Ramsey-type results for unions of comparability graphs

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Abstract

It is well known that the comparability graph of any partially ordered set of n elements contains either a clique or an independent set of size at least $\sqrt{n}$. In this note we show that any graph of n vertices which is the union of two comparability graphs on the same vertex set, contains either a clique or an independent set of size at least $n^{\frac{1}{3}}$. On the other hand, there exist such graphs for which the size of any clique or independent set is at most $n^{0.4118}$. Similar results are obtained for graphs which are unions of a fixed number k comparability graphs. We also show that the same bounds hold for unions of perfect graphs.

1 Introduction

Let $S = (V, \prec)$ be a partially ordered set on $V = \{v_1, v_2, \ldots, v_n\}$. The *comparability graph* of S, $G = (V, E)$ is a graph such that $(v_i, v_j) \in E$ if and only if either $v_i \prec v_j$ or $v_j \prec v_i$.

A graph G is called *perfect*, if for any spanned subgraph $G' \subset G$, its chromatic number is equal to the size of its largest clique (see also [B73], [L93]).

Dilworth's Theorem. [D50] *Let S be a partially ordered set containing no chain (totally ordered subset) of size $k+1$. Then P can be covered by k antichains (subsets of pairwise incomparable elements).*

It follows directly from Dilworth's theorem that comparability graphs are perfect. Any perfect graph of n vertices contains either a clique or an independent set of size at least $\sqrt{n}$. It is easy to see that this bound can not be improved even for comparability graphs.

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In this note we obtain similar results for unions of two or more comparability (resp. perfect) graphs on the same vertex set.

Definition. Let $f_k(n)$ (resp. $g_k(n)$) be the largest number such that any graph G of n vertices, which is the union of k comparability graphs (resp. perfect graphs) on the same vertex set, contains either a clique or an independent set of size at least $f_k(n)$ (resp. $g_k(n)$).

Theorem 1.

$$n^{\frac{1}{k+1}} \le g_k(n) \le f_k(n) \le n^{\frac{1+\log k}{k}}.$$

In particular, for $k = 2$,

Theorem 2.

$$n^{\frac{1}{3}} \le g_2(n) \le f_2(n) \le n^{0.4118}.$$

Unions of comparability graphs are often used in combinatorial geometry, for proving Ramsey-type results ([LMPT94], [PT94], [KPT97], [KK97], [TV98], [PT99], [T99], see also the Remarks).

Let $\omega(G)$, $\alpha(G)$, and $\chi(G)$ denote the clique number, independence number, and the chromatic number of a graph G, respectively. Obviously, $n/\alpha(G) \le \chi(G)$, therefore the inequality $n^{\frac{1}{k+1}} \le f_k(n)$ is a direct consequence of the following stronger result, which is essentially tight.

Theorem 3. *If G is the union of k perfect graphs, then*

$$\chi(G) \le \omega^k(G).$$

On the other hand, for any k there is a graph G which is the union of k comparability graphs, with

$$\chi(G) \ge (\omega(G))^{k(1-o(1))}.$$

2 Proof of Theorems 1 and 2

All comparability graphs are perfect [L93], therefore $g_k(n) \le f_k(n)$.

We prove the lower bound by induction on k. For $k = 1$, the statement is a direct consequence of the definition of perfect graphs [L93]. Suppose that we have already proved the statement for $k - 1$ and for all n. Let $G_i(V, E_i)$ $(1 \le i \le k)$ be perfect graphs, $V = \{v_1, v_2, \ldots, v_n\}$. Suppose for simplicity that $m = n^{1/(k+1)}$ is an integer. Suppose that the size of any clique in $G(V, E)$, $E = \cup_{i=1}^{n} E_i$, is less than m. Since $G_1(V, E_1)$ is perfect and $\omega(G_1) < m$, it can be colored by less than m colors, so it contains an independent set of size at least $n/m = m^k$, say, $V' = \{v_1, v_2, \ldots, v_{m^k}\}$.

For $i = 2, \ldots, k$, let G_i' be the subgraph of G_i spanned by V'. The graphs G_i are perfect, hence G_i' are also perfect. By the induction hypothesis and the assumption, there is a set of size m, which is an independent set in each G_i', $i = 2, \ldots, n$. But then it is an independent set in $G(V, E)$, proving the lower bound.

Definition. Let $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ be two (directed) graphs. The *ordered product* $G(V, E) = G_1 \times G_2$ is a (directed) graph with vertex set $V = V_1 \times V_2$, and edge set

$$E(G) = \{((x_1, x_2), (y_1, y_2)) \mid x_1, y_1 \in V_1, x_2, y_2 \in V_2, \text{ and}$$
$$\text{either } (x_1, y_1) \in E_1 \text{ or } x_1 = y_1 \text{ and } (x_2, y_2) \in E_2\}.$$

Let k be a fixed number. By known bounds for Ramsey numbers [AS92], we know that there is a graph G such that $|V(G)| = 2^k$, and $\omega(G), \alpha(G) < 2k$.

Proposition. Every graph G is the union of $\lceil \log \chi(G) \rceil$ bipartite graphs.

Proof. Take a $\chi(G)$-coloring of G. Clearly, there exists a bipartite graph $B_1 \subset G$ on V, such that the chromatic number of $G \setminus B_1$ is at most $\lceil \chi(G)/2 \rceil$. Similarly, we can take a bipartite graph $B_2 \subset G \setminus B_1$, such that the chromatic number of $G \setminus B_1 \setminus B_2$ is at most $\lceil \chi(G)/4 \rceil$. After $\lceil \log \chi(G) \rceil$ analogous staps, the remaining graph is 1-colorable, that is, we decomposed the edge set of G into $\lceil \log \chi(G) \rceil$ bipartite graphs.

Definition. Let $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ be two (directed) graphs. The *ordered product* $G(V, E) = G_1 \times G_2$ is a (directed) graph with vertex set $V = V_1 \times V_2$, and edge set

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Let k be a fixed number. By known bounds for Ramsey numbers [AS92], we know that there is a graph G such that $|V(G)| = 2^k$, and $\omega(G), \alpha(G) < 2k$.

By the Proposition, the edge set of G can be decomposed into k bipartite graphs. Since bipartite graphs are comparability graphs, G is the union of k comparability graphs. Let

$$G^i = \overbrace{G \times \cdots \times G}^{i \text{ times}}.$$

It is easy to see that G^i is also a union of k comparability graphs. $|V(G^i)| = 2^{ik}$, and

$$\omega(G^i), \alpha(G^i) < (2k)^i = |V(G^i)|^{\frac{1+\log k}{k}}.$$

Now we show the upper bound of Theorem 2. We define two graphs, H_1 and H_2. $V(H_1) = \{v_1, v_2, \dots v_{13}\}$, $(v_i, v_j) \in E(H_1)$ if and only if

$$i - j \equiv 1,\ 5,\ 8,\ \text{or}\ \ 12 \pmod{13}.$$

See Fig. 1. By Brooks' theorem [B41] H_1 is four-colorable, therefore by the Proposition, it is the union of two comparability graphs. Such a decomposition is given on Fig. 1.

It is easy to see that $\omega(H_1) = 2$ and $\alpha(H_1) = 4$.

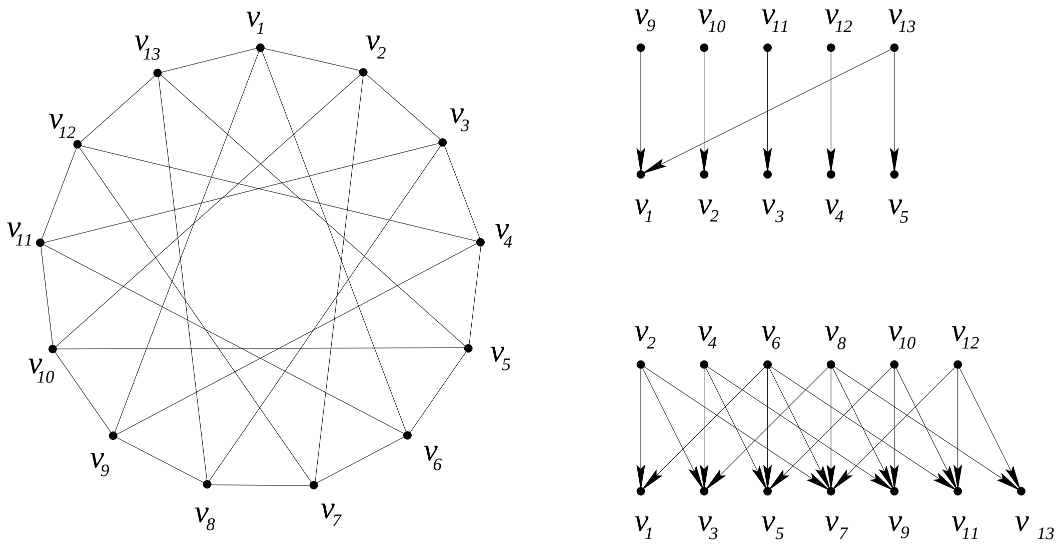

Figure 1.

For H_2, consider the set $\mathcal{S}$ of 12 chords of a cycle, shown on Fig. 2. There are no three pairwise intersecting and no five pairwise disjoint chords in $\mathcal{S}$. The vertices of H_2 represent the chords in $\mathcal{S}$, and two vertices are connected if and only if the corresponding chords are disjoint. Then $|V(H_2)| = 12$, $\omega(H_2) = 4$ and $\alpha(H_2) = 2$. We show that H_2 is the union of two comparability graphs. Suppose the endpoints of the chords in $\mathcal{S}$ are denoted by $1, 2, \dots, 24$, in clockwise direction (starting with an arbitrary endpoint), $(a, b), (c, d) \in \mathcal{S}$, $a < b$, $c < d$. Then let $(a, b) \prec_1 (c, d)$ if and only if $a < c < d < b$, and let $(a, b) \prec_2 (c, d)$ if and only if $a < b < c < d$. Clearly, both $\prec_1$ and $\prec_2$ are partial orderings, and two chords are comparable by one of them if and only if they are disjoint. This implies a decomposition of H_2 into two comparability graphs.

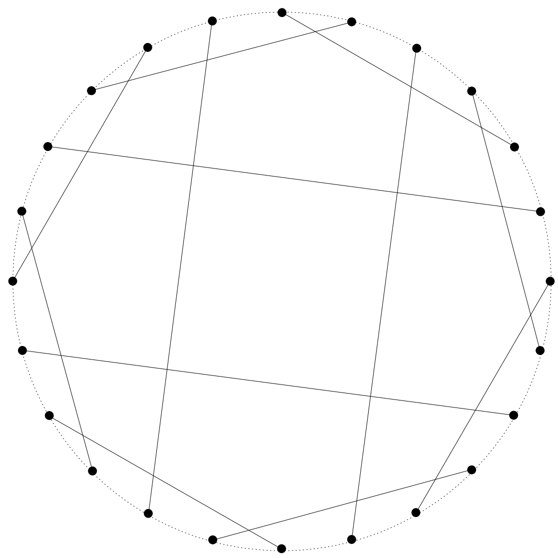

Figure 2.

Let $G = H_1 \times H_2$. G has 156 vertices, $\omega(G) = \alpha(G) = 8$, G is the union of two comparability graphs. Finally, let

$$G^i = \overbrace{G \times \cdots \times G}^{i \text{ times}}.$$

Then G^i is still the union of two comparability graphs, $|V(G^i)| = 156^i$, and

$$\omega(G^i), \alpha(G^i) = 8^i = |V(G^i)|^{\frac{\log 8}{\log 156}} < |V(G^i)|^{0.4118}.$$

3 Proof of Theorem 3

Suppose that G is the union of k perfect graphs, $G_1, G_2, \ldots, G_k$, all on the same vertex set, and let $m = \omega(G)$. Since each G_i can be colored by m colors, the product of these colorings is a proper coloring of G, with at most $m^k = \omega^k(G)$ colors.

To establish the second statement of the theorem, take a triangle-free graph H with t vertices and $\alpha(H) \leq \sqrt{t} \log t$. The existence of such a graph was proved by Erdős [ES74].

Claim. *For a triangle-free graph G, $\chi(G) \leq \alpha(G) + 1$.*

Proof of Claim. Let $\Delta(G)$ be the maximum degree in G. For any graph, $\chi(G) \leq \Delta(G) + 1$, and since G is triangle-free, the neighborhood of any vertex is an independent set, consequently, $\Delta(G) + 1 \leq \alpha(G) + 1$.

By the Claim, $\chi(H) \le \sqrt{t}\log t + 1$. Therefore, by the Proposition, we can decompose H into $\lceil \log(\chi(H)) \rceil \le (1/2 + o(1))\log t$ bipartite graphs. Since all bipartite graphs are comparability graphs, H is the union of $k = (1/2 + o(1))\log t$ comparability graphs. Let

$$G^r = \overbrace{H \times \cdots \times H}^{r \text{ times}}.$$

G^r has t^r vertices, $\omega(G^r) = 2^r$, $\alpha(G^r) \le (\sqrt{t}\log t)^r$ and $\chi(G^r) \ge t^r/\alpha(G^r) \ge (\sqrt{t}/\log t)^r$. Since the product of comparability graphs is also a comparability graph, G^r is the union of $k = (1/2 + o(1))\log t$ comparability graphs. Thus, we have,

$$\chi(G^r) \ge (\sqrt{t}/\log t)^r \ge (\omega(G^r))^{k(1-o(1))}.$$

4 Remarks

Very recently Theorem 2 has been improved by Tibor Szabó, showing that $f_2(n) \le n^{0.3878}$ [S00].

Probably the first Ramsey-type result in geometry that used unions of comparability graphs is the result of Larman et. al.

Theorem [LMPT94] *Among any n convex sets in the plane, there are either $n^{1/5}$ pairwise intersecting or $n^{1/5}$ pairwise disjoint.*

This result is very likely not the best possible. The best known result from the other direction is given by Károlyi et. al.

Theorem [KPT97] *For any n large enough, there exists a collection of n convex sets in the plane such that no $n^{0.4207}$ of them are pairwise intersecting or pairwise disjoint.*

For some special classes of convex sets there are even stronger results.

Theorem [FK93], [LMPT94] *Any collection of n axis-parallel rectangles contains $\sqrt{n/\log n}$ pairwise intersecting or pairwise disjoint.*

Theorem [KK97] *Any collection of n chords of a circle contains $(1 + o(1))\sqrt{2n/\log n}$ pairwise intersecting or pairwise disjoint.*

Theorem [PT99] *Among any n translates of a convex set in the plane there are either $c\sqrt{n}$ pairwise intersecting or $c\sqrt{n}$ pairwise disjoint.*

It is the consequence of the following more general result. We say that a convex planar set *K-fat*, if the ratio of the radii of the smallest covering disc and the largest inscribed circle $R/r < K$.

Theorem [P80] *For any $K > 0$, any collection of n K-fat convex sets in the plane contains $c_K\sqrt{n}$ pairwise intersecting or pairwise disjoint.*

A *geometric graph* is a graph drawn in the plane so that the vertices are represented by points in general position, the edges are represented by straight line segments connecting the corresponding points. Using comparability graphs, Pach and Törőcsik obtained the following result.

Theorem [PT94] *Any geometric graph of n vertices and more than $k^4 n$ edges contains $k+1$ pairwise disjoint edges.*

Using similar methods, this bound was improved in [TV98] and recently further improved in [T99].

Theorem [T99] *Any geometric graph of n vertices and more than $2^9 k^2 n$ edges contains $k+1$ pairwise disjoint edges.*

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