A Ramsey-type bound for rectangles

Géza Tóth
Courant Institute, New York University, New York, NY
E-mail: toth@cims.nyu.edu

Abstract

It is proved that for any rectangle T and for any 2-coloring of the points of the 5-dimensional Euclidean space, one can always find a rectangle T' congruent to T, all of whose vertices are of the same color. We also show that for any k-coloring of the $k^2 + o(k^2)$–dimensional space, there is a monochromatic rectangle congruent to any given rectangle.

1 Introduction

Throughout this paper by a rectangle we always mean the vertex set of a rectangle. By a coloring of the Euclidean space we mean a coloring of the points of the Euclidean space.

In a general paper about Euclidean Ramsey theory [4], Erdős, Graham, Montgomery, Rothschild, Spencer, and Straus proved that for any rectangle T and any 2-coloring of the 8-dimensional Euclidean space, one can always find a monochromatic rectangle T' congruent to T. Recently, Cantwell [2] proved that the same statement for squares is already true in the 4-dimensional space.

Here we show that for rectangles 5 dimensions are sufficient. We also investigate colorings with many colors and prove that in any k-coloring of the $k^2 + o(k^2)$–dimensional space there is a monochromatic rectangle congruent to any given rectangle.

Theorem 1. *For any rectangle T and for any 2-coloring of the 5-dimensional Euclidean space, one can find a monochromatic copy of T.*

Theorem 2. *For any rectangle T and for any k-coloring of the $k^2 + o(k^2)$-dimensional Euclidean space, one can find a monochromatic copy of T.*

2 Proof of Theorem 1.

Lemma 1 *For any given rectangle T of sides $a \geq b$ and for any red-blue coloring of the 5-dimensional space, one can find either a rectangle T' congruent to T, all of whose vertices are red, or a 3-dimensional regular simplex of side a, all of whose four vertices are blue.*

The proof is straightforward: Suppose, there is no rectangle congruent to T, all of whose vertices are red. The radius of the circumscribing circle around T is $(\sqrt{a^2+b^2})/2 \leq a/\sqrt{2}$.

There must be a blue point, A. If the sphere of radius a around A is entirely red, there is a red rectangle congruent to T on it, because $a/\sqrt{2} < a$. So, there is a blue point B at distance a from A.

The locus of the third vertex of an equilateral triangle whose two vertices are A and B, is a 4-dimensional sphere S of radius $a\sqrt{3}/2$ around the midpoint of AB. Since $a\sqrt{3}/2 > a/\sqrt{2}$, the whole sphere cannot be red, otherwise we could find a red rectangle congruent to T. Let C be a blue point on S. So, A, B and C form a regular triangle of side a.

Finally, the locus of the fourth vertex of a regular simplex whose three vertices are A, B and C, is a 3-dimensional sphere S' of radius $a\sqrt{2}/\sqrt{3}$ (the altitude of the simplex) around the center of the triangle ABC.

Since $a\sqrt{2}/\sqrt{3} > a/\sqrt{2}$, by the same argument we can find on S' the fourth blue vertex of the regular simplex of side a. □

Proof of Theorem 1: Let $a \geq b$ be the sides of T. By Lemma 1, if there is no rectangle T' congruent to T, all of whose vertices are red, there is a regular simplex $ABCD$ of side a, all of whose vertices are blue. Let $\boldsymbol{S}$ be a 3-dimensional subspace containing the simplex $ABCD$. Since we are in a 5-dimensional space, there is a 2-dimensional subspace $\boldsymbol{P}$ through A, orthogonal to $\boldsymbol{S}$. Let A_1, A_2 be two points on $\boldsymbol{P}$, such that AA_1A_2 is an equilateral triangle of side b. In this case all the three edges of the triangle AA_1A_2 are perpendicular to all six edges of the simplex $ABCD$. Translate the simplex $ABCD$ so that A moves to A_1. Let the images of B, C, and D be denoted by B_1, C_1, and D_1, respectively. Similarly, by moving A to A_2, we obtain the points B_2, C_2, and D_2.

Now we have three regular simplices and any two vertices of a simplex with the corresponding two vertices of another simplex form a rectangle congruent to T.

A, B, C, and D are blue. So if two of A_1, B_1, C_1, and D_1 or two of A_2, B_2, C_2, and D_2 are blue, there is a blue rectangle congruent to T. Otherwise, at most one of A_1, B_1, C_1, and D_1, say, A_1 can be blue, and at most one of B_2, C_2 and D_2, say, B_2 can be blue. In this case, C_1, D_1, C_2, and D_2 form a red rectangle congruent to T. $\square$

3 Proof of Theorem 2.

Let $t \geq 2$ be an integer, and let

$$m = \left\lceil \frac{\binom{kt}{2}}{\binom{t}{2}} \right\rceil + 1.$$

Lemma 2. *Let $X_1, X_2, \dots X_{kt}$ be different points. If we have m k-colorings of $X_1, X_2, \dots X_{kt}$, there are two points and two colorings such that both points in both of the colorings have the same color.*

Proof: For a fixed coloring of $X_1, X_2, \dots X_{kt}$, two points are said to form a "good" pair if they get the same color. First we give a lower bound on the number of "good" pairs in the above coloring. Denote by a_i the number of points of the ith color. Obviously, $a_1 + a_2 + \ldots + a_k = kt$. So we have $\binom{a_1}{2} + \binom{a_2}{2} + \ldots + \binom{a_k}{2}$ "good" pairs. By Jensen's inequality,

$$\binom{a_1}{2} + \binom{a_2}{2} + \ldots + \binom{a_k}{2} \geq \binom{\frac{a_1+a_2+\ldots+a_k}{k}}{2} k = \binom{t}{2} k.$$

But there are $\binom{kt}{2}$ possible "good" pairs altogether, and a "good" pair can be colored with k different colors.

Thus, as long as

$$m\binom{t}{2}k > \binom{kt}{2}k$$

holds, among any m colorings of $X_1, X_2, \dots X_{kt}$ with k colors we can always find two sharing a common "good" pair which receives the same color in both colorings. $\square$

Proof of Theorem 2: For a fixed k, let t and m be as above. Consider a $kt+m-2$-dimensional space. Let $\boldsymbol{S}$ and $\boldsymbol{P}$ be two complementary orthogonal subspaces of dimension $kt-1$ and $m-1$, and let A_{11} denote their intersection. Let $M_1 = A_{1,1}A_{1,2}\dots A_{1,kt}$ be a regular simplex of side a in $\boldsymbol{S}$, and let $A_{1,1}A_{2,1}\dots A_{m,1}$ be a regular simplex of side b in $\boldsymbol{P}$. For any $1 < i \leq m, 1 <$

$j \leq kt$, define the point $A_{i,j}$ as the image of $A_{1,j}$ under a translation taking $A_{1,1}$ into $A_{i,1}$. Denote the simplex $A_{i,1}A_{i,2}\ldots A_{i,kt}$ by M_i. Now we have m translated copies, $M_1, M_2, \ldots M_m$, of the simplex M_1, including the original one, and any two vertices of any of them with the corresponding two vertices of another form a rectangle T' congruent to T.

Consider the colorings of the simplices $M_1, M_2, \ldots M_m$. They correspond to m colorings of the simplex M_1 so that the ith coloring corresponds to the coloring of M_i. By Lemma 2, there is a good pair that occurs at least twice with the same color. That is, in the pth and qth colorings, say, $A_{1,i}$ and $A_{1,j}$ have both the lth color. Then $A_{p,i}$, $A_{q,i}$, $A_{p,j}$, $A_{q,j}$ form a rectangle congruent to T, all of whose vertices are of the same color.

So, the dimension of the space in which we were able to find a monochromatic copy of T, is

$$d = m + kt - 2 < kt + \frac{\binom{kt}{2}}{\binom{t}{2}} = k^2 + \frac{k^2 - k}{t - 1} + kt.$$

Put $t = \left[\sqrt{k}\right]$. Then

$$d < k^2 + k\left[\sqrt{k}\right] + \frac{k^2 - k}{\left[\sqrt{k}\right] - 1} = k^2 + o(k^2).$$

□

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