an even number of times. Then G can be redrawn so that the elements of E_0 are not involved in any crossing. Finally, we show that the determination of each of these parameters is an NP-hard problem and it is NP-complete in the case of the crossing number and the odd-crossing number.

1 Introduction

The crossing number of a graph G is usually defined as "the minimum number of edge crossings in any drawing of G in the plane" [BL84]. However, one has to be careful with this definition, because it can be interpreted in several ways. Sometimes it is assumed that in a proper drawing no two edges cross more than once, and if two edges share an endpoint, they cannot have another point in common ([WB78], [B91]). Many authors do not make this assumption ([T70], [GJ83], [SSSV97]). If two edges are allowed to cross several times, we may count their intersections with multiplicity or without. We may also wish to impose some further restrictions on the drawings (e.g., the edges

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must be straight-line segments [J71], or polygonal paths of length at most k [BD93]). No matter what definition we use, the determination of the crossing number of a graph appears to be an extremely difficult task ([GJ83], [B91]). In fact, we do not even know the asymptotic value of *any* of the above quantities for the complete graph K_n with n vertices and for the complete bipartite graph $K_{n,n}$ with $2n$ vertices, as n tends to infinity [RT97]. The latter question, raised more than fifty years ago, is often referred to as Turán's Brick Factory Problem [T77] or as Zarankiewicz's problem [G69].

In the present paper, we investigate the relationship between various crossing numbers. First we agree on the terminology.

A *drawing* of a simple undirected graph is a mapping f that assigns to each vertex a distinct point in the plane and to each edge uv a continuous arc (i.e., a homeomorphic image of a closed interval) connecting $f(u)$ and $f(v)$, not passing through the image of any other vertex. For simplicity, the arc assigned to uv is called an *edge* of the drawing, and if this leads to no confusion, it is also denoted by uv. We assume that no three edges have an interior point in common, and if two edges share an interior point p, then they cross at p. We also assume that any two edges of a drawing have a only a finite number of *crossings* (common interior points). A common endpoint of two edges does not count as a crossing.

Definition. Let G be a simple undirected graph.

(i) The *rectilinear crossing number* of G, LIN-CR(G), is the minimum number of crossings in any drawing of G, in which every edge is represented by a straight-line segment.

(ii) The *crossing number* of G, CR(G), is the minimum number of edge crossings in any drawing of G.

(iii) The *pairwise crossing number* of G, PAIR-CR(G), is the minimum number of pairs of edges (e, e') such that e and e' determine at least one crossing, over all drawings of G. (That is, now crossings are counted *without* multiplicities.)

(iv) The *odd-crossing number* of G, ODD-CR(G), is the minimum number of pairs of edges (e, e') such that e and e' cross an odd number of times.

Clearly, we have

$$\text{ODD-CR}(G) \leq \text{PAIR-CR}(G) \leq \text{CR}(G) \leq \text{LIN-CR}(G),$$

It was shown by Bienstock and Dean [BD93] that there are graphs with crossing number 4, whose rectilinear crossing numbers are arbitrarily large. On the other hand, we cannot rule out the possibility that

$$\text{ODD-CR}(G) = \text{PAIR-CR}(G) = \text{CR}(G)$$

for every graph G. The only result in this direction is the following remarkable theorem of Hanani and Tutte (see also [LPS97]).

Theorem A. [Ch34], [T70] *If a graph G can be drawn in the plane so that any two edges which do not share an endpoint cross an even number of times, then G is planar.*

For a generalization of this result to other surfaces, see [CN99].

In a fixed drawing of a graph G, an edge is called *even* if it crosses every other edge an *even* number of times. It follows from Theorem A that if all edges of G are even, i.e., if $\text{ODD-CR}(G) = 0$, then $\text{CR}(G) = 0$. (In this case, by Fáry's theorem [F48], we also have $\text{LIN-CR}(G) = 0$.) In the next section, we establish the following generalization of this statement.

Theorem 1. *For a fixed drawing of a graph G, let $G_0 \subseteq G$ denote the subgraph formed by all even edges.*

Then G can be drawn in such a way that the edges belonging to G_0 are not involved in any crossing.

At the end of the next section, we show how Theorem 1 implies that if the odd-crossing number of a graph is bounded, then its crossing number cannot be arbitrarily large. More precisely, we prove

Theorem 2. *The crossing number of any graph G satisfies*

$$\text{CR}(G) \leq 2\left(\text{ODD-CR}(G)\right)^2.$$

It was discovered by Leighton [L84] that the crossing number can be used to obtain a lower bound on the chip area required for the VLSI circuit layout of a graph. For this purpose, he proved the following general lower bound for $\text{CR}(G)$, which was discovered independently by Ajtai, Chvátal, Newborn, and Szemerédi. The best known constant, $1/33.75$, in the theorem is due to Pach and Tóth.

Theorem B. [ACNS82], [L84], [PT97] *Let G be a graph with vertex set $V(G)$ and edge set $E(G)$ such that $|E(G)| \geq 7.5|V(G)|$. Then we have*

$$\text{CR}(G) \geq \frac{1}{33.75} \frac{|E(G)|^3}{|V(G)|^2}.$$

In Section 3, we prove that a similar inequality holds for the odd-crossing number.

Theorem 3. *Let G be a graph with vertex set $V(G)$ and edge set $E(G)$ such that $|E(G)| \geq 4|V(G)|$. Then we have*

$$\text{ODD-CR}(G) \geq \frac{1}{64} \frac{|E(G)|^3}{|V(G)|^2}.$$

It was shown by Garey and Johnson [GJ83] that, given a graph G and an integer K, it is an NP-complete problem to decide whether $\text{CR}(G) \leq K$. In the last section we show that the same is true for the odd-crossing number.

Theorem 4. *Given a graph G and an integer K, it is an NP-complete problem to decide whether* $\textsc{odd-cr}(G) \le K$.

We can not prove the same for the pair-crossing number. (See Remark at the end of Section 4.)

2 Proofs of Theorems 1 and 2

First we establish Theorem 1. The proof somewhat resembles a proof of Kuratowski's theorem (see [BM76]).

Suppose that Theorem 1 is false. Then there exists a graph G with vertex set $V(G) = V$ and edge set $E(G) = E$, and there is a subset $E_0 \subseteq E$ such that G has a drawing, in which every edge in E_0 is even, but there is no drawing, in which none of these edges is involved in any crossing. Let us fix a *minimal counterexample* to Theorem 1, i.e., a pair (G, E_0) such that there exists no other pair $(\overline{G}, \overline{E}_0)$, $\overline{E}_0 \subseteq \overline{E}$, with the above property, for which the triple $(|\overline{E}|, |\overline{E}_0|, |\overline{V}|)$ would precede $(|E|, |E_0|, |V|)$ in the lexicographic ordering. In particular, it follows from the minimality of (G, E_0) that G is connected.

If it leads to no confusion, throughout this section G will stand both for the graph and for a particular drawing, in which all edges of E_0 are even. Let $G_0 = (V, E_0)$. A path (resp. cycle) in G is said to be an *E_0-path* (resp. *E_0-cycle*), if all of its edges belong to E_0. Two edges are called *independent*, if they do not share an endpoint.

Claim 1. *G and $G_0 = (V, E_0)$ satisfy the following properties.*
(i) *There is no vertex of degree 1 in G_0.*
(ii) *There are no two adjacent vertices of degree 2 in G_0.*
(iii) *In any subdivision of K_5 or $K_{3,3}$ contained in G, there are two paths representing independent edges, such that neither of them is an E_0-path.*

Proof. If v has degree 1 in $G_0 = (V, E_0)$, and $uv \in E_0$, then $(G, E_0 \setminus \{uv\})$ is another counterexample, (lexicographically) smaller than (G, E_0). If u, v both have degree 2 in G_0 and $uv \in E_0$, then contract the edge uv and remove all multiple edges (that is, keep only one copy of each edge), to obtain a smaller counterexample. Finally, part (iii) is an immediate corollary to Theorem A. □

Let C be any E_0-cycle of G. A connected subgraph $B \subset G$ is a *bridge* of C (in G) if it consists of either a single edge whose endpoints belong to $V(C)$, or of a connected component of $G - V(C)$ together with all edges connecting it to C. The endpoints of these edges in C are called the *endpoints of* bridge B. (See also [BM76].) In the following, $P(x, y)$ will always denote a path in G between two vertices, x and y.

Claim 2. *G contains an E_0-cycle which has at least two bridges.*

Proof. First we show that there is an E_0-cycle with a chord which is either a single E_0-edge or an E_0-path of length two.

Delete all isolated vertices of G_0. For every vertex v, which is adjacent to exactly two vertices, u and w, in G_0, replace uv, vw, and v with the single edge uw. Call the resulting multigraph $\widehat{G}_0$. By Claim 1, the degree of every vertex of $\widehat{G}_0$ is at least 3.

Let $P = x_0x_1 \dots x_m$ be a longest path in $\widehat{G}_0$. Vertex x_0 has at least 3 neighbors, and, by the maximality of the path, all of them are on P. Hence, for some $1 < i < j$, x_0x_i and x_0x_j are edges of $\widehat{G}_0$. Then $x_0x_1 \dots x_j$ is a cycle with chord x_0x_i in $\widehat{G}_0$. Since every edge of $\widehat{G}_0$ arose from either an edge or a path of length two in G_0, the corresponding edges of G_0 form a cycle C with a chord c which is either a single edge or an E_0-path of length 2.

If C has at least two bridges, then we are done. Assume it has only one bridge, B. Now c is not a single edge, otherwise B would be identical with c, and $G = G_0 = C \cup c$ is *not* a counterexample. Therefore, we can assume that c is an E_0-path xvy of length 2.

The points x and y divide C into two complementary paths (arcs). If two vertices of C, a and b (different from x and y) do not belong to the same arc, we say that the pair $\{x, y\}$ *separates* a from b on C. Equivalently, the pair $\{a, b\}$ separates x from y.

We distinguish three cases.

CASE 1. *B has no two endpoints separated by the pair $\{x, y\}$.*

Let $P(x, y)$ denote the arc of C containing no endpoint of B in its interior. Let G' be the graph obtained from G by replacing $P(x, y)$ with a single edge xy, and let $E_0' = E_0 \cup \{xy\}$. It is easy to see that (G', E_0') is also a counterexample. By the minimality of (G, E_0), we have that $G = G'$, i.e., $P(x, y)$ is a single edge $xy \in E_0$.

Swapping xy with the chord xvy, we obtain an E_0-cycle C' with a chord xy. Therefore, C' has at least two bridges, and Claim 2 is true.

CASE 2. *There is a path $P(a, b) \subset B$, not passing through v, which connects two points, a and $b \in V(C)$, separated by the pair $\{x, y\}$.*

Since v and $P(a, b)$ belong to the same bridge, there is a path $P(v, q) \subset B$ connecting v to an interior point q of $P(a, b)$. Then G contains a subdivision of $K_{3,3}$ with vertex classes $\{x, y, q\}$ and $\{a, b, v\}$. Moreover, all paths representing the edges of $K_{3,3}$ belong to E_0, with the possible exceptions of those adjacent to q. This contradicts Claim 1 (iii), which shows that this case cannot occur.

CASE 3. *Every path in B, whose endpoints are separated on C by the pair $\{x, y\}$, passes through v.*

Let $P_1(x, y)$ and $P_2(x, y)$ denote the two complementary arcs of C, and let B_i be the union of all paths in B, which connect an internal point of $P_i(x, y)$ to x, v, or y.

Suppose first that $B = B_1 \cup B_2$. Then, by the minimality (G, E_0), $G - B_i$, for $i = 1, 2$, has a drawing where no edge belonging to E_0 is involved in any crossing. In particular, in this drawing, xvy and the edges of C are not crossed by any edge, so we can assume that all curves representing the edges of B_i lie in the region bounded by $P_i(x, y)$ and xvy $(i = 1, 2)$. Redrawing $G - B_2$, if necessary, so that C and xvy are mapped to exactly the same curves as in the drawing of $G - B_1$, the two drawings can be combined to give a drawing of G, contradicting our assumption that (G, E_0) is a counterexample.

We are left with the case when $B \neq B_1 \cup B_2$. Then there is a vertex s of B which can not be reached from any internal point of $P_i(x, y)$ without passing through x, v, or y $(i = 1, 2)$. Swapping $P_1(x, y)$ with xvy, we obtain an E_0-cycle C' with a chord $P_1(x, y)$, which can be arbitrarily long. C' has at least two bridges, because $P_1(x, y)$ and s do not be in the same bridge. □

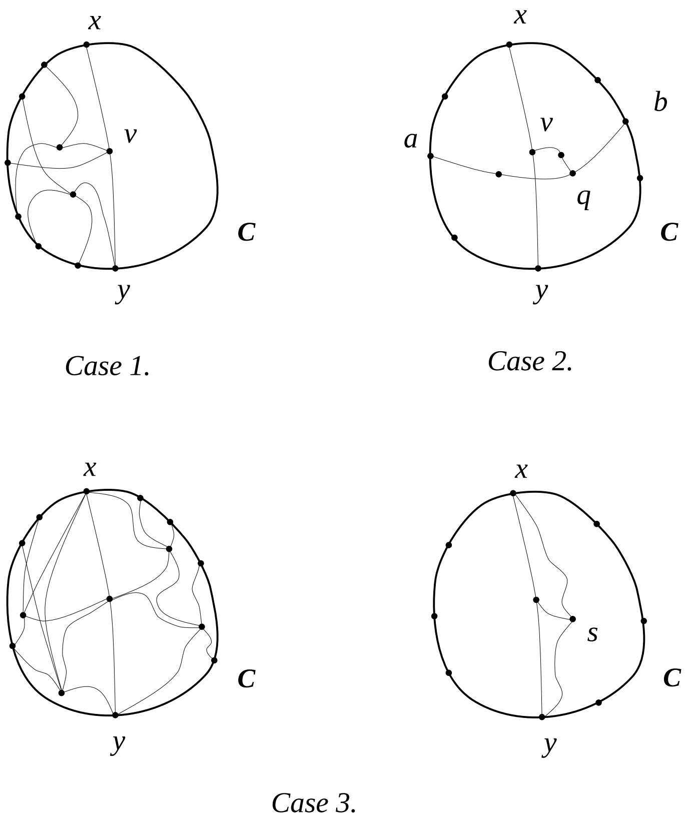

Figure 1.

In the sequel, let C denote a fixed E_0-cycle of G which has at least two bridges.

Claim 3. *C has at least three bridges.*

Proof. Suppose there are only two bridges of C, B_1 and B_2. By the minimality of G, $G - B_1$ (resp. $G - B_2$) can be drawn in the plane so that none of its edges belonging to E_0 is involved in any crossing. In particular, in this drawing none of the edges of C is involved in any crossing, therefore B_2 (resp. B_1) lies entirely on one side of C, say, in its interior (resp. exterior). But then we can combine the two drawings and get a drawing of G. It is a contradiction since G is assumed to be a counterexample. □

Let B_1 and B_2 be two bridges of C. By the minimality of (G, E_0), the graph $C \cup B_1 \cup B_2$ can be drawn in the plane so that none of its edges belonging to E_0 participates in any crossing. If in all such drawings B_1 and B_2 are on different sides of C, then B_1 and B_2 are said to be *conflicting.*

Claim 4. *C has exactly three bridges, at least one of which is a single edge.*

Proof. Construct a graph Γ whose vertices correspond to the bridges of C, and two vertices are connected by an edge if and only if the corresponding bridges are conflicting. By the minimality of (G, E_0), after the removal of any bridge the remaining graph can be drawn in the plane so that none of its edges belonging to E_0 is involved in any crossing. In other words, if we delete any vertex of Γ , it becomes two-colorable (the two colors correspond to the bridges inside and outside C). Therefore, any odd cycle of Γ passes through every vertex of Γ, hence Γ itself is an odd cycle.

Fix now any drawing of G, in which all edges belonging to E_0 are even. The closed curve representing C divides the plane into connected cells. Color them with black and white so that no two cells that share a boundary arc receive the same color.

Let B_i be a bridge of C. We need the following observation, which is an immediate consequence of the fact that every edge of B_i crosses all edges of C an even number of times. Assume that in a small neighborhood of one of its endpoints some edge of B_i runs in the black (white) region. Then *every* edge of B_i is black (resp. white) in a sufficiently small neighborhood of *both* of its endpoints. In this case, B_i is said to be a *black* (resp. *white*) *bridge.* Every non-endpoint of a black (white) bridge must lie in the black (resp. white) region.

Since Γ is an odd cycle, it has two consecutive vertices such that the corresponding bridges, say, B_1 and B_2, are conflicting and they are of the same color, say, black. We will specify two edges, $b_1 \in E(B_1)$ and $b_2 \in E(B_2)$. We distinguish two cases.

Suppose first that B_1 and B_2 have a common endpoint v. In a small neighborhood of v, all edges of B_1 and B_2 emanating from v are disjoint and run in the black region. Therefore, we can find two consecutive edges, b_1 and b_2, in the cyclic order around v such that $b_i \in B_i,\ i = 1, 2$. In this case, set $w_1 = w_2 = v$.

Suppose next that B_1 and B_2 do not have a common endpoint. Let $v_i v_{i+1} \dots v_j$ be a piece of C such that v_i is an endpoint of B_1, v_j is an endpoint of B_2, and no v_k $(i < k < j)$ is an endpoint of either B_1 or B_2. There may be several edges of B_1 adjacent to v_i, which lie in the black region in a small neighborhood of v_i; let b_1 denote the *last* one in the cyclic order from the initial piece of $v_i v_{i-1}$ to that of $v_i v_{i+1}$. Similarly, let b_2 denote the *first* edge of B_2 emanating from v_j in the cyclic order from the initial piece of $v_j v_{j-1}$ to that of $v_j v_{j+1}$. Now set $w_1 = v_i$ and $w_2 = v_j$.

Consider the drawing of $C \cup B_1 \cup B_2$ inherited from the original drawing of G. In this drawing, all edges belonging to $E_0 \cap (E(C) \cup E(B_1) \cup E(B_2))$ are even. We distinguish three cases depending on whether B_1 and B_2 are single edges, and in each case we slightly modify the graph $C \cup B_1 \cup B_2$

and its drawing. The modified graph and its drawing will be denoted by $\overline{G} = (\overline{V}, \overline{E})$, and we will also specify a set of edges $\overline{E}_0 \subseteq \overline{E}$.

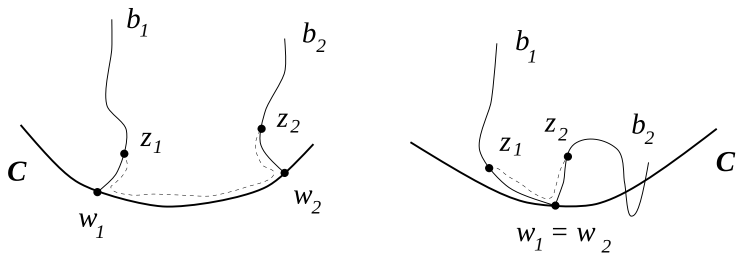

Figure 2.

CASE 1. *Both B_1 and B_2 are single edges.*

Then $E(B_i) = \{b_i\} = \{w_i u_i\}$, $i = 1, 2$. Split b_i into two edges by adding an extra vertex z_i very close to w_i, $i = 1, 2$. Connect z_1 and z_2 by an edge running very close to the path $z_1 w_1 ... w_2 z_2$, but not intersecting it (see Fig. 2), and denote the resulting graph drawing by $\overline{G}$. Since b_1 and b_2 are conflicting, at least one of them (say, b_1) belongs to E_0. Then set $\overline{E}_0 = E(C) \cup \{w_1 z_1, z_1 u_1\}$.

CASE 2. *B_1 is a single edge, B_2 is not.*

Then $E(B_1) = \{b_1\} = \{w_1 u_1\}$, $E(B_2) \supset \{b_2\} = \{w_2 z_2\}$, where $u_1 \in V(C)$ and $z_2 \notin V(C)$. Split b_1 into two edges by adding a vertex z_1 very close to w_1. As before, connect z_1 and z_2 by an edge running very close to the path $z_1 w_1 ... w_2 z_2$, and denote the resulting graph drawing by $\overline{G}$. If $b_1 \in E_0$ then set $\overline{E}_0 = E(C) \cup \{w_1 z_1, z_1 u_1\}$. Otherwise, let $\overline{E}_0 = E_0 \cap (E(C) \cup E(B_2))$, i.e., we leave the set of specified edges unchanged.

CASE 3. *Neither B_1 nor B_2 is a single edge.*

Then $E(B_i) \supset \{b_i\} = \{w_i z_i\}$, where $z_i \notin V(C)$, for $i = 1, 2$. Connect z_1 and z_2 by an edge running very close to the path $z_1 w_1 ... w_2 z_2$, and denote the resulting graph drawing by $\overline{G}$. As in the previous case, let us leave the set of specified edges unchanged, i.e., set $\overline{E}_0 = E_0 \cap (E(C) \cup E(B_1) \cup E(B_2))$.

It follows from the construction that in the above drawing of $\overline{G}$, every edge belonging to $\overline{E}_0$ is even. Recall that B_1 and B_2 were conflicting (see the last paragraph before Claim 4), which implies that in every drawing of $\overline{G}$ with the property that no edge in $\overline{E}_0$ is involved in any crossing, z_1 and z_2 lie on different sides of C. However, $z_1 z_2 \in E(\overline{G}) = \overline{E}$, proving that $(\overline{G}, \overline{E}_0)$ is also a counterexample to Theorem 1.

Suppose, to obtain a contradiction, that C has more than three bridges in G. Since Γ is an odd cycle, the number of bridges is odd, i.e., C has at least five bridges. In the construction of $\overline{G}$, we kept only two of these bridges, so we deleted at least three bridges, hence at least three edges.

In Cases 1 and 2, we added at most two new edges. Thus, in these cases, $|E(\overline{G})| = |\overline{E}| < |E|$, contradicting our assumption that (G, E_0) is a minimal counterexample.

The only remaining possibility is that C has exactly five bridges, all of which are single edges. It follows from the structure of Γ that at least three of these bridges (edges) belong to E_0. On the other hand, $\overline{G}$ has only two edges not in C that belong to $\overline{E}_0$. Thus, in this case, $|\overline{E}| = |E|$, but $|\overline{E}_0| < |E_0|$. This again contradicts the minimality of our counterexample.

Therefore, we can assume that C has exactly three bridges in G, B_1, B_2, and B_3. If none of them is a single edge, then we can add one edge (as in Case 3) and delete a bridge, which contains more than one edge, to obtain a counterexample smaller than (G, E_0). □

Claim 5. *C has at least two bridges which are single edges.*

Proof. Assume, to obtain a contradiction, that C has only one bridge which consists of a single edge. Take a closer look at the transformation in the proof of Claim 4. By deleting B_3 and adding one, two, or three edges, we obtained another counterexample $(\overline{G}, \overline{E_0})$.

If B_1 or B_2 was the bridge consisting of a single edge, then we added two edges (cf. Case 2 in the proof of Claim 4) and deleted B_3, which had at least three edges. This contradicts the assumption that (G, E_0) was a minimal counterexample.

Therefore, we can assume that B_3 consists of a single edge xy. Then, during the above transformation we deleted B_3 and added an edge that does not belong to $\overline{E_0}$ (cf. Case 3). Therefore, using the minimality of (G, E_0) again, we obtain that $xy \notin E_0$.

Since B_1 and B_3 are conflicting, it follows that there is an E_0-path $P(a, b) \subset B_1$ whose endpoints, a and b, separate x and y on C. Let $P_x(a, b)$ and $P_y(a, b)$ denote the two complementary arcs of C between a and b, containing x and y, respectively.

We distinguish two cases.

CASE 1. *All endpoints of B_2 belong to the same arc, $P_x(a, b)$ or $P_y(a, b)$.*

By symmetry, we can assume that all endpoints of B_2 are on $P_x(a, b)$. Then all endpoints of B_1 must also belong to $P_x(a, b)$. Indeed, if an endpoint of B_1 did not lie on this arc, then we could delete all edges of B_1 adjacent to it and obtain a smaller counterexample.

Consider the graph $\overline{G}$ constructed in the proof of Claim 4. In this graph, y is adjacent to only two vertices, y' and y'', both of which belong to C. Let G' denote the graph obtained from $\overline{G}$ by deleting y and replacing the E_0-path $y'yy''$ by a single edge $y'y''$. Set $E_0' = E_0 \setminus \{yy', yy''\} \cup \{y'y''\}$. Clearly, (G', E_0') is a counterexample to Theorem 1, which precedes (G, E_0), contradicting the minimality of (G, E_0).

CASE 2. *There exists a path $P(p, q) \subseteq B_2$ such that p and q are interior points of $P_x(a, b)$ and $P_y(a, b)$, respectively.*

Consider again the graph $\overline{G}$. Clearly, B_1 contains a path connecting b_1 to some internal point r of $P(a, b)$. (Note that r may be an endpoint of b_1. Moreover, b_1 may belong to $P(a, b)$.) Similarly, B_2 contains a path connecting b_2 to some internal point s of $P(p, q)$. However, in this case, $\overline{G}$ contains a subdivision of $K_{3,3}$ with vertex classes $\{a, b, s\}$ and $\{p, q, r\}$. Furthermore, with the exception of the paths incident to s, all paths representing the edges of $K_{3,3}$ belong to E_0. However, this contradicts Claim 1 (iii). □

Now we can complete the proof of Theorem 1. By Claims 4 and 5, C has precisely three pairwise conflicting bridges B_i, $(i = 1, 2, 3)$ in G. Two of them, say, B_1 and B_2, are single edges, xy and ab, respectively. Since B_1 and B_2 are conflicting, at least one of them, say xy, is in E_0.

Using the fact that B_3 is in conflict with $xy \in E_0$, we obtain that it contains a path connecting a pair of points $\{p, q\} \subset V(C)$ which separates x from y. Similarly, since B_3 is in conflict with ab, it also contains a path connecting a pair of points $\{p', q'\} \subset V(C)$ which separates a from b, and this path belongs to E_0 unless $ab \in E_0$. According to the position of these paths, we can distinguish four different cases up to symmetry (see Fig. 3). $P(p, q)$ always stands for a path connecting p and q, whose internal vertices do not belong to C.

CASE 1. *B_3 contains a path $P(p, q)$; $p, q \in V(C)$, such that the pair $\{p, q\}$ separates a from b and x from y, and ab or $P(p, q)$ belongs to E_0.*

Then G has a subdivision of $K_{3,3}$ with vertex classes $\{a, p, y\}$ and $\{b, q, x\}$. Moreover, with the exception of ab or $P(p, q)$, all paths representing the edges of $K_{3,3}$ belong to E_0. This contradicts Claim 1 (iii).

CASE 2. *B_3 contains three internally disjoint paths, $P(a, r)$, $P(p, r)$ and $P(q, r)$, such that r does not belong to C; the pair $\{p, q\}$ separates b from the set $\{a, x, y\}$; and ab or $P(p, r) \cup P(q, r)$ belongs to E_0.*

Then G *properly* contains a subdivision of $K_{3,3}$ with vertex classes $\{x, r, b\}$, and $\{a, p, q\}$. It is easy to see that deleting from G the arc of C between a and y which does not contain $\{x, p, b, q\}$, we obtain a smaller counterexample. Thus, this case cannot occur.

CASE 3. *B_3 contains three internally disjoint paths, $P(p, r)$, $P(q, r)$, and $P(y, r)$, such that r does not belong to C; the pair $\{p, q\}$ separates x from the set $\{a, b, y\}$; and at least one of ab, $P(p, r) \cup P(y, r)$ and $P(q, r) \cup P(y, r)$ belongs to E_0.*

Then G *properly* contains a subdivision of $K_{3,3}$ with vertex classes $\{x, r, b\}$, and $\{y, p, q\}$. If ab belongs to E_0, then deleting from G the arc of C between a and y which does not contain $\{p, x, q, b\}$, we obtain a smaller counterexample. If ab does not belong to E_0, but, say, $P(p, r) \cup P(y, r)$ does, then, by the minimality of (G, E_0), all paths depicted in Fig. 3 (3) are single edges, and G has no further edges. However, this case cannot occur, because here b and q are two adjacent vertices of degree 2 in G_0, contradicting Claim 1 (ii).

CASE 4. *The endpoints of B_3 are a, b, x, y.*

Since B_2, and B_3 are conficting, B_3 contains two intersecting paths, $P(a, b)$ and $P(x, y)$, such that either ab or $P(x, y)$ belongs to E_0. It follows from the minimality of our counterexample that $P(a, b)$ and $P(x, y)$ have only one vertex in common. Denoting it with r, we can write

$P(a,b) = P(a,r) \cup P(b,r)$ and $P(x,y) = P(x,r) \cup P(y,r)$. Then G contains a subdivision of K_5 induced by a,b,x,y,r. Moreover, with the exception of ab, $P(a,r)$, and $P(b,r)$, all paths representing the edges of $K_{3,3}$ belong to E_0. This contradicts Claim 1 (iii).

In each case, we arrived at a contradiction. Thus, there exists no (minimal) counterexample (G, E_0) to Theorem 1. The proof of Theorem 1 is complete. □

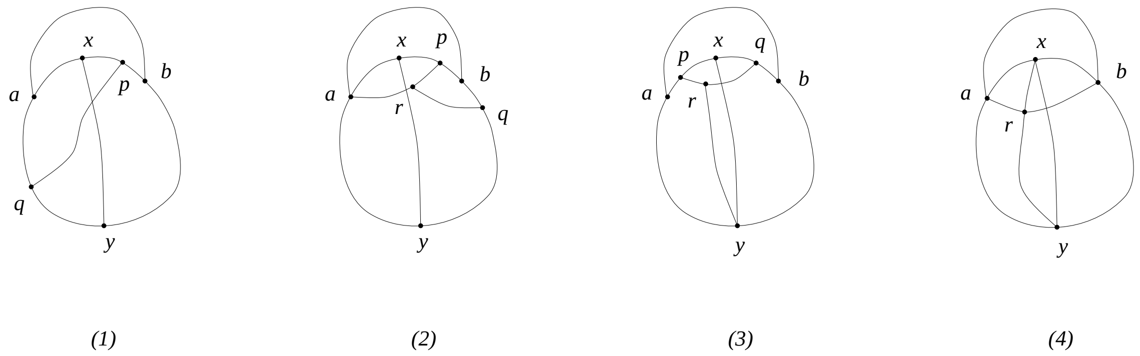

Figure 3.

Theorem 2 is an easy corollary to Theorem 1. Let $G = (V, E)$ be a simple graph drawn in the plane with $\lambda = \text{ODD-CR}(G)$ pairs of edges that cross an odd number of times. Let $E_0 \subset E$ denote the set of *even* edges in this drawing. Since every edge not in E_0 crosses at least one other edge an odd number of times, we obtain that

$$|E \setminus E_0| \leq 2\lambda.$$

By Theorem 1, there exists a drawing of G, in which no edge of E_0 is involved in any crossing. Pick a drawing with this property such that the total number of crossing points between all pairs of edges not in E_0 is minimal. Notice that in this drawing, any two edges cross at most once. Therefore, the number of crossings is at most

$$\binom{|E \setminus E_0|}{2} \leq \binom{2\lambda}{2} \leq 2\lambda^2,$$

and Theorem 2 follows.

3 Proof of Theorem 3

The proofs of Theorem B readily generalize to this case. We include a short argument, for completeness.

First, we show that for any graph G,

$$\text{ODD-CR}(G) \geq |E(G)| - 3|V(G)|. \tag{1}$$

If $|E(G)| \leq 3|V(G)|$, then (1) is trivially true. Let $|E(G)| > 3|V(G)|$ and suppose that (1) holds for any graph with $|V(G)|$ vertices and less than $|E(G)|$ edges. Consider a drawing of G with exactly ODD-CR(G) pairs of edges crossing an odd number of times. Since $|E(G)| > 3|V(G)|$, G is not planar, so by Theorem A, ODD-CR$(G) \geq 1$. Let $\overline{G}$ denote the the graph obtained from G by deleting one edge that crosses at least one other edge an odd number of times. Applying the induction hypothesis to $\overline{G}$, we get

$$\text{ODD-CR}(G) \geq \text{ODD-CR}(\overline{G}) + 1 \geq |E(\overline{G})| - 3|V(\overline{G})| + 1 = |E(G)| - 3|V(G)|,$$

as required.

To prove Theorem 3, fix a drawing of G with exactly ODD-CR(G) pairs of edges crossing an odd number of times, and suppose that $|E(G)| \geq 4|V(G)|$. Construct a *random* subgraph $G' \subseteq G$ by selecting each vertex of G independently with probability p, and letting G' be the subgraph induced by the selected vertices. The expected number of vertices of G', $\text{Exp}[|V(G')|] = p|V(G)|$. Similarly, $\text{Exp}[|E(G')|] = p^2|E(G)|$. The expected number of pairs of edges that cross an odd number of times in the drawing of G' inherited from G is p^4ODD-CR(G), hence the expected value of the odd-crossing number of G' cannot be larger than this.

By (1), ODD-CR$(G') \geq |E(G')| - 3|V(G')|$ for every particular G'. Taking expectations,

$$p^4\text{ODD-CR}(G) \geq \text{Exp}[\text{ODD-CR}(G')] \geq \text{Exp}[|E(G')|] - 3\text{Exp}[|V(G')|] = p^2|E(G)| - 3p|V(G)|.$$

Setting $p = 4|V(G)|/|E(G)|$ we obtain

$$\text{ODD-CR}(G) \geq \frac{1}{64}\frac{|E(G)|^3}{|V(G)|^2}, \tag{2}$$

whenever $|E(G)| \geq 4|V(G)|$. □

Remarks. 1. In case $|E(G)| \geq 6|V(G)|$, Theorem 2 trivially follows from Theorem 3. Indeed, for any graph G,

$$\text{CR}(G) \leq \binom{|E(G)|}{2} < |E(G)|^2/2.$$

If $|E(G)| \geq 6|V(G)|$ then Theorem 3 implies

$$2(\text{ODD-CR}(G))^2 \geq 2 \cdot \left(\frac{1}{64}\frac{|E(G)|^3}{|V(G)|^2}\right)^2 \geq \frac{|E(G)|^2}{2} > \text{CR}(G).$$

2. Using the fact that Theorem A guarantees, in any non-planar graph, the existence of two *independent* edges that cross an odd number of times, the above proof gives the same lower bound, $(1/64)|E(G)|^3/|V(G)|^2$, for the minimum number of pairs of *independent* edges that cross an odd number of times. This result is somewhat stronger than Theorem 3, because here we do not count any odd crossing between two edges that share an endpoint.

4 Proof of Theorem 4

First, we prove that the ODD CROSSING NUMBER PROBLEM, ODD-CR$(G) \le K$, is in NP, and then we show that there is an NP-complete problem that can be reduced to it in polynomial time.

Fix a graph G with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and edge set E. Every drawing $\mathcal{D}$ of G can be represented by an $\binom{|E|}{2}$-dimensional $(0,1)$-vector $\bar{X}_{\mathcal{D}}(G)$, in which each coordinate is assigned to an unordered pair of edges $\{e, f\} \subseteq E$, and is equal to 1 if and only if e and f cross an odd number of times. That is,

$$\bar{X}_{\mathcal{D}}(G) = (x_{\mathcal{D}}\{e, f\})_{e \neq f; e, f \in E},$$

where, for every $e, f \in E$,

$$x_{\mathcal{D}}\{e, f\} = \begin{cases} 0 \text{ if } e \text{ and } f \text{ cross an even number of times,} \\ 1 \text{ if } e \text{ and } f \text{ cross an odd number of times.} \end{cases}$$

We say that two drawings of G, $\mathcal{D}$ and $\mathcal{D}'$ are *equivalent* if they are represented by the same vector, i.e., if $\bar{X}_{\mathcal{D}}(G) = \bar{X}_{\mathcal{D}'}(G)$. An $\binom{|E|}{2}$-dimensional $(0,1)$-vector $\bar{X}$ is said to be *realizable* if there exists a drawing $\mathcal{D}$ of G such that $\bar{X}_{\mathcal{D}}(G) = \bar{X}$

Using an idea of Tutte [T70], it is not hard to describe the set of all realizable vectors of G. We need some further notation. For any $v \in V$, $g \in E$, let

$$\bar{Y}_{v,g} = (y\{e, f\})_{e \neq f; e, f \in E},$$

where

$$y\{e, f\} = \begin{cases} 1 \text{ if } e = g \text{ and } f \text{ is adjacent to } v, \text{ or } f = g \text{ and } e \text{ is adjacent to } v, \\ 0 \text{ otherwise.} \end{cases}$$

Let Φ denote the vector space over GF(2) generated by the vectors $\bar{Y}_{v,g}$, i.e.,

$$\Phi = \langle \bar{Y}_{v,g} \mid v \in V,\ g \in E \rangle_{\text{gen}} \subset \{0, 1\}^{\binom{|E|}{2}}.$$

Place the vertices $v_1, v_2, \ldots, v_n$ on a circle in this clockwise order so that they form a regular n-gon, and connect v_i and v_j $(i \neq j)$ by a straight-line segment if and only if $v_iv_j \in E$. This drawing is said to be the *convex* drawing of G, and is denoted by $\mathcal{C}$.

For any $1 \leq i \leq n$ let d_i be the degree of v_i and let $e_1^i, e_2^i, \ldots, e_{d_i}^i$ be the list of edges adjacent to v_i, in clockwise in the convex drawing of G. Let $\sigma_i : \{1, 2, \ldots, d_i\} \to \{1, 2, \ldots, d_i\}$ be any permutation. Define

$$\bar{Z}_{v_i,\sigma_i} = (z\{e, f\})_{e \neq f; e,f \in E}\,,$$

where

$$z\{e, f\} = \begin{cases} 1 \text{ if } e = e_\alpha^i,\ f = e_\beta^i \text{ and } (\alpha - \beta)(\sigma_i(\alpha) - \sigma_i(\beta)) < 0, \\ 0 \text{ otherwise.} \end{cases}$$

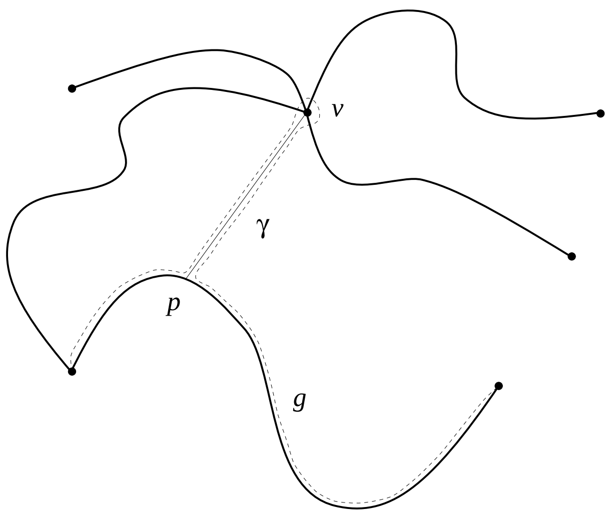

Figure 4.

Lemma 4.1. *Let Φ denote the vector space over* GF(2) *generated by the vectors $\bar{Y}_{v,g}$, $v \in V$, $g \in E$, let $\bar{X}_{\mathcal{C}}(G)$ be the $(0,1)$-vector representing the convex drawing of G, and let*

$$\Gamma = \left\{ \sum_{i=1}^{n} \bar{Z}_{v_i,\sigma_i} \mid \sigma_i \textit{ is any permutation } \{1, 2, \ldots, d_i\} \to \{1, 2, \ldots, d_i\} \right\}.$$

Then the set of all realizable vectors of G is

$$\Psi = \bar{X}_{\mathcal{C}}(G) + \Gamma + \Phi,$$

where the sum is taken mod 2.

Proof. Let $\mathcal{D}$ be any drawing of G, let $v \in V, g \in E$. Consider the following two operations:

(i) Choose a simple smooth arc γ connecting any internal point p of g to v such that it does not pass through any vertex, is not tangent to any edge, and crosses every edge a finite number of times. Replace a small piece of g containing p by a path going around v and running extremely close to γ (see Fig. 4). The $(0,1)$-vector representing this new drawing is

$$\bar{X}_{\mathcal{E}}(G) = \bar{X}_{\mathcal{D}}(G) + \bar{Y}_{v,g} \qquad (\text{mod } 2).$$

(ii) Let σ_i be the clockwise order of $e_1^i, e_2^i, \ldots, e_{d_i}^i$ as they emanate from v_i in drawing $\mathcal{D}$. Change the clockwise order of edges as they emanate from v_i to $e_1^i, e_2^i, \ldots, e_{d_i}^i$ in a small neighborhood of v_i. (See Fig. 5.) The $(0,1)$-vector representing this new drawing is

$$\bar{X}_{\mathcal{F}}(G) = \bar{X}_{\mathcal{D}}(G) + \bar{Z}_{v_i,\sigma_i} \qquad (\text{mod } 2).$$

This shows that any vector in Ψ is realizable.

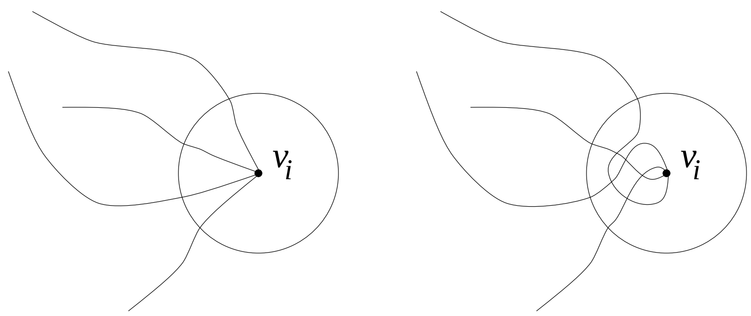

Figure 5.

Next we prove that $\bar{X}_{\mathcal{D}}(G) \in \Psi$, for any drawing $\mathcal{D}$ of G. Using a topological transformation of the plane, if necessary, we can assume without loss of generality that the vertices of G, $v_1, v_2, \ldots, v_n$, form a regular n-gon, in this clockwise order. First, for every $1 \le i \le n$, in a small neighborhood of v_i, change the clockwise order of edges as they emanate from v_i to $e_1^i, e_2^i, \ldots, e_{d_i}^i$ such that in a very small neighborhood of v_i, each egde v_iv_j is represented by the corresponding part of the segment v_iv_j.

Then, pick an edge $g = v_iv_j$, and transform it into the straight-line segment between v_i and v_j, by continuous deformation. Performing this operation for all edges, one by one, we obtain $\mathcal{C}$ the convex drawing of G.

Let $\mathcal{D}'$ denote the drawing after the first step. Then,

$$\bar{X}_{\mathcal{D}'}(G) = \bar{X}_{\mathcal{D}}(G) + \sum_{i=1}^{n} \bar{Z}_{v_i,\sigma_i} \pmod 2$$

for some permutations $\sigma^1, \sigma^2, \dots, \sigma^n$.

During the second step, the representation vector of the drawing changes whenever a deforming edge g hits a vertex v. Let $\mathcal{E}$ and $\mathcal{F}$ denote the drawing immediately before and after this event. Clearly,

$$\bar{X}_{\mathcal{F}}(G) = \bar{X}_{\mathcal{E}}(G) + \bar{Y}_{v,g} \pmod 2.$$

Finally, we obtain

$$\bar{X}_{\mathcal{C}}(G) = \bar{X}_{\mathcal{D}}(G) + \bar{Y} \pmod 2,$$

for some $\bar{Y} \in \Phi$, hence

$$\bar{X}_{\mathcal{D}}(G) \in \bar{X}_{\mathcal{C}}(G) + \bar{Y} = \Psi. \qquad \square$$

Now we are in a position to prove that the ODD CROSSING NUMBER PROBLEM is in NP. Suppose that ODD-CR$(G) \leq K$. Then, by Lemma 4.1, there is a realizable vector $\bar{Y} \in \Psi$ such that all but at most K coordinates of $\bar{Y}$ are 0. We can give the vector $\bar{Y}$ in the form

$$\bar{Y} = \bar{X}_{\mathcal{C}}(G) + \sum_{i=1}^{n} \bar{Z}_{v_i,\sigma_i} + \sum_{v \in V, g \in E} \alpha_{(v,g)} \bar{Y}_{v,g} \pmod 2,$$

where $\alpha_{(v,g)} \in \{0,1\}$ and $\sigma_i : \{1,2,\dots,d_i\} \to \{1,2,\dots,d_i\}$ are permutations. Clearly, the correctness of this equation can be checked in polynomial time. Thus, the ODD CROSSING NUMBER PROBLEM is in NP.

The OPTIMAL LINEAR ARRANGEMENT PROBLEM is the following. Given a graph $G = (V, E)$ and an integer K, is there a one-to one function $\sigma : V \to \{1, 2, \dots, |V|\}$ such that $\sum_{uv \in E} |\sigma(u) - \sigma(v)| \leq K$?

Notice that the ODD CROSSING NUMBER PROBLEM for simple graphs is equivalent to the same problem for *multigraphs*, i.e., when the graph G may have multiple (parallel) edges. Indeed, we can remove all multiplicities by introducing new vertices along the edges of G. For any graph $\overline{G}$ obtained from G by subdividing one (or more) of its edges, we have

$$\text{ODD-CR}(\overline{G}) = \text{ODD-CR}(G).$$

Lemma 4.2. *The* OPTIMAL LINEAR ARRANGEMENT PROBLEM *can be reduced to the* ODD CROSSING NUMBER PROBLEM *in polynomial time.*

Proof. Suppose we are given an instance $G = (V, E)$, K, and we want to decide if there exists a one-to one function $\sigma : V \to \{1, 2, \dots, |V|\}$ such that $\sum_{uv \in E} |\sigma(u) - \sigma(v)| \leq K$. Let $V = \{v_1, v_2, \dots, v_n\}$

and assume without loss of generality that G is connected. We construct a multigraph G'_K and a number K' such that the answer to our OPTIMAL LINEAR ARRANGEMENT PROBLEM is affirmative if and only if $\text{ODD-CR}(G'_K) \le K'$.

Let $G'_K = (V', E')$, where $V' = V_1 \cup V_2 \cup \{u, w\}$, $E = E_1 \cup E_2 \cup E_3$,

$$V_1 = \{u_i \mid 1 \le i \le n\}, \quad V_2 = \{w_i \mid 1 \le i \le n\},$$

$$E_1 = \{|E|^2 \text{ copies of } u_i w_i \mid 1 \le i \le n\},$$

$$E_2 = \{u_i w_j \mid i < j \text{ and } v_i w_j \in E\},$$

$$E_3 = \{K^2 |E|^2 \text{ copies of } uw,\ uu_i,\ ww_i,\ 1 \le i \le n\},$$

and let

$$K' = |E|^2 (K - |E|) + |E|^2 - 1.$$

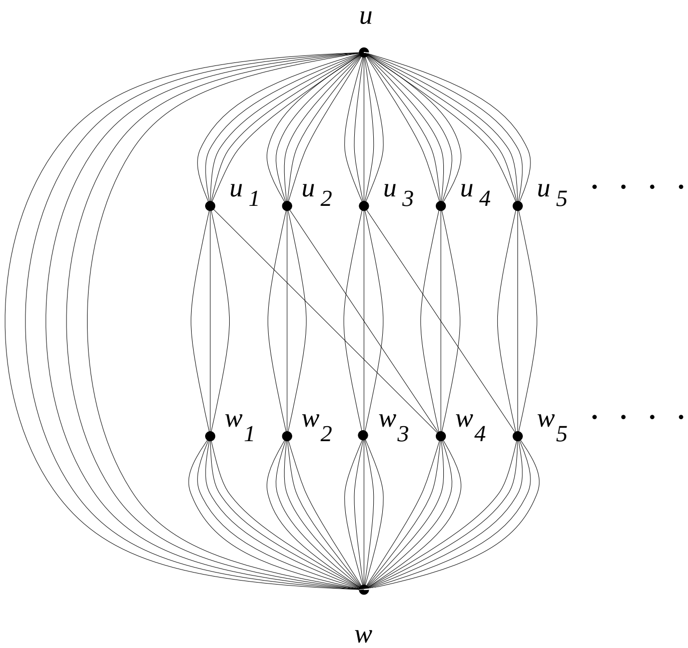

Figure 6.

Suppose first that there exists a bijection $\sigma : V \to \{1, 2, \ldots, |V|\}$ such that $\sum_{uv \in E} |\sigma(u) - \sigma(v)| \le K$. We construct a drawing of G' with at most K' pairs of crossing edges. Place u_i at $(1, \sigma(v_i))$, w_i at $(0, \sigma(v_i))$, u at $(2, 0)$, and w at $(-1, 0)$. Represent all single edges by straight-line segments and all multiple edges by pairwise disjoint curves running very close to the corresponding straight line segment. It is easy to see that the total number of crossing pairs of edges is at most

$$\sum_{uv \in E} (|\sigma(u) - \sigma(v)| - 1)|E|^2 + |E|^2 - 1 \leq |E|^2(K - |E|) + |E|^2 - 1 = K'.$$

Next, suppose that ODD-CR$(G'_K) \leq K'$. We show, using some simple transformations, that there is another drawing of G' generated by a function σ in the way described above, which has at most K' pairs of edges that cross an odd number of times. Consider a drawing of G'_K with at most K' pairs of edges that cross an odd number of times.

(a) We can assume that any two parallel edges, e and f, are drawn very close to each other, so that they are openly disjoint, and any other edge crosses both of them the same number of times. Indeed, if e and f are drawn differently, then replacing either e by an arc running very close to f, or f by an arc running very close to e, we obtain a new drawing of G which has at most as many pairs of edges that cross an odd number of times as the original drawing.

(b) Any two edges $e, f \in E_1 \cup E_3$ must cross an even number of times. Indeed, otherwise, by (a), we can assume that each of the at least $|E|^2$ edges parallel (or identical) to e crosses each of the at least $|E|^2$ edges parallel (or identical) to f an odd number of times. This implies that the number of edge pairs that cross an odd number of times is at least $|E|^4 > K'$, a contradiction.

(c) No edge of G'_K can cross any edge between u and w an odd number of times. Otherwise, by (a), the number of pairs of edges that cross an odd number of times would be at least $K^2|E|^2 > K'$, which is impossible.

(d) Let e be any edge between u and w, and let f_i (resp. g_i) be any edge whose endpoints are u and u_i (resp. w and w_i), $1 \leq i \leq n$. If for some $i \neq j$, the edges (e, f_i, f_j) emanate from u in clockwise order, then (e, g_i, g_j) must emanate from v in counter-clockwise order.

To see this, consider a cycle C formed by f_i, e, g_i, and any edge connecting u_i and w_i. The closed curve representing this cycle divides the plane into connected cells. As in the proof of Theorem 1, color these cells with black and white so that no two cells that share a boundary arc receive the same color. Let P be a path formed by f_j, g_j, and any edge between u_j and w_j. Suppose that in a small neighborhood of u, f_j is in, say, the black region. Then, in a small neighborhood of w, g_j must also lie in the black region, because, by (b), every edge of P crosses (every edge of) C an even number of times.

(e) Suppose that $e, f_1, f_2, \ldots, f_n$ emanate from u in the clockwise order $e, f_{\alpha(1)}, f_{\alpha(2)}, \ldots, f_{\alpha(n)}$. Then, by (d), $e, g_1, g_2, \ldots, g_n$ must emanate from w in the reverse order $e, g_{\alpha(n)}, g_{\alpha(n-1)}, \ldots, g_{\alpha(1)}$. Let $\sigma(v_i) = \alpha^{-1}(i)$, $1 \leq i \leq n$.

We *claim* that for every $u_i w_j \in E_2$, there are at least $(|\sigma(v_i) - \sigma(v_j)| - 1)|E|^2$ edges in G'_K that cross $u_i w_j$ an odd number of times. To see this, it is enough to show that for every $r < s < t$, if $v_{\alpha(r)} v_{\alpha(t)} \in E$, then the edge $e_{rt} := u_{\alpha(r)} w_{\alpha(t)}$ must cross the path $P_s := f_{\alpha(s)} \cup e_{\alpha(s)} \cup g_{\alpha(s)}$ an odd number of times, where $e_{\alpha(s)}$ denotes any edge between $u_{\alpha(s)}$ and $w_{\alpha(s)}$. As before, color the cells determined by the closed curve $P_s \cup e$ with black and white. It follows from (d) that if in a small neighborhood of u, $f_{\alpha(r)} \cup e_{rt} \cup g_{\alpha(t)}$ is in the black region, then in a small neighborhood of w it is in the white region. In view of (b) and (c), this implies that e_{rt} crosses at least one of the edges $f_{\alpha(s)}$, $e_{\alpha(s)}$, and $g_{\alpha(s)}$ an odd number of times. In each case, we are done, and our claim is true.

Therefore, we have

$$\sum_{uv \in E} (|\sigma(u) - \sigma(v)| - 1)|E|^2 \leq \text{ODD-CR}(G'_K) \leq K' = |E|^2(K - |E|) + |E|^2 - 1,$$

which implies that

$$\sum_{uv \in E} (|\sigma(u) - \sigma(v)| \leq K,$$

as desired. □

With Lemma 4.2, the proof of Theorem 4 (ii) is complete, because the OPTIMAL LINEAR ARRANGEMENT PROBLEM is known to be NP-complete [GJS76].

Remark. We can prove that the PAIR CROSSING NUMBER PROBLEM, $\text{PAIR-CR}(G) \leq K$, is NP-hard. The proof is analogous to the proofs of the corresponding results for the crossing number (see [GJ83]) and for the odd-crossing number (see Lemma 4.2). On the other hand, we are unable to prove that the PAIR CROSSING NUMBER PROBLEM is in NP, that is, we can not generalize Lemma 4.1 for $\text{PAIR-CR}(G)$.

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