



Fixed Parameter Algorithms

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Classical complexity

A brief review:

- ⑥ We usually aim for **polynomial-time** algorithms: the running time is $O(n^c)$, where n is the input size.
- ⑥ Classical polynomial-time algorithms: shortest path, matching, minimum spanning tree, 2SAT, convex hull, planar drawing, linear programming, etc.
- ⑥ It is unlikely that polynomial-time algorithms exist for **NP-hard** problems.
- ⑥ Unfortunately, many problems of interest are NP-hard: Hamiltonian cycle, 3-coloring, 3SAT, etc.
- ⑥ We expect that these problems can be solved only in exponential time (i.e., c^n).

Can we say anything nontrivial about NP-hard problems?

Parameterized complexity

Main idea: Instead of expressing the running time as a function $T(n)$ of n , we express it as a function $T(n, k)$ of the input size n and some parameter k of the input.

In other words: we do not want to be efficient on all inputs of size n , only for those where k is small.

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In other words: we do not want to be efficient on all inputs of size n , only for those where k is small.

What can be the parameter k ?

- ⑥ The size k of the solution we are looking for.
- ⑥ The maximum degree of the input graph.
- ⑥ The diameter of the input graph.
- ⑥ The length of clauses in the input Boolean formula.
- ⑥ ...

Parameterized complexity

Problem:

MINIMUM VERTEX COVER

MAXIMUM INDEPENDENT SET

Input:

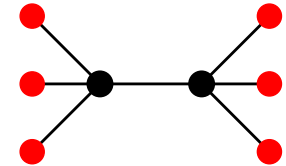
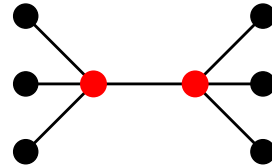
Graph G , integer k

Graph G , integer k

Question:

Is it possible to cover the edges with k vertices?

Is it possible to find k independent vertices?

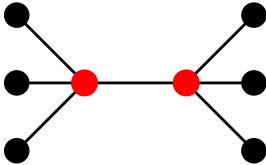
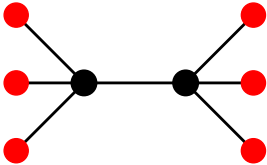


Complexity:

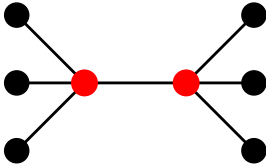
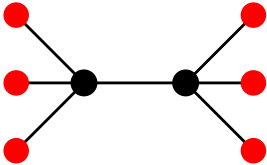


NP-complete

NP-complete

Parameterized complexity

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Complexity:	NP-complete	NP-complete
Complete enumeration:	$O(n^k)$ possibilities	$O(n^k)$ possibilities

Parameterized complexity

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Question:	Is it possible to cover the edges with k vertices?	Is it possible to find k independent vertices?
		
Complexity:	NP-complete	NP-complete
Complete enumeration:	$O(n^k)$ possibilities $O(2^k n^2)$ algorithm exists	$O(n^k)$ possibilities No $n^{o(k)}$ algorithm known
		

Bounded search tree method

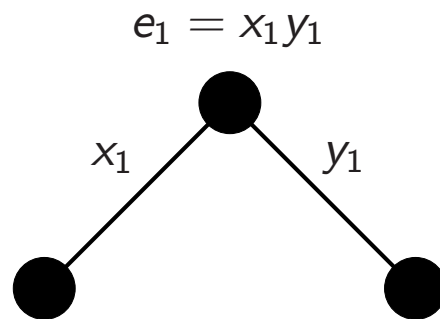
Algorithm for MINIMUM VERTEX COVER:

$$e_1 = x_1 y_1$$



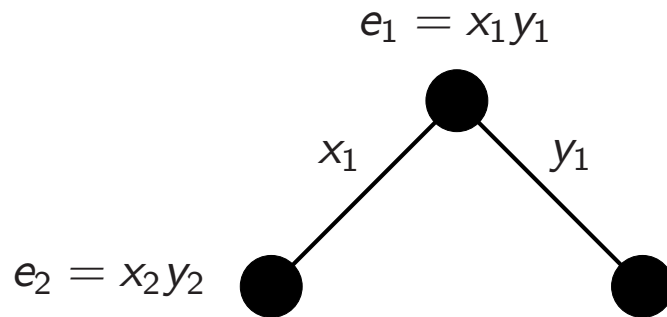
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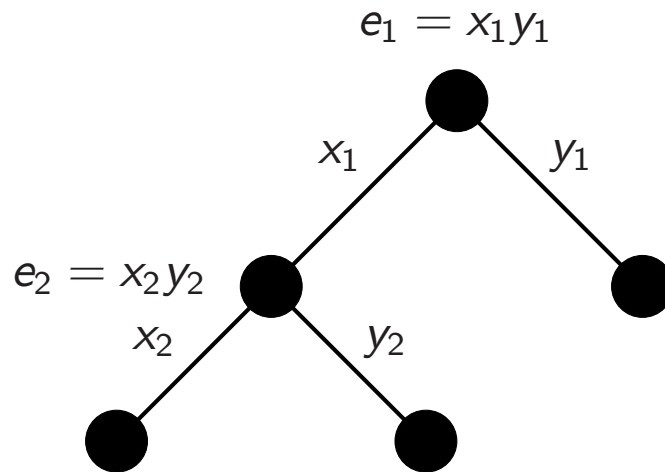
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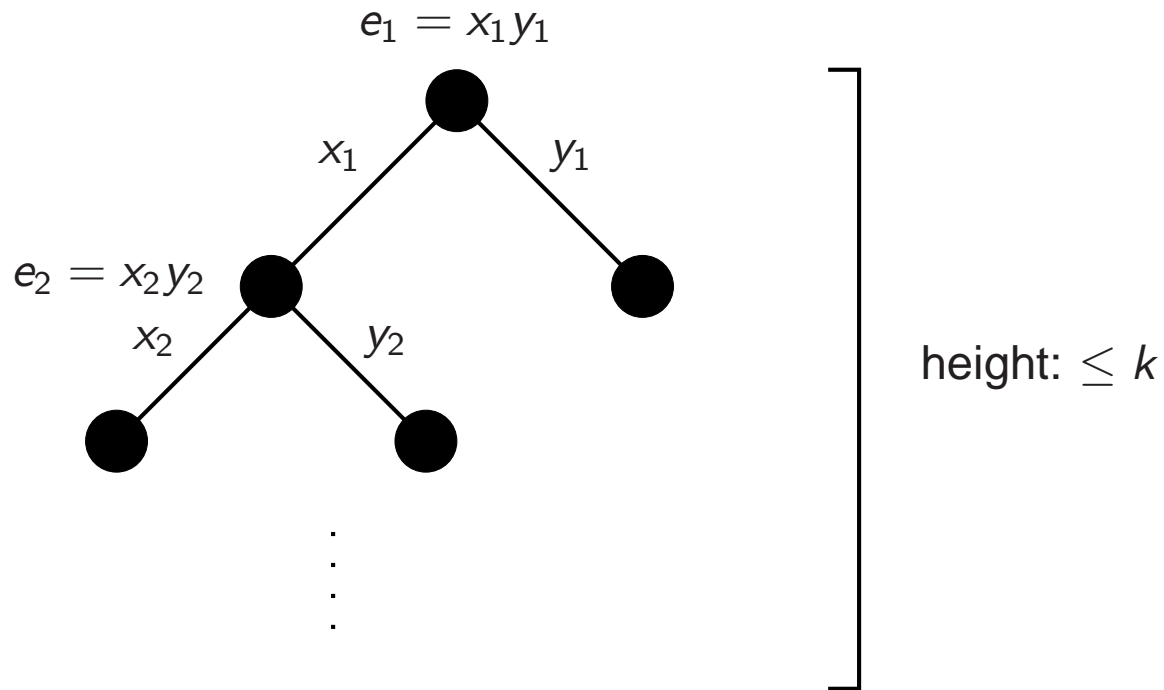
Bounded search tree method

Algorithm for MINIMUM VERTEX COVER:



Bounded search tree method

Algorithm for MINIMUM VERTEX COVER:



Height of the search tree is $\leq k \Rightarrow$ number of leaves is $\leq 2^k \Rightarrow$ complete search requires $2^k \cdot \text{poly}$ steps.

Fixed-parameter tractability

Definition: A **parameterization** of a decision problem is a function that assigns an integer parameter k to each input instance x .

The parameter can be

- ⑥ explicit in the input (for example, if the parameter is the integer k appearing in the input (G, k) of VERTEX COVER), or
- ⑥ implicit in the input (for example, if the parameter is the diameter d of the input graph G).

Main definition:

A parameterized problem is **fixed-parameter tractable (FPT)** if there is an $f(k)n^c$ time algorithm for some constant c .

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Example: MINIMUM VERTEX COVER parameterized by the required size k is FPT: we have seen that it can be solved in time $O(2^k + n^2)$.

Better algorithms are known: e.g, $O(1.2832^k k + k|V|)$.

Main goal of parameterized complexity: to find FPT problems.

FPT problems

Examples of NP-hard problems that are FPT:

- ⑥ Finding a vertex cover of size k .
- ⑥ Finding a path of length k .
- ⑥ Finding k disjoint triangles.
- ⑥ Drawing the graph in the plane with k edge crossings.
- ⑥ Finding disjoint paths that connect k pairs of points.
- ⑥ ...

FPT algorithmic techniques

- ⑥ Significant advances in the past 20 years or so (especially in recent years).
- ⑥ Powerful toolbox for designing FPT algorithms:

Bounded Search Tree

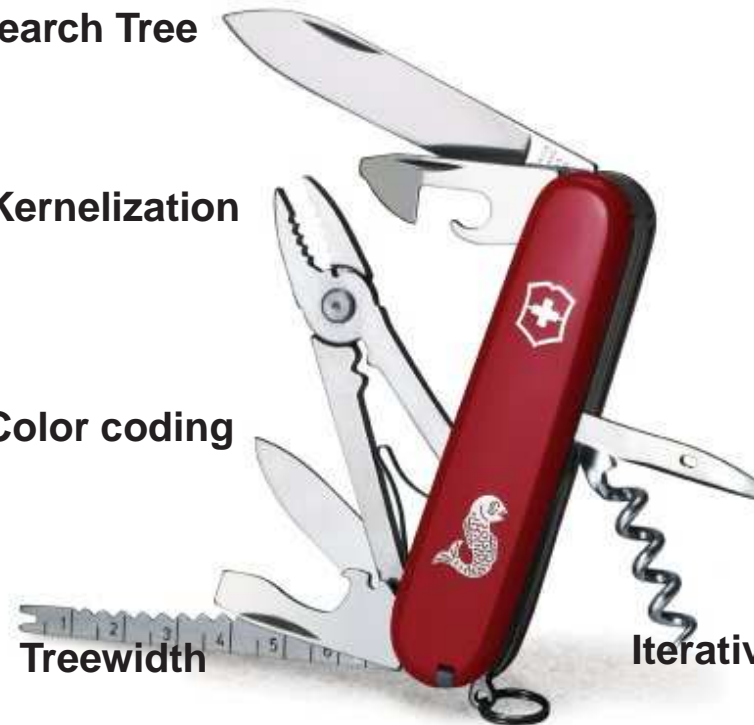
Kernelization

Color coding

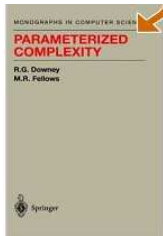
Treewidth

Graph Minors Theorem

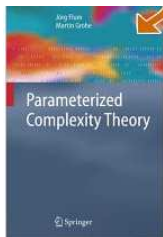
Iterative compression



Books



Downey-Fellows: Parameterized Complexity, Springer, 1999



Flum-Grohe: Parameterized Complexity Theory, Springer, 2006



Niedermeier: Invitation to Fixed-Parameter Algorithms, Oxford University Press, 2006.



Goals of the course

- ⑥ Demonstrate techniques that were successfully used in the analysis of parameterized problems.
 - △ Determine quickly if a problem is FPT.
 - △ Design fast algorithms (improve the function $f(k)$).
- ⑥ Introduce the basics of parameterized hardness theory (W[1]-hardness).

- ⑥ **Warning:** The results presented for particular problems are not necessarily the best known results or the most useful approaches for these problems.
- ⑥ Conventions:
 - △ Unless noted otherwise, k is the parameter.
 - △ O^* notation: $O^*(f(k))$ means $O(f(k) \cdot n^c)$ for some constant c .
 - △ Citations are mostly omitted (only for classical results).
 - △ We gloss over the difference between decision and search problems.

Kernelization



Kernelization

Definition: **Kernelization** is a polynomial-time transformation that maps an instance (I, k) to an instance (I', k') such that

- ⑥ (I, k) is a yes-instance if and only if (I', k') is a yes-instance,
- ⑥ $k' \leq k$, and
- ⑥ $|I'| \leq f(k)$ for some function $f(k)$.

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Simple fact: If a problem has a kernelization algorithm, then it is FPT.

Proof: Solve the instance (I', k') by brute force.

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Simple fact: If a problem has a kernelization algorithm, then it is FPT.

Proof: Solve the instance (I', k') by brute force.

Converse: Every FPT problem has a kernelization algorithm.

Proof: Suppose there is an $f(k)n^c$ algorithm for the problem.

- ⑥ If $f(k) \leq n$, then solve the instance in time $f(k)n^c \leq n^{c+1}$, and output a trivial yes- or no-instance.
- ⑥ If $n < f(k)$, then we are done: a kernel of size $f(k)$ is obtained.

Kernelization for VERTEX COVER

General strategy: We devise a list of reduction rules, and show that if none of the rules can be applied and the size of the instance is still larger than $f(k)$, then the answer is trivial.

Reduction rules for VERTEX COVER instance (G, k) :

Rule 1: If v is an isolated vertex $\Rightarrow (G \setminus v, k)$

Rule 2: If $d(v) > k \Rightarrow (G \setminus v, k - 1)$

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If neither Rule 1 nor Rule 2 can be applied:

- ⑥ If $|V(G)| > k(k + 1) \Rightarrow$ There is no solution (every vertex should be the neighbor of at least one vertex of the cover).
- ⑥ Otherwise, $|V(G)| \leq k(k + 1)$ and we have a $k(k + 1)$ vertex kernel.

Kernelization for VERTEX COVER

Let us add a third rule:

Rule 1: If v is an isolated vertex $\Rightarrow (G \setminus v, k)$

Rule 2: If $d(v) > k \Rightarrow (G \setminus v, k - 1)$

Rule 3: If $d(v) = 1$, then we can assume that its neighbor u is in the solution $\Rightarrow (G \setminus (u \cup v), k - 1)$.

If none of the rules can be applied, then every vertex has degree at least 2.

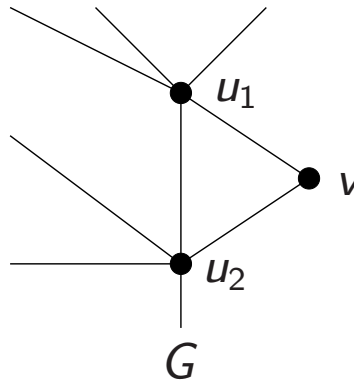
$$\Rightarrow |V(G)| \leq |E(G)|$$

- ⑥ If $|E(G)| > k^2 \Rightarrow$ There is no solution (each vertex of the solution can cover at most k edges).
- ⑥ Otherwise, $|V(G)| \leq |E(G)| \leq k^2$ and we have a k^2 vertex kernel.

Kernelization for VERTEX COVER

Let us add a fourth rule:

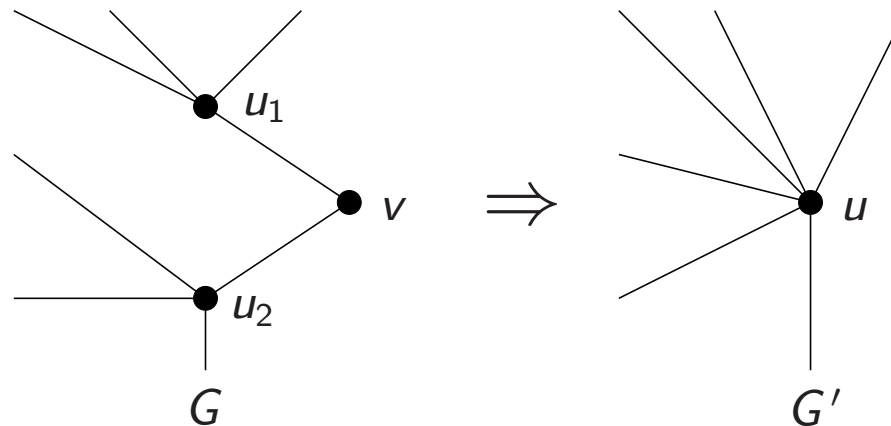
Rule 4a: If v has degree 2, and its neighbors u_1 and u_2 are adjacent, then we can assume that u_1, u_2 are in the solution $\Rightarrow (G \setminus \{u_1, u_2, v\}, k - 2)$.



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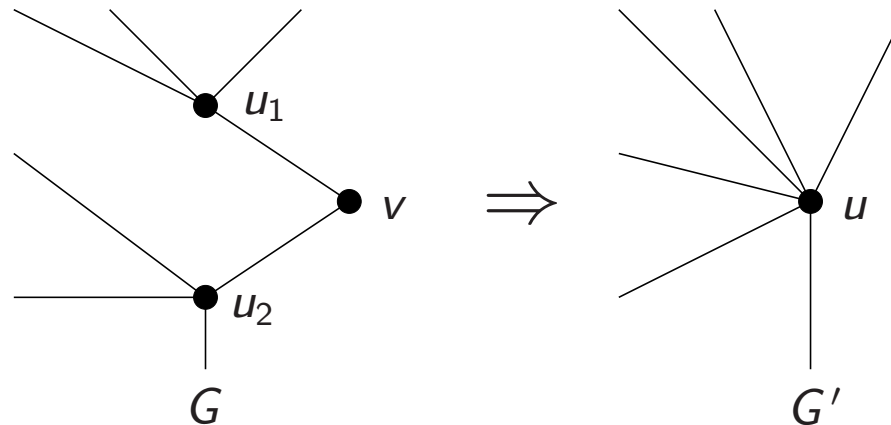
Rule 4b: If v has degree 2, then G' is obtained by identifying the two neighbors of v and deleting $v \Rightarrow (G', k - 1)$.



Kernelization for VERTEX COVER

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Correctness:

Let S' be a vertex cover of size $k - 1$ for G' .

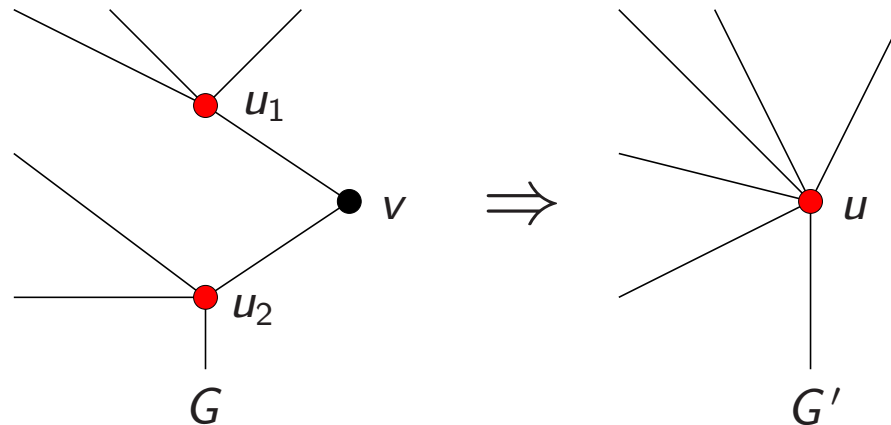
If $u \in S' \Rightarrow (S' \setminus u) \cup \{u_1, u_2\}$ is a vertex cover of size k for G .

If $u \notin S' \Rightarrow S' \cup v$ is a vertex cover of size k for G .

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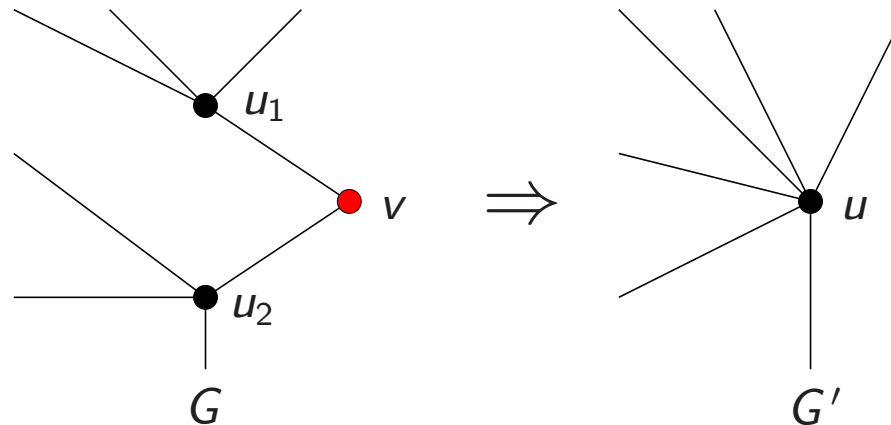
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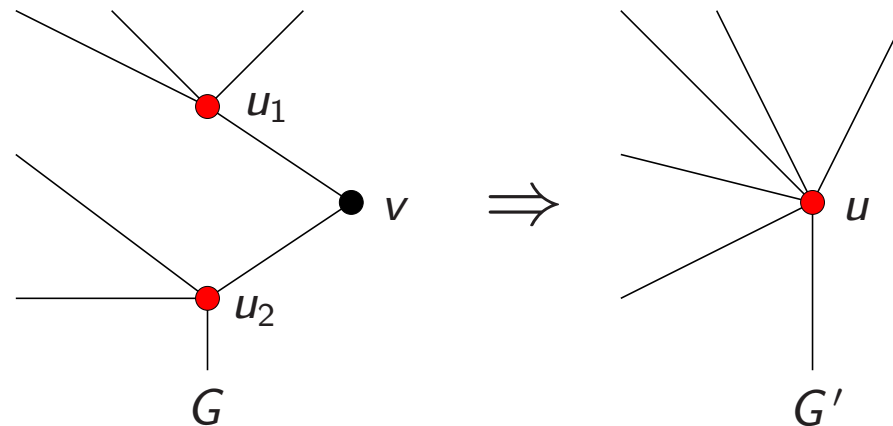
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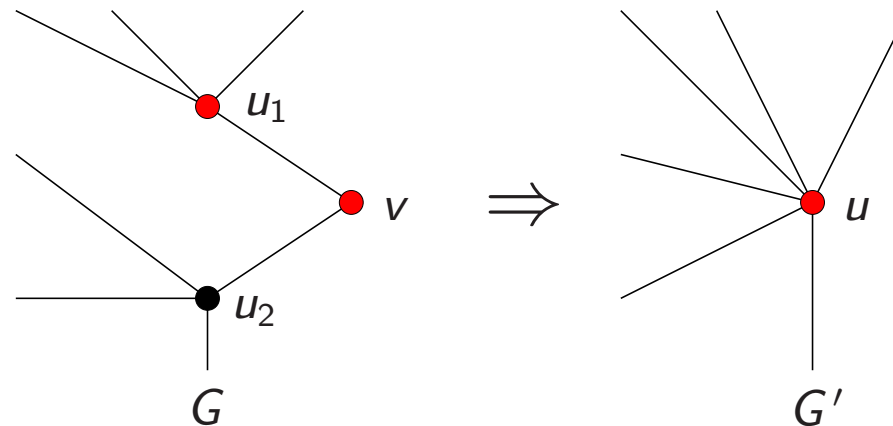
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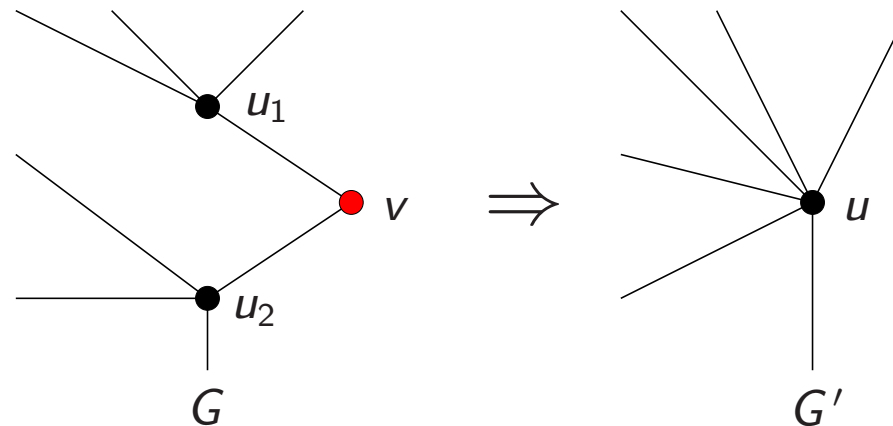
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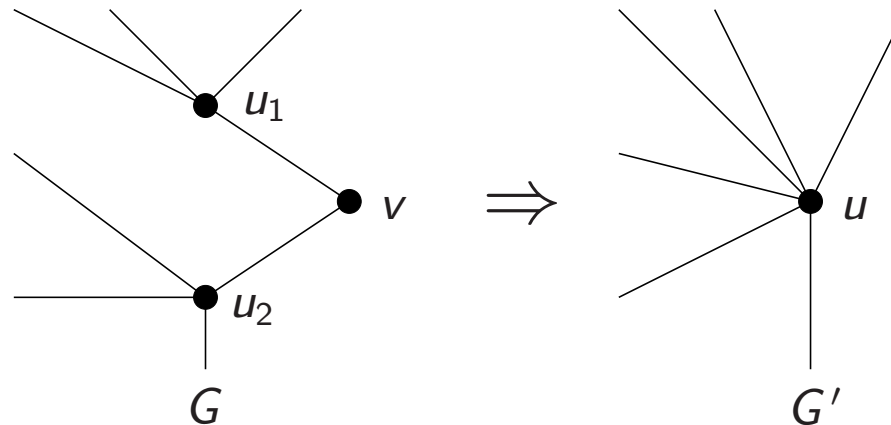
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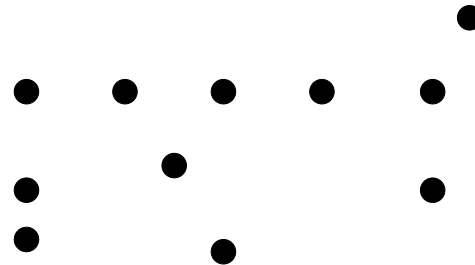


Kernel size:

- ⑥ If $|E(G)| > k^2 \Rightarrow$ There is no solution (each vertex of the solution can cover at most k edges).
- ⑥ Otherwise, $|V(G)| \leq 2|E(G)|/3 \leq \frac{2}{3}k^2$ and we have a $\frac{2}{3}k^2$ vertex kernel.

COVERING POINTS WITH LINES

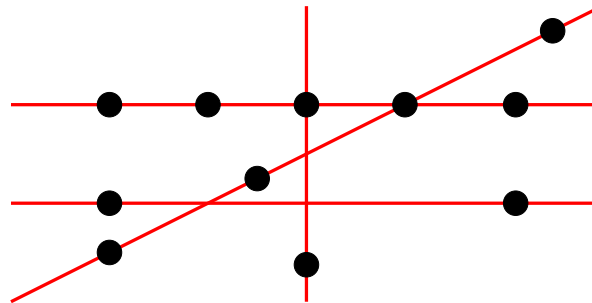
Task: Given a set P of n points in the plane and an integer k , find k lines that cover all the points.



Note: We can assume that every line of the solution covers at least 2 points, thus there are at most n^2 candidate lines.

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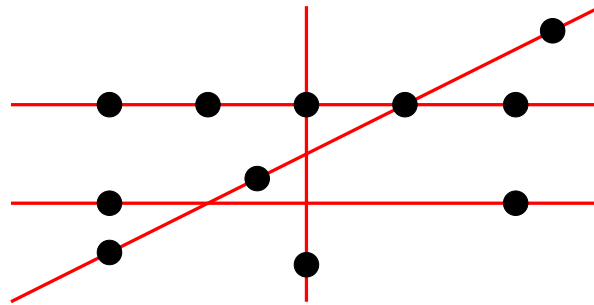
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Reduction Rule:

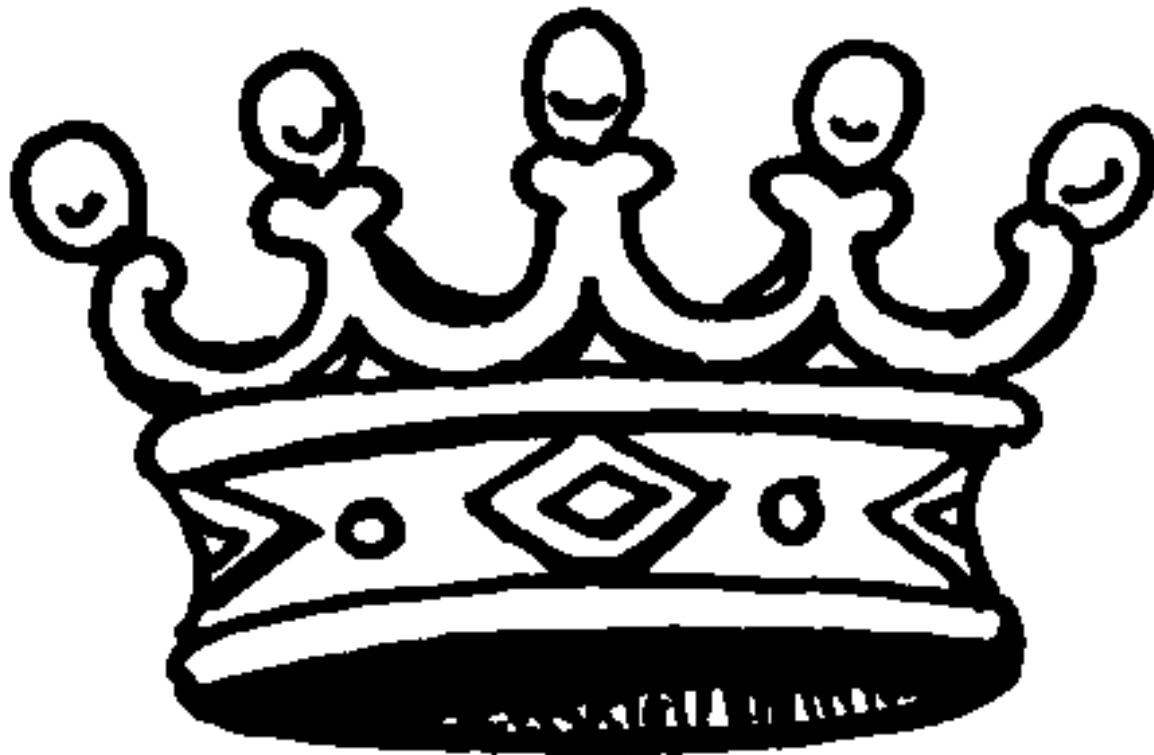
If a candidate line covers a set S of more than k points $\Rightarrow (P \setminus S, k - 1)$.

If this rule cannot be applied and there are still more than k^2 points, then there is no solution \Rightarrow Kernel with at most k^2 points.

Kernelization

- ⑥ Kernelization can be thought of as a polynomial-time preprocessing before attacking the problem with whatever method we have. “It does no harm” to try kernelization.
- ⑥ Some kernelizations use lots of simple reduction rules and require a complicated analysis to bound the kernel size...
- ⑥ ... while other kernelizations are based on surprising nice tricks (Next: Crown Reduction and the Sunflower Lemma).
- ⑥ Possibility to prove lower bounds.

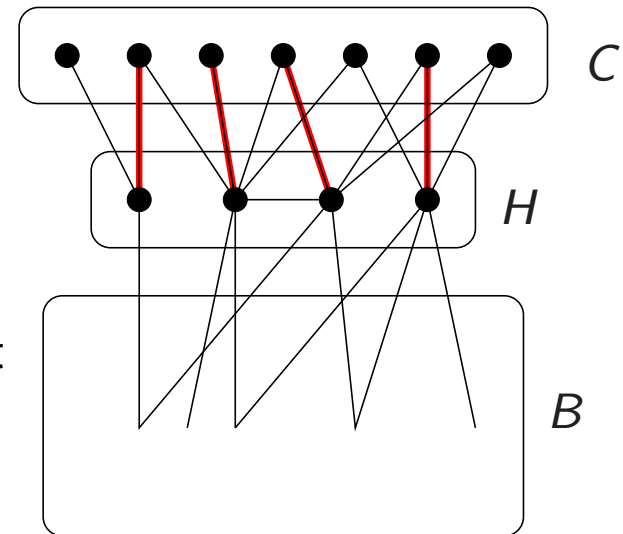
Crown Reduction



Crown Reduction

Definition: A **crown decomposition** is a partition $C \cup H \cup B$ of the vertices such that

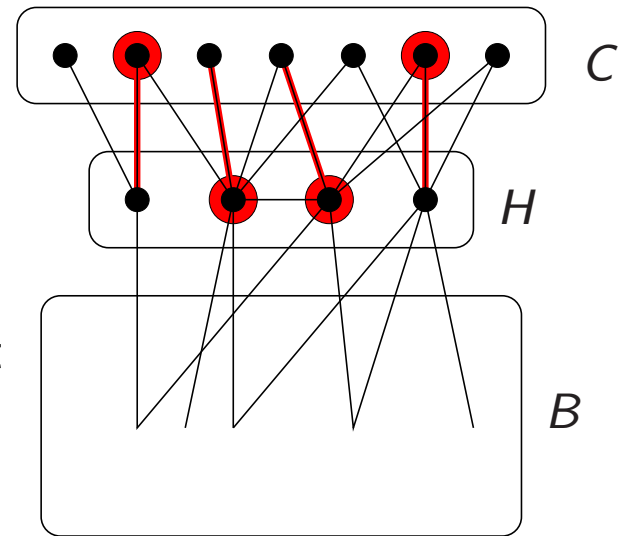
- ⑥ C is an independent set,
- ⑥ there is no edge between C and B ,
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Crown rule for VERTEX COVER:

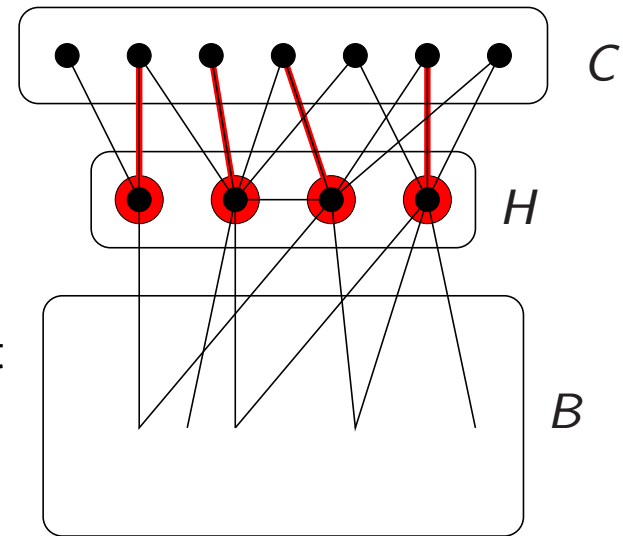
The matching needs to be covered and we can assume that it is covered by H (makes no sense to use vertices of C)

$\Rightarrow (G \setminus (H \cup C), k - |H|)$.

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Crown Reduction

Key lemma:

Lemma: Given a graph G without isolated vertices and an integer k , in polynomial time we can either

- ⑥ find a matching of size $k + 1$,
- ⑥ find a crown decomposition,
- ⑥ or conclude that the graph has at most $3k$ vertices.

Crown Reduction

Key lemma:

Lemma: Given a graph G without isolated vertices and an integer k , in polynomial time we can either

- ⑥ find a matching of size $k + 1$, \Rightarrow **No solution!**
- ⑥ find a crown decomposition, \Rightarrow **Reduce!**
- ⑥ or conclude that the graph has at most $3k$ vertices.
 \Rightarrow **$3k$ vertex kernel!**

This gives a $3k$ vertex kernel for VERTEX COVER.

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For the proof, we need the classical König's Theorem.

$\tau(G)$: size of the minimum vertex cover

$\nu(G)$: size of the maximum matching (independent set of edges)

Theorem: [Kőnig, 1931] If G is **bipartite**, then

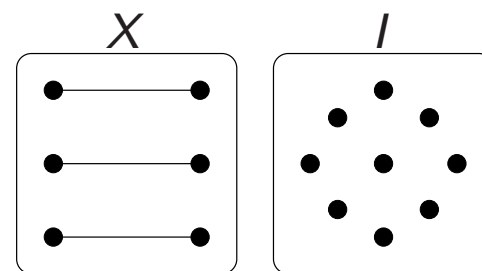
$$\tau(G) = \nu(G)$$

Proof

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Proof: Find (greedily) a maximal matching; if its size is at least $k + 1$, then we are done. The rest of the graph is an independent set I .



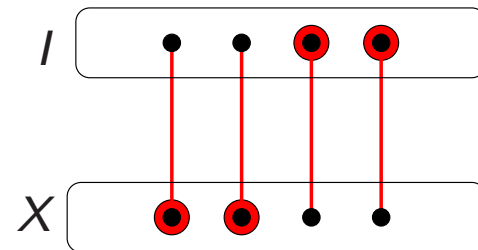
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Find a maximum matching/minimum vertex cover in the bipartite graph between X and I .



Proof

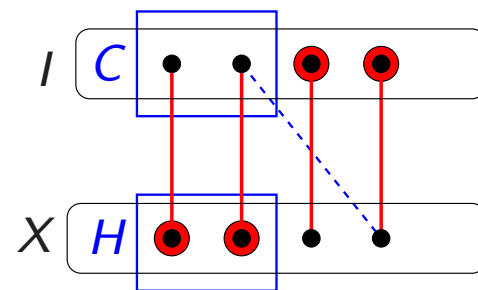
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- ⑥ find a matching of size $k + 1$,
- ⑥ find a crown decomposition,
- ⑥ or conclude that the graph has at most $3k$ vertices.

Proof:

Case 1: The minimum vertex cover contains at least one vertex of X

⇒ There is a crown decomposition.



Proof

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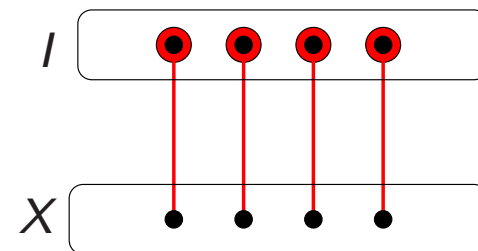
Proof:

Case 1: The minimum vertex cover contains at least one vertex of X

⇒ There is a crown decomposition.

Case 2: The minimum vertex cover contains only vertices of I ⇒ It contains every vertex of I

⇒ There are at most $2k + k$ vertices.



DUAL OF VERTEX COLORING

Parameteric dual of k -COLORING. Also known as SAVING k COLORS.

Task: Given a graph G and an integer k , find a vertex coloring with $|V(G)| - k$ colors.

Crown rule for DUAL OF VERTEX COLORING:

DUAL OF VERTEX COLORING

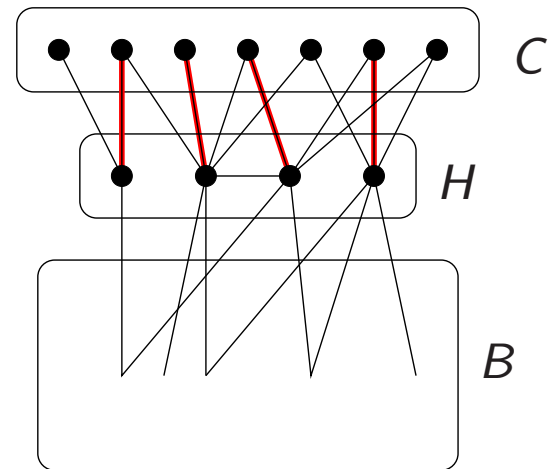
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Crown rule for DUAL OF VERTEX COLORING:

Suppose there is a crown decomposition for the **complement graph** \overline{G} .

- ⑥ C is a clique in G : each vertex needs a distinct color.
- ⑥ Because of the matching, it is possible to color H using only these $|C|$ colors.
- ⑥ These colors cannot be used for B .
- ⑥ $(G \setminus (H \cup C), k - |H|)$



DUAL OF VERTEX COLORING

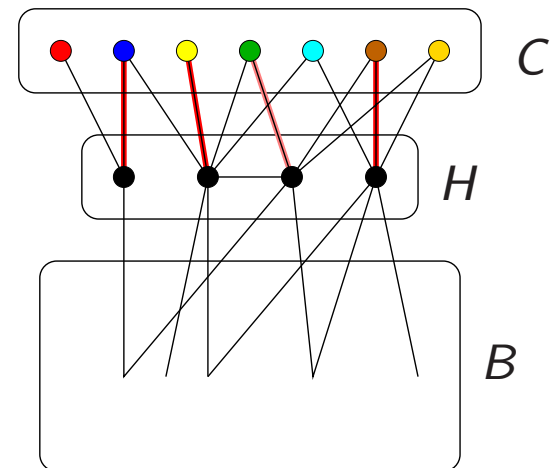
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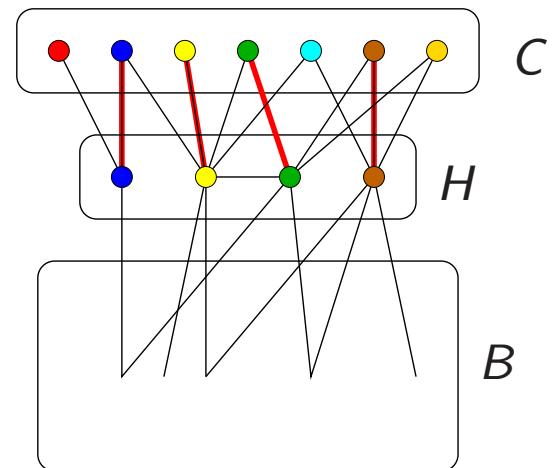
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Crown Reduction for DUAL OF VERTEX COLORING

Use the key lemma for the complement \overline{G} of G :

Lemma: Given a graph G without isolated vertices and an integer k , in polynomial time we can either

- ⑥ find a matching of size $k + 1$, \Rightarrow YES: we can save k colors!
- ⑥ find a crown decomposition, \Rightarrow Reduce!
- ⑥ or conclude that the graph has at most $3k$ vertices.
 \Rightarrow $3k$ vertex kernel!

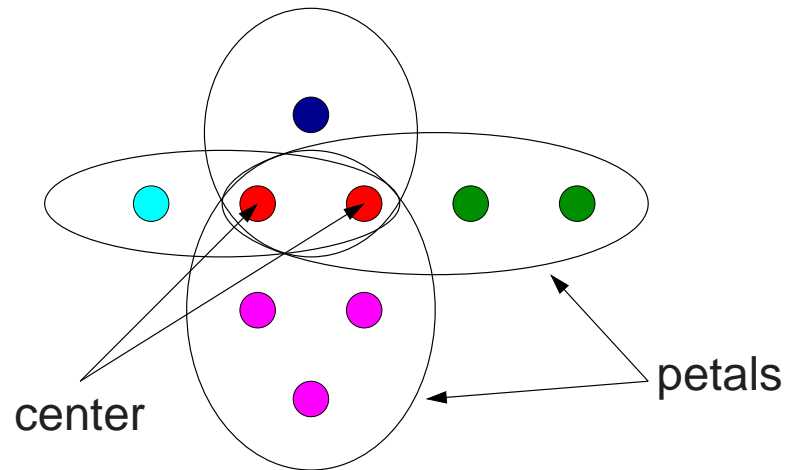
This gives a $3k$ vertex kernel for DUAL OF VERTEX COLORING.

Sunflower Lemma



Sunflower lemma

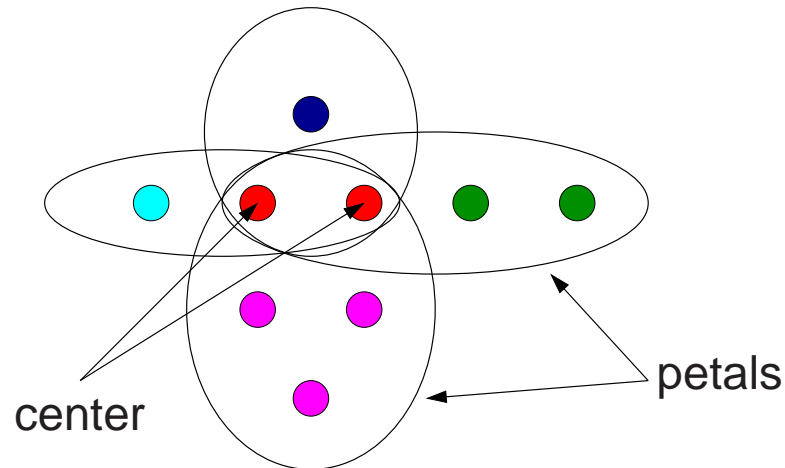
Definition: Sets S_1, S_2, \dots, S_k form a **sunflower** if the sets $S_i \setminus (S_1 \cap S_2 \cap \dots \cap S_k)$ are disjoint.



Lemma: [Erdős and Rado, 1960] If the size of a set system is greater than $(p - 1)^d \cdot d!$ and it contains only sets of size at most d , then the system contains a sunflower with p petals. Furthermore, in this case such a sunflower can be found in polynomial time.

Sunflowers and d -HITTING SET

d -HITTING SET: Given a collection \mathcal{S} of sets of size at most d and an integer k , find a set S of k elements that intersects every set of \mathcal{S} .



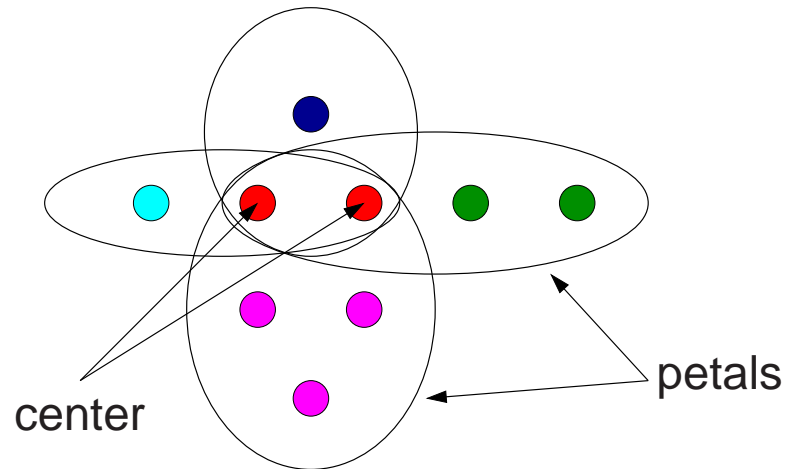
Reduction Rule: If $k + 1$ sets form a sunflower, then remove these sets from \mathcal{S} and add the center C to \mathcal{S} (S does not hit one of the petals, thus it has to hit the center).

Note: if the center is empty (the sets are disjoint), then there is no solution.

If the rule cannot be applied, then there are at most $O(k^d)$ sets.

Sunflowers and d -HITTING SET

d -HITTING SET: Given a collection \mathcal{S} of sets of size at most d and an integer k , find a set S of k elements that intersects every set of \mathcal{S} .



Reduction Rule (variant): Suppose more than $k + 1$ sets form a sunflower.

- ⑥ If the sets are disjoint \Rightarrow No solution.
- ⑥ Otherwise, keep only $k + 1$ of the sets.

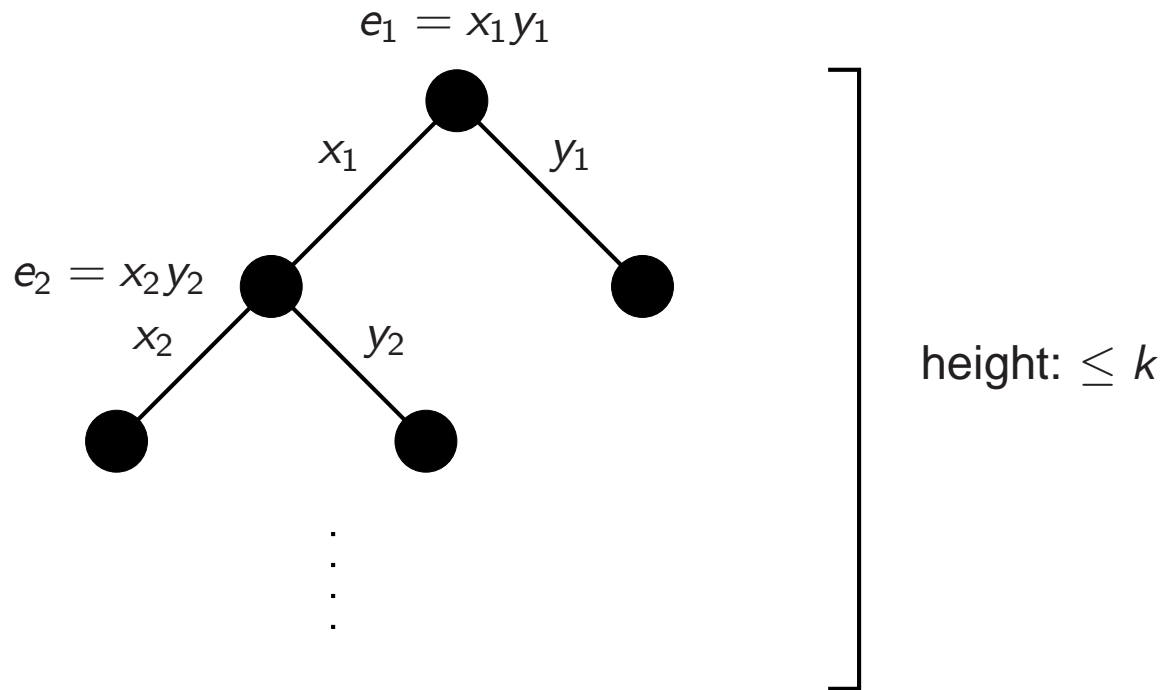
If the rule cannot be applied, then there are at most $O(k^d)$ sets.

Branching and bounded search trees



Bounded search tree method

Recall how we solved MINIMUM VERTEX COVER:



Bounded search tree method

We solve the problem by one or more **branching rules**.

Each rule makes a “guess” in such a way that at least one guess will lead to a correct solution.

If we have branching rules such that

- ⑥ each rule branches into at most $b(k)$ directions, and
- ⑥ applying a rule decreases the parameter,

then the problem can be solved in time $O^*(b(k)^k)$.

In many cases, this crude upper bound can be improved by better analysis.

VERTEX COVER

Improved algorithm for VERTEX COVER.

- ⑥ If every vertex has degree ≤ 2 , then the problem can be solved in polynomial time.
- ⑥ **Branching rule:** If there is a vertex v with at least 3 neighbors, then
 - △ either v is in the solution,
 - △ or every neighbor of v is in the solution.

Crude upper bound: $O^*(2^k)$, since the branching rule decreases the parameter.

VERTEX COVER

Improved algorithm for VERTEX COVER.

- ⑥ If every vertex has degree ≤ 2 , then the problem can be solved in polynomial time.
- ⑥ **Branching rule:** If there is a vertex v with at least 3 neighbors, then
 - △ either v is in the solution, $\Rightarrow k$ decreases by 1
 - △ or every neighbor of v is in the solution. $\Rightarrow k$ decreases by at least 3

Crude upper bound: $O^*(2^k)$, since the branching rule decreases the parameter.

But it is somewhat better than that, since in the second branch, the parameter decreases by at least 3.

Better analysis

Let $t(k)$ be the maximum number of leaves of the search tree if the parameter is at most k (let $t(k) = 1$ for $k \leq 0$).

$$t(k) \leq t(k-1) + t(k-3)$$

There is a standard technique for bounding such functions asymptotically.

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There is a standard technique for bounding such functions asymptotically.

We prove by induction that $t(k) \leq c^k$ for some $c > 1$ as small as possible.

What values of c are good? We need:

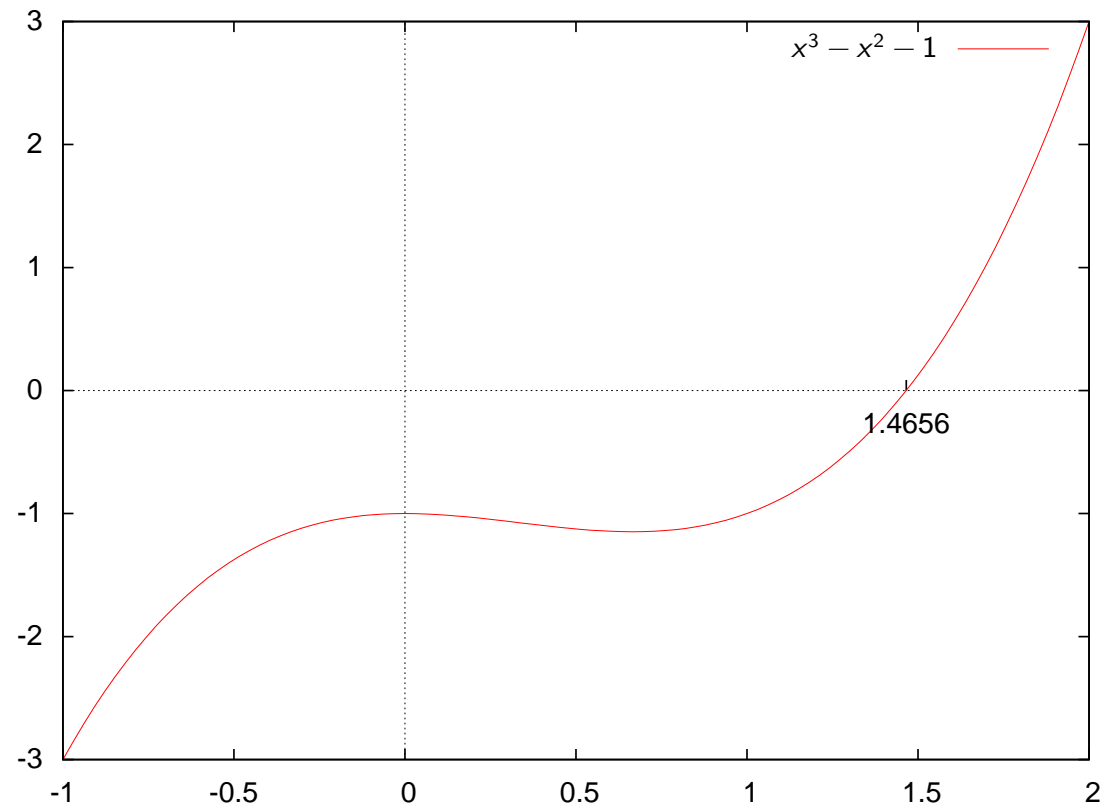
$$c^k \geq c^{k-1} + c^{k-3}$$

$$c^3 - c^2 - 1 \geq 0$$

We need to find the roots of the **characteristic equation** $c^3 - c^2 - 1 = 0$.

Note: it is always true that such an equation has a unique positive root.

Better analysis



$c = 1.4656$ is a good value $\Rightarrow t(k) \leq 1.4656^k$

\Rightarrow We have a $O^*(1.4656^k)$ algorithm for VERTEX COVER.

Better analysis

We showed that if $t(k) \leq t(k-1) + t(k-3)$, then $t(k) \leq 1.4656^k$ holds.

Is this bound tight? There are two questions:

- ⑥ Can the function $t(k)$ be that large?
Yes (ignoring rounding problems).
- ⑥ Can the search tree of the VERTEX COVER algorithm be that large?
Difficult question, hard to answer in general.

Branching vectors

The **branching vector** of our $O^*(1.4656^k)$ VERTEX COVER algorithm was $(1, 3)$.

Example: Let us bound the search tree for the branching vector $(2, 5, 6, 6, 7, 7)$.
(2 out of the 6 branches decrease the parameter by 7, etc.).

Branching vectors

The **branching vector** of our $O^*(1.4656^k)$ VERTEX COVER algorithm was $(1, 3)$.

Example: Let us bound the search tree for the branching vector $(2, 5, 6, 6, 7, 7)$.
(2 out of the 6 branches decrease the parameter by 7, etc.).

The value $c > 1$ has to satisfy:

$$c^k \geq c^{k-2} + c^{k-5} + 2c^{k-6} + 2c^{k-7}$$

$$c^7 - c^5 - c^2 - 2c - 2 \geq 0$$

Unique positive root of the characteristic equation: $1.4483 \Rightarrow t(k) \leq 1.4483^k$.

It is hard to compare branching vectors intuitively.

Branching vectors

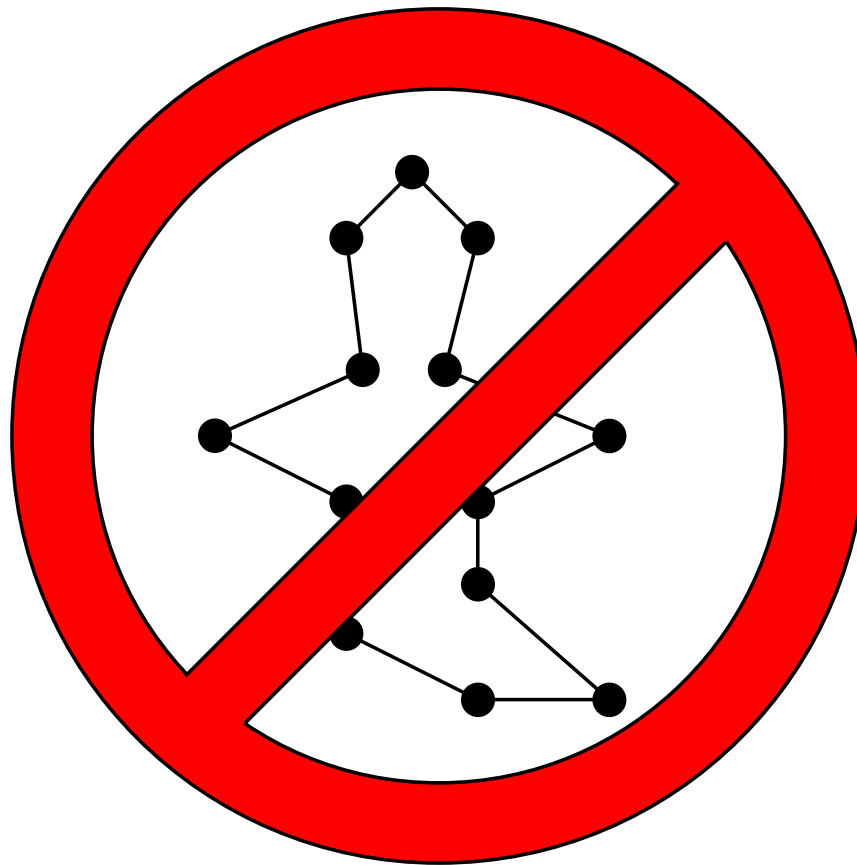
Example: The roots for branching vector (i, j) ($1 \leq i, j \leq 6$).

$$t(k) \leq t(k-i) + t(k-j) \Rightarrow c^k \geq c^{k-i} + c^{k-j}$$
$$c^j - c^{j-i} - 1 \geq 0$$

We compute the unique positive root.

	1	2	3	4	5	6
1	2.0000	1.6181	1.4656	1.3803	1.3248	1.2852
2	1.6181	1.4143	1.3248	1.2721	1.2366	1.2107
3	1.4656	1.3248	1.2560	1.2208	1.1939	1.1740
4	1.3803	1.2721	1.2208	1.1893	1.1674	1.1510
5	1.3248	1.2366	1.1939	1.1674	1.1487	1.1348
6	1.2852	1.2107	1.1740	1.1510	1.1348	1.1225

Forbidden subgraphs



Forbidden subgraphs

General problem class: Given a graph G and an integer k , transform G with at most k modifications (add/remove vertices/edges) into a graph having property \mathcal{P} .

Example:

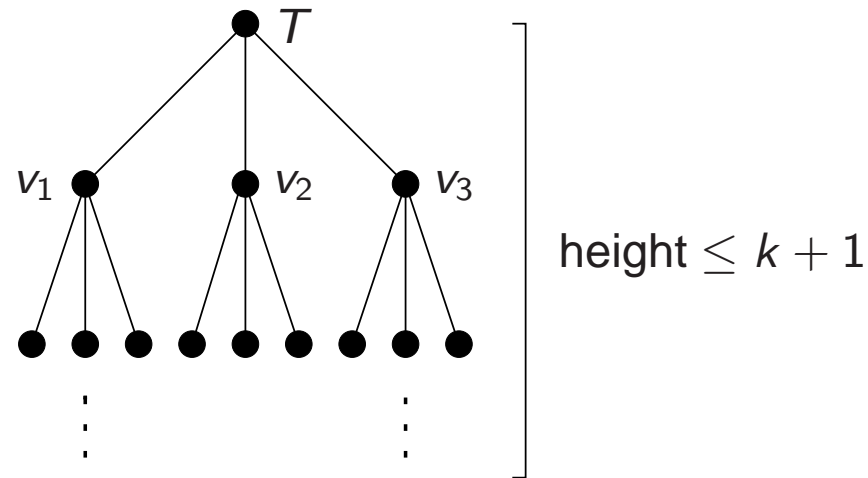
TRIANGLE DELETION: make the graph triangle-free by deleting at most k vertices.

Branching algorithm:

- ⑥ If the graph is triangle-free, then we are done.
- ⑥ **Branching rule:** If there is a triangle $v_1 v_2 v_3$, then at least one of v_1, v_2, v_3 has to be deleted \Rightarrow We branch into 3 directions.

TRIANGLE DELETION

Search tree:



The search tree has at most 3^k leaves and the work to be done is polynomial at each step $\Rightarrow O^*(3^k)$ time algorithm.

Note: If the answer is “NO”, then the search tree has **exactly** 3^k leaves.

Hereditary properties

Definition: A graph property \mathcal{P} is **hereditary** if for every $G \in \mathcal{P}$ and induced subgraph G' of G , we have $G' \in \mathcal{P}$ as well.

Examples: triangle-free, bipartite, interval graph, planar

Observation: Every hereditary property \mathcal{P} can be characterized by a (finite or infinite) set \mathcal{F} of forbidden induced subgraphs:

$$G \in \mathcal{P} \iff \forall H \in \mathcal{F}, H \not\subseteq_{\text{ind}} G$$

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Hereditary properties

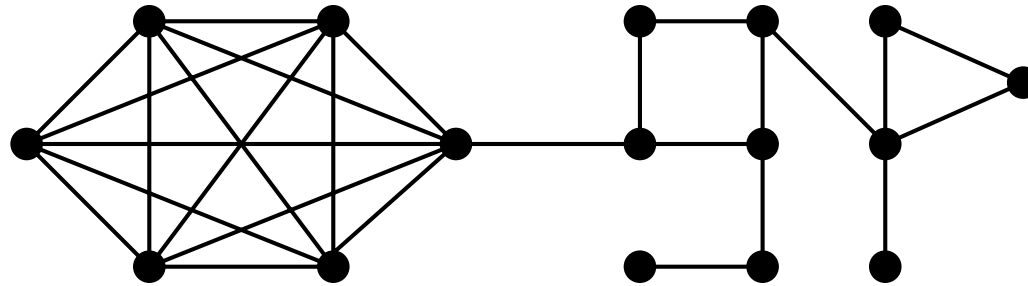
Theorem: If \mathcal{P} is hereditary and can be characterized by a **finite** set \mathcal{F} of forbidden induced subgraphs, then the graph modification problems corresponding to \mathcal{P} are FPT.

Proof:

- ⑥ Suppose that every graph in \mathcal{F} has at most r vertices. Using brute force, we can find in time $O(n^r)$ a forbidden subgraph (if exists).
- ⑥ If a forbidden subgraph exists, then we have to delete one of the at most r vertices or add/delete one of the at most $\binom{r}{2}$ edges
⇒ Branching factor is a constant c depending on \mathcal{F} .
- ⑥ The search tree has at most c^k leaves and the work to be done at each node is $O(n^r)$.

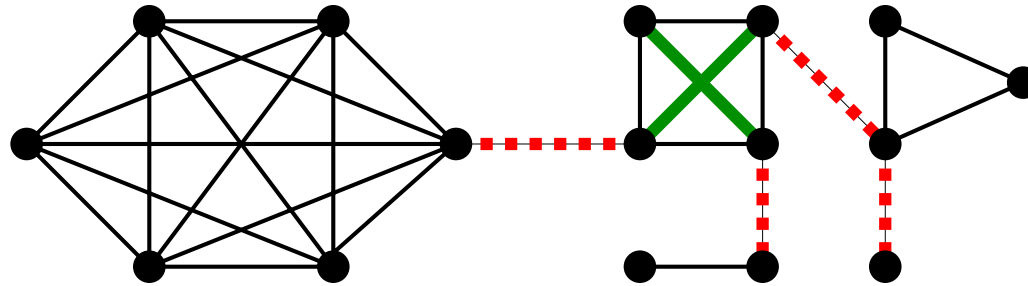
CLUSTER EDITING

Task: Given a graph G and an integer k , add/remove at most k edges such that every component is a clique in the resulting graph.



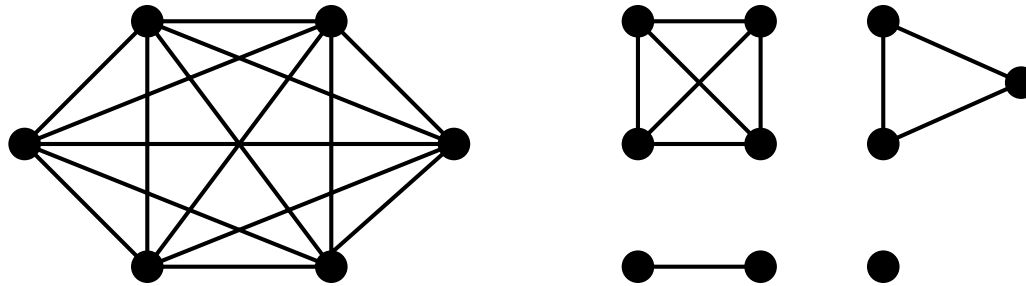
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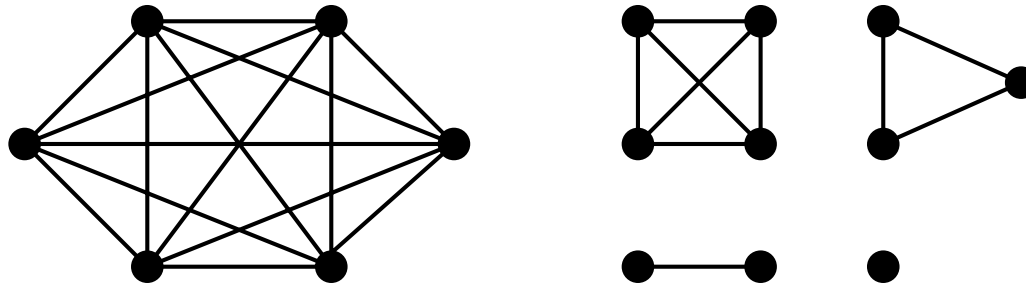
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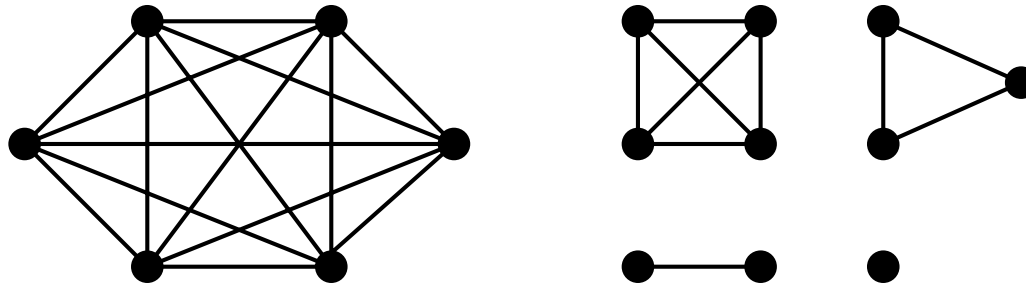
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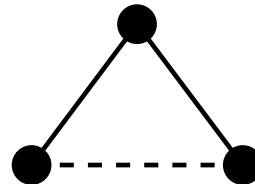
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Forbidden induced subgraph:



$O^*(3^k)$ time algorithm.

CHORDAL COMPLETION

Definition: A graph is **chordal** if it does not contain an induced cycle of length greater than 3.

CHORDAL COMPLETION: Given a graph G and an integer k , add at most k edges to G to make it a chordal graph.

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The forbidden induced subgraphs are the cycles of length greater 3

⇒ Not a finite set!

CHORDAL COMPLETION

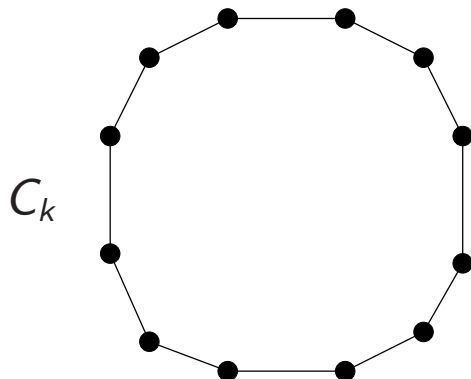
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Lemma: At least $k - 3$ edges are needed to make a k -cycle chordal.

Proof: By induction. $k = 3$ is trivial.



CHORDAL COMPLETION

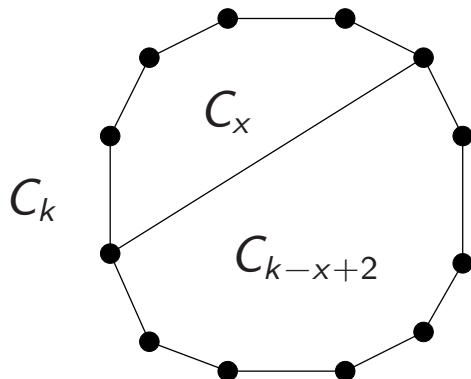
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C_x : $x - 3$ edges

C_{k-x+2} : $k - x - 1$ edges

C_k : $(x-3) + (k-x-1) + 1 = k-3$
edges

CHORDAL COMPLETION

Algorithm:

- ⑥ Find an induced cycle C of length at least 4 (can be done in polynomial time).
- ⑥ If no such cycle exists \Rightarrow **Done!**
- ⑥ If C has more than $k + 3$ vertices \Rightarrow **No solution!**
- ⑥ Otherwise, one of the

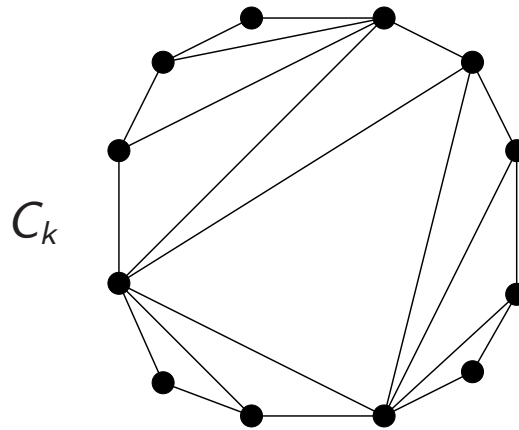
$$\binom{|C|}{2} - |C| \leq (k + 3)(k + 2)/2 - k = O(k^2)$$

missing edges has to be added \Rightarrow **Branch!**

Size of the search tree is $k^{O(k)}$.

CHORDAL COMPLETION – *more efficiently*

Definition: Triangulation of a cycle.



Lemma: Every chordal supergraph of a cycle C contains a triangulation of the cycle C .

Lemma: The number of ways a cycle of length k can be triangulated is exactly the $(k - 2)$ -nd Catalan number

$$C_{k-2} = \frac{1}{k-1} \binom{2(k-2)}{k-2} \leq 4^{k-3}.$$

CHORDAL COMPLETION – *more efficiently*

Algorithm:

- ⑥ Find an induced cycle C of length at least 4 (can be done in polynomial time).
- ⑥ If no such cycle exists \Rightarrow **Done!**
- ⑥ If C has more than $k + 3$ vertices \Rightarrow **No solution!**
- ⑥ Otherwise, one of the $\leq 4^{|C|-3}$ triangulations has to be in the solution \Rightarrow **Branch!**

Claim: Search tree has at most $T_k = 4^k$ leaves.

Proof: By induction. Number of leaves is at most

$$T_k \leq 4^{|C|-3} \cdot T_{k-(|C|-3)} \leq 4^{|C|-3} \cdot 4^{k-(|C|-3)} = 4^k.$$

Iterative compression



Iterative compression

- ⑥ A surprising small, but very powerful trick.
- ⑥ Most useful for deletion problems: delete k things to achieve some property.
- ⑥ Demonstration: ODD CYCLE TRANSVERSAL aka BIPARTITE DELETION aka GRAPH BIPARTIZATION: Given a graph G and an integer k , delete k vertices to make the graph bipartite.
- ⑥ Forbidden induced subgraphs: odd cycles. There is no bound on the size of odd cycles.

BIPARTITE DELETION

Solution based on iterative compression:

⑥ Step 1:

Solve the **annotated problem** for bipartite graphs:

Given a **bipartite** graph G , two sets $B, W \subseteq V(G)$, and an integer k , find a set S of at most k vertices such that $G \setminus S$ has a 2-coloring where $B \setminus S$ is black and $W \setminus S$ is white.

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⑥ Step 2:

Solve the **compression problem** for general graphs:

Given a graph G , an integer k , and **a set S' of $k + 1$ vertices such that $G \setminus S'$ is bipartite**, find a set S of k vertices such that $G \setminus S$ is bipartite.

BIPARTITE DELETION

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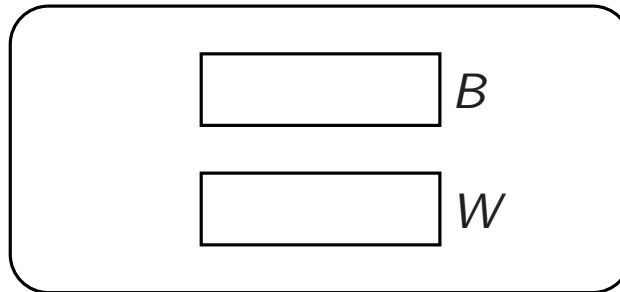
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⑥ Step 3:

Apply the magic of iterative compression...

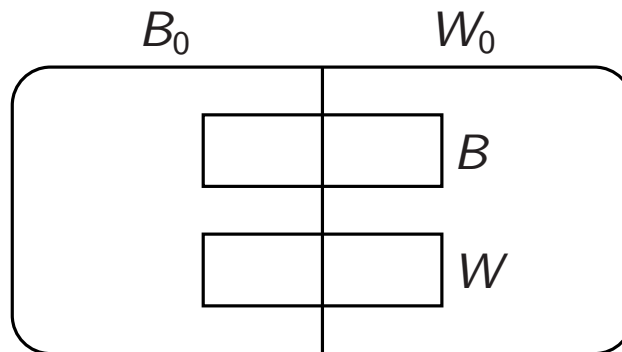
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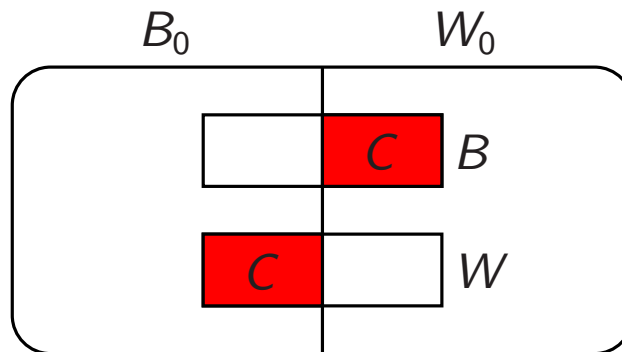
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Find an arbitrary 2-coloring (B_0, W_0) of G .

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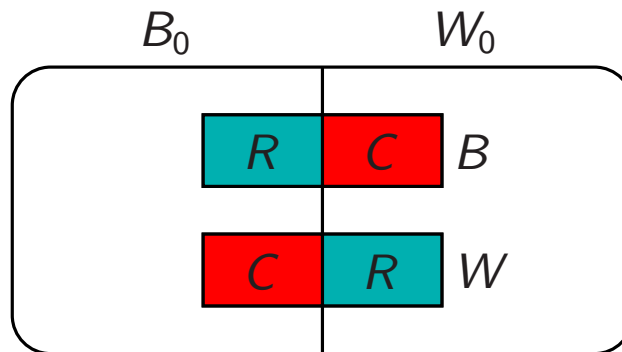


Find an arbitrary 2-coloring (B_0, W_0) of G .

$C := (B_0 \cap W) \cup (W_0 \cap B)$ should change color, while $R := (B_0 \cap B) \cup (W_0 \cap W)$ should remain the same color.

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Lemma: $G \setminus S$ has the required 2-coloring if and only if S separates C and R , i.e., no component of $G \setminus S$ contains vertices from both $C \setminus S$ and $R \setminus S$.

Step 1: The annotated problem

Lemma: $G \setminus S$ has the required 2-coloring if and only if S separates C and R , i.e., no component of $G \setminus S$ contains vertices from both $C \setminus S$ and $R \setminus S$.

Proof:

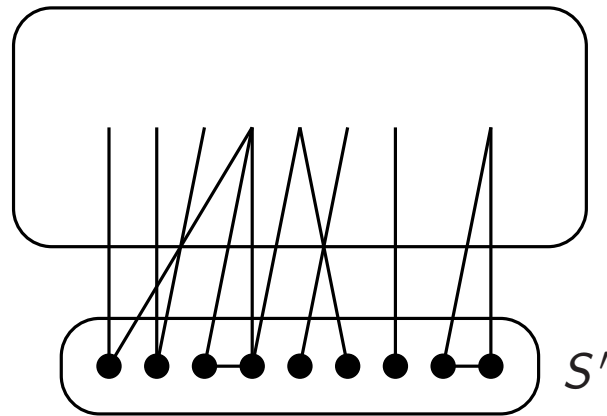
\Rightarrow In a 2-coloring of $G \setminus S$, each vertex either remained the same color or changed color. Adjacent vertices do the same, thus every component either changed or remained.

\Leftarrow Flip the coloring of those components of $G \setminus S$ that contain vertices from $C \setminus S$. No vertex of R is flipped.

Algorithm: Using max-flow min-cut techniques, we can check if there is a set S that separates C and R . It can be done in time $O(k|E(G)|)$ using k iterations of the Ford-Fulkerson algorithm.

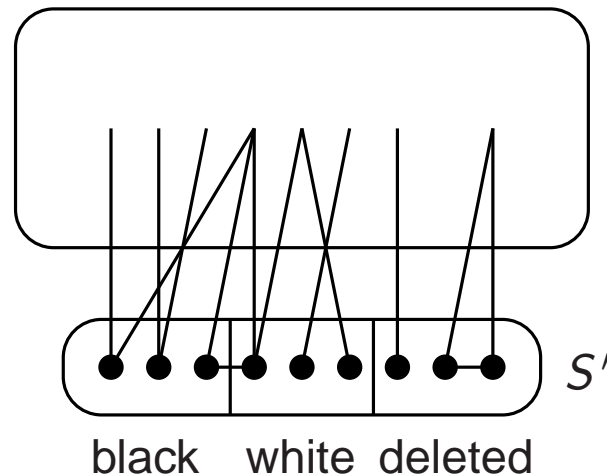
Step 2: The compression problem

Given a graph G , an integer k , and a set S' of $k + 1$ vertices such that $G \setminus S'$ is bipartite, find a set S of k vertices such that $G \setminus S$ is bipartite.



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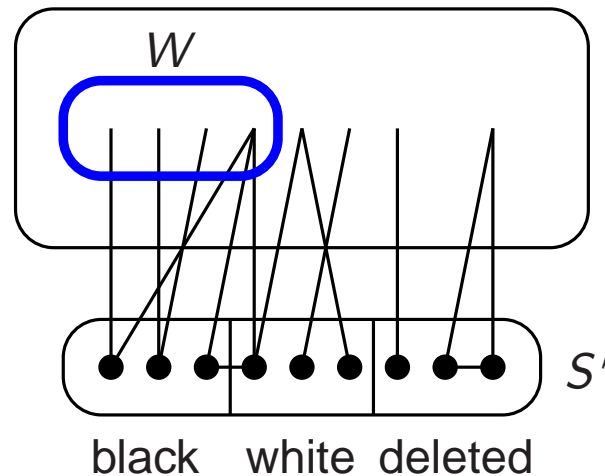
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Branch into 3^{k+1} cases: each vertex of S' is either black, white, or deleted.
Trivial check: no edge between two black or two white vertices.

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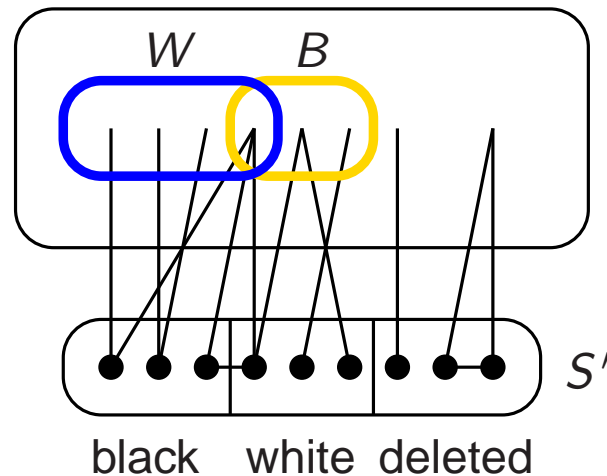
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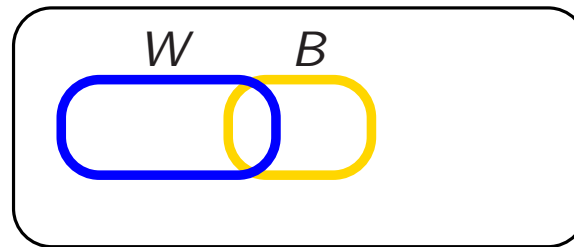
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The vertices of S' can be disregarded. Thus we need to solve the annotated problem on the bipartite graph $G \setminus S'$.

Running time: $O(3^k \cdot k|E(G)|)$ time.

Step 3: Iterative compression

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Let $V(G) = \{v_1, \dots, v_n\}$ and let G_i be the graph induced by $\{v_1, \dots, v_i\}$.

For every i , we find a set S_i of size k such that $G_i \setminus S_i$ is bipartite.

- ⑥ For G_k , the set $S_k = \{v_1, \dots, v_k\}$ is a trivial solution.
- ⑥ If S_{i-1} is known, then $S_{i-1} \cup \{v_i\}$ is a set of size $k + 1$ whose deletion makes G_i bipartite \Rightarrow We can use the compression algorithm to find a suitable S_i in time $O(3^k \cdot k|E(G_i)|)$.

Step 3: Iterative Compression

Bipartite-Deletion(G, k)

1. $S_k = \{v_1, \dots, v_k\}$
2. for $i := k + 1$ to n
3. Invariant: $G_{i-1} \setminus S_{i-1}$ is bipartite.
4. Call Compression($G_i, S_{i-1} \cup \{v_i\}$)
5. If the answer is “NO” \Rightarrow return “NO”
6. If the answer is a set $X \Rightarrow S_i := X$
7. Return the set S_n

Running time: the compression algorithm is called n times and everything else can be done in linear time

$\Rightarrow O(3^k \cdot k|V(G)| \cdot |E(G)|)$ time algorithm.

Graph Minors



Neil Robertson



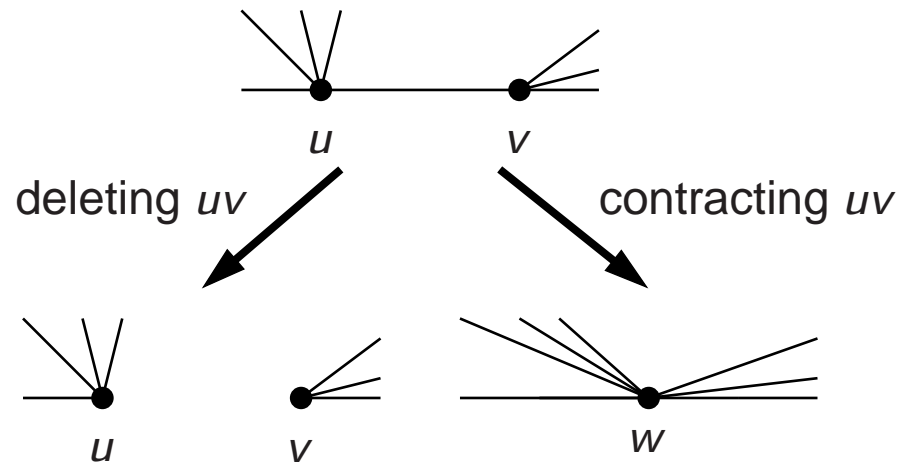
Paul Seymour

Graph Minors

- ⑥ Some consequences of the Graph Minors Theorem give a quick way of showing that certain problems are FPT.
- ⑥ However, the function $f(k)$ in the resulting FPT algorithms can be HUGE, completely impractical.
- ⑥ History: motivation for FPT.
- ⑥ Parts and ingredients of the theory are useful for algorithm design.
- ⑥ New algorithmic results are still being developed.

Graph Minors

Definition: Graph H is a **minor** G ($H \leq G$) if H can be obtained from G by deleting edges, deleting vertices, and contracting edges.

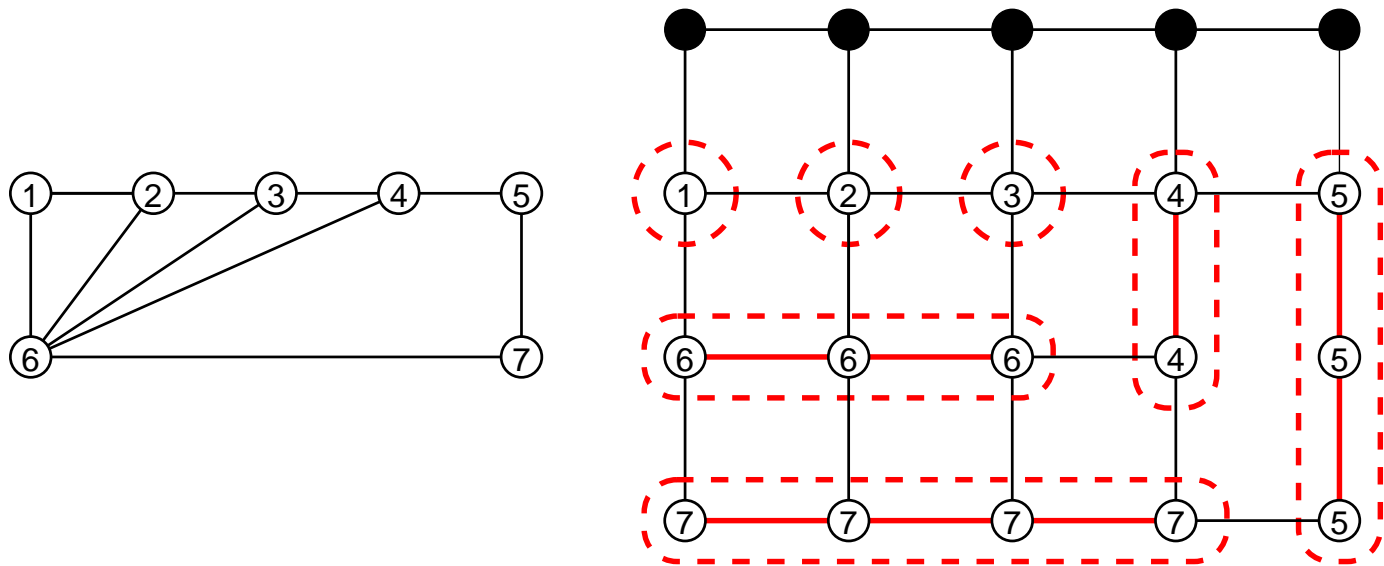


Example: A triangle is a minor of a graph G if and only if G has a cycle (i.e., it is not a forest).

Graph minors

Equivalent definition: Graph H is a **minor** of G if there is a mapping ϕ that maps each vertex of H to a connected subset of G such that

- ⑥ $\phi(u)$ and $\phi(v)$ are disjoint if $u \neq v$, and
- ⑥ if $uv \in E(G)$, then there is an edge between $\phi(u)$ and $\phi(v)$.



Minor closed properties

Definition: A set \mathcal{G} of graphs is **minor closed** if whenever $G \in \mathcal{G}$ and $H \leq G$, then $H \in \mathcal{G}$ as well.

Examples of minor closed properties:

- planar graphs
- acyclic graphs (forests)
- graphs having no cycle longer than k
- empty graphs

Examples of **not** minor closed properties:

- complete graphs
- regular graphs
- bipartite graphs

Forbidden minors

Let \mathcal{G} be a minor closed set and let \mathcal{F} be the set of “minimal bad graphs”: $H \in \mathcal{F}$ if $H \notin \mathcal{G}$, but every proper minor of H is in \mathcal{G} .

Characterization by forbidden minors:

$$G \in \mathcal{G} \iff \forall H \in \mathcal{F}, H \not\preceq G$$

The set \mathcal{F} is the **obstruction set** of property \mathcal{G} .

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The set \mathcal{F} is the **obstruction set** of property \mathcal{G} .

Theorem: [Wagner] A graph is planar if and only if it does not have a K_5 or $K_{3,3}$ minor.

In other words: the obstruction set of planarity is $\mathcal{F} = \{K_5, K_{3,3}\}$.

Does every minor closed property have such a finite characterization?

Graph Minors Theorem

Theorem: [Robertson and Seymour] Every minor closed property \mathcal{G} has a finite obstruction set.

Note: The proof is contained in the paper series “Graph Minors I–XX”.

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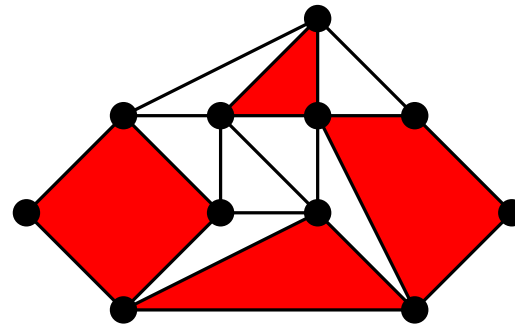
Note: The size of the obstruction set can be astronomical even for simple properties.

Theorem: [Robertson and Seymour] For every fixed graph H , there is an $O(n^3)$ time algorithm for testing whether H is a minor of the given graph G .

Corollary: For every minor closed property \mathcal{G} , there is an $O(n^3)$ time algorithm for testing whether a given graph G is in \mathcal{G} .

Applications

PLANAR FACE COVER: Given a graph G and an integer k , find an embedding of planar graph G such that there are k faces that cover all the vertices.



One line argument:

For every fixed k , the class \mathcal{G}_k of graphs of yes-instances is minor closed.

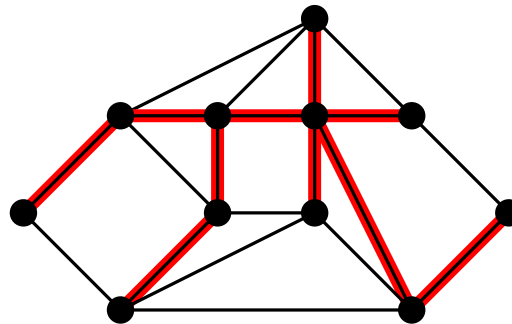


For every fixed k , there is a $O(n^3)$ time algorithm for PLANAR FACE COVER.

Note: non-uniform FPT.

Applications

k -LEAF SPANNING TREE: Given a graph G and an integer k , find a spanning tree with **at least** k leaves.



Technical modification: Is there such a spanning tree for at least one component of G ?

One line argument:

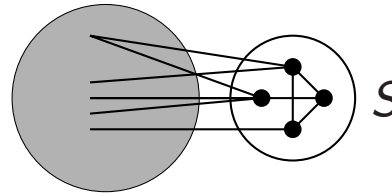
For every fixed k , the class \mathcal{G}_k of no-instances is minor closed.



For every fixed k , k -LEAF SPANNING TREE can be solved in time $O(n^3)$.

$\mathcal{G} + k$ vertices

Let \mathcal{G} be a graph property, and let $\mathcal{G} + kv$ contain graph G if there is a set $S \subseteq V(G)$ of k vertices such that $G \setminus S \in \mathcal{G}$.

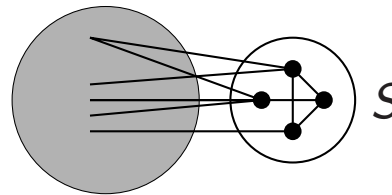


Lemma: If \mathcal{G} is minor closed, then $\mathcal{G} + kv$ is minor closed for every fixed k .

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- ⑥ If $\mathcal{G} = \text{forests}$ $\Rightarrow \mathcal{G} + kv = \text{graphs that can be made acyclic by the deletion of } k \text{ vertices}$ \Rightarrow FEEDBACK VERTEX SET is FPT.
- ⑥ If $\mathcal{G} = \text{planar graphs}$ $\Rightarrow \mathcal{G} + kv = \text{graphs that can be made planar by the deletion of } k \text{ vertices (} k\text{-apex graphs)}$ $\Rightarrow k\text{-APEX GRAPH is FPT.}$
- ⑥ If $\mathcal{G} = \text{empty graphs}$ $\Rightarrow \mathcal{G} + kv = \text{graphs with vertex cover number at most } k$ \Rightarrow VERTEX COVER is FPT.

Color coding



Color coding

- ⑥ Works best when we need to ensure that a small number of “things” are disjoint.
- ⑥ We demonstrate it on the problem of finding an s - t path of length **exactly** k .
- ⑥ Randomized algorithm, but can be derandomized using a standard technique.
- ⑥ Very robust technique, we can use it as an “opening step” when investigating a new problem.

k -PATH

Task: Given a graph G , an integer k , two vertices s, t , find a **simple** s - t path with exactly k internal vertices.

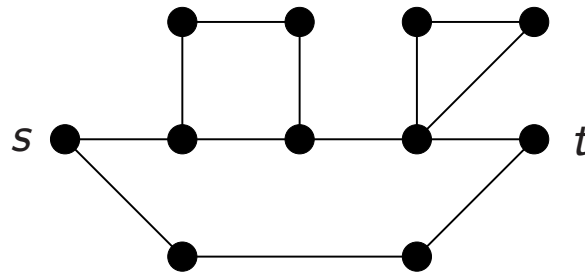
Note: Finding such a **walk** can be done easily in polynomial time.

Note: The problem is clearly NP-hard, as it contains the s - t HAMILTONIAN PATH problem.

The k -PATH algorithm can be used to check if there is a cycle of length exactly k in the graph.

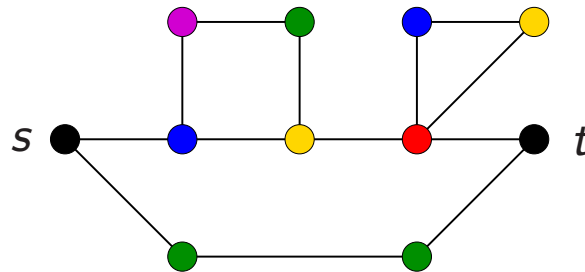
k -PATH

- ⑥ Assign colors from $[k]$ to vertices $V(G) \setminus \{s, t\}$ uniformly and independently at random.



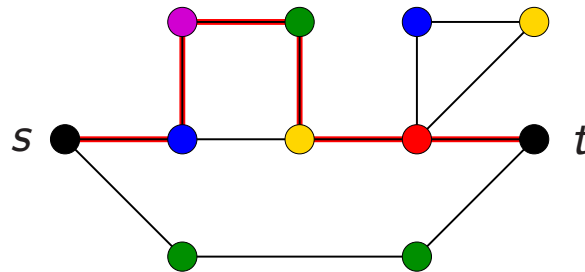
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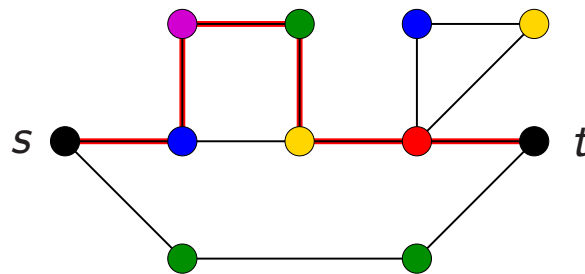
- Assign colors from $[k]$ to vertices $V(G) \setminus \{s, t\}$ uniformly and independently at random.



- Check if there is a **colorful** s - t path: a path where each color appears exactly once on the internal vertices; output “YES” or “NO”.

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- Check if there is a **colorful** s - t path: a path where each color appears exactly once on the internal vertices; output “YES” or “NO”.
 - △ If there is no s - t k -path: no such colorful path exists \Rightarrow “NO”.
 - △ If there is an s - t k -path: the probability that such a path is colorful is

$$\frac{k!}{k^k} > \frac{\left(\frac{k}{e}\right)^k}{k^k} = e^{-k},$$

thus the algorithm outputs “YES” with at least that probability.

Error probability

- ⑥ **Useful fact:** If the probability of success is at least p , then the probability that the algorithm **does not** say “YES” after $1/p$ repetitions is at most

$$(1 - p)^{1/p} < (e^{-p})^{1/p} = 1/e \approx 0.38$$

- ⑥ Thus if $p > e^{-k}$, then error probability is at most $1/e$ after e^k repetitions.
- ⑥ Repeating the whole algorithm a constant number of times can make the error probability an arbitrary small constant.
- ⑥ For example, by trying $100 \cdot e^k$ random colorings, the probability of a wrong answer is at most $1/e^{100}$.

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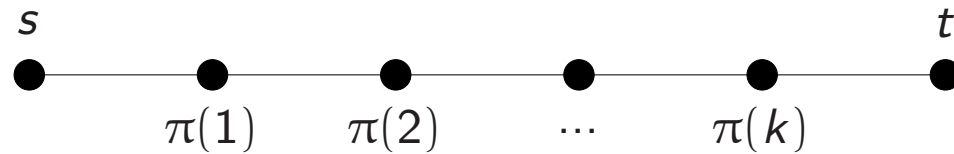
It remains to see how a colorful s - t path can be found.

Method 1: Trying all permutations.

Method 2: Dynamic programming.

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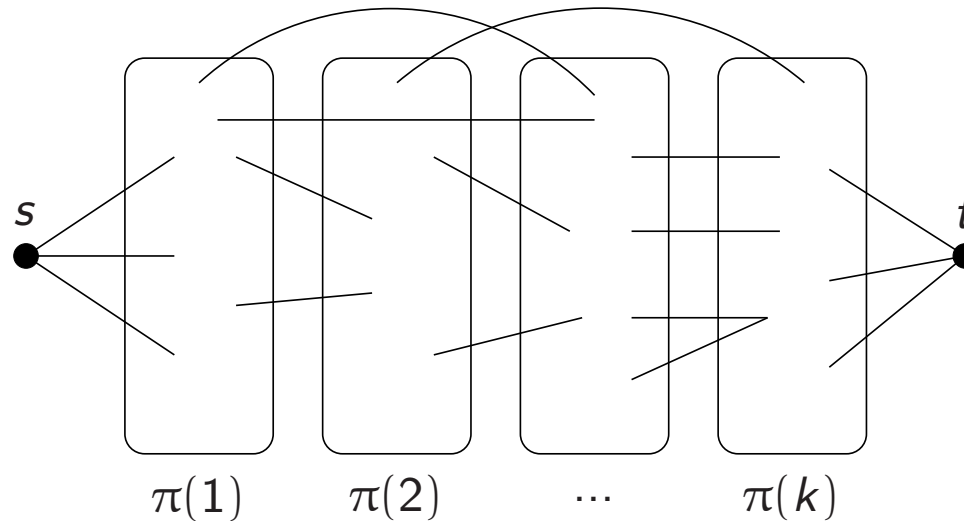
The colors encountered on a colorful s - t path form a permutation π of $\{1, 2, \dots, k\}$:



We try all possible $k!$ permutations. For a fixed π , it is easy to check if there is a path with this order of colors.

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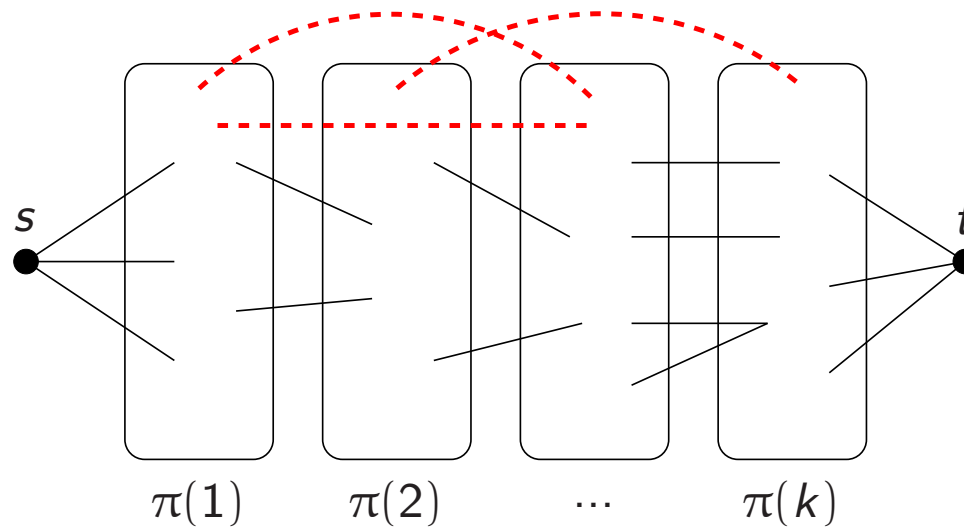
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- ⑥ Edges connecting nonadjacent color classes are removed.
- ⑥ The remaining edges are directed.
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- ⑥ Running time is $O(k! \cdot |E(G)|)$.

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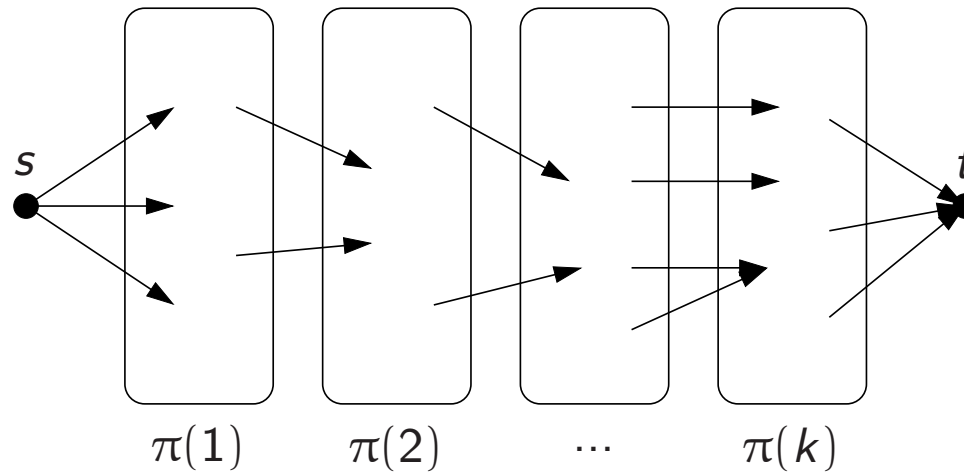
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We introduce $2^k \cdot |V(G)|$ Boolean variables:

$x(v, C) = \text{TRUE}$ for some $v \in V(G)$ and $C \subseteq [k]$



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If we know every $x(v, C)$ with $|C| = i$, then we can determine every $x(v, C)$ with $|C| = i + 1 \Rightarrow$ All the values can be determined in time $O(2^k \cdot |E(G)|)$.

There is a colorful s - t path $\iff x(v, [k]) = \text{TRUE}$ for some neighbor of t .

Derandomization

Using Method 2, we obtain a $O^*((2e)^k)$ time algorithm with constant error probability. How to make it deterministic?

Definition: A family \mathcal{H} of functions $[n] \rightarrow [k]$ is a **k -perfect** family of hash functions if for every $S \subseteq [n]$ with $|S| = k$, there is a $h \in \mathcal{H}$ such that $h(x) \neq h(y)$ for any $x, y \in S, x \neq y$.

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Instead of trying $O(e^k)$ random colorings, we go through a k -perfect family \mathcal{H} of functions $V(G) \rightarrow [k]$. If there is a solution \Rightarrow The internal vertices S are colorful for at least one $h \in \mathcal{H} \Rightarrow$ Algorithm outputs “YES”.

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Theorem: There is a k -perfect family of functions $[n] \rightarrow [k]$ having size $2^{O(k)} \log n$ (and can be constructed in time polynomial in the size of the family).

\Rightarrow There is a **deterministic** $2^{O(k)} \cdot n^{O(1)}$ time algorithm for the k -PATH problem.

Cut problems

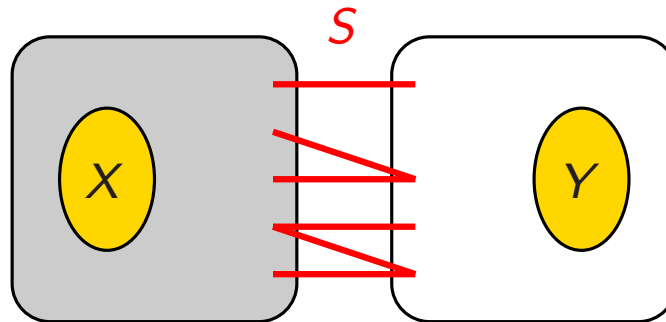


MULTIWAY CUT

Task: Given a graph G , a set T of vertices, and an integer k , find a set S of at most k edges that separates T (each component of $G \setminus S$ contains at most one vertex of T).

Polynomial for $|T| = 2$, but NP-hard for $|T| = 3$.

Theorem: MULTIWAY CUT is FPT parameterized by k .



$\delta(R)$: set of edges leaving R

$\lambda(X, Y)$: minimum number of edges in an (X, Y) -separator

Submodularity

Fact: The function δ is **submodular**: for arbitrary sets A, B ,

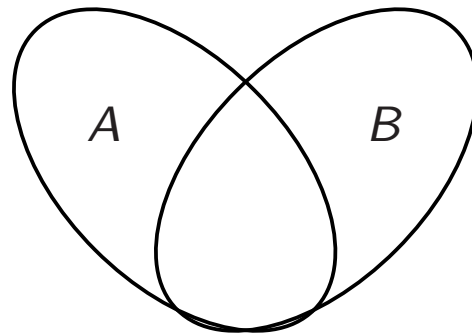
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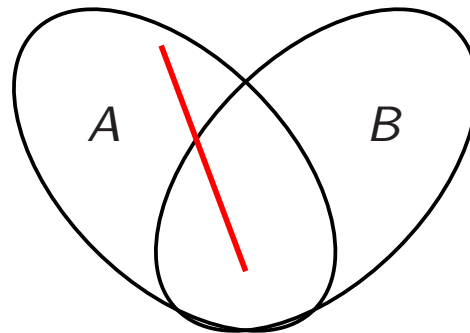
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0 1 1 0

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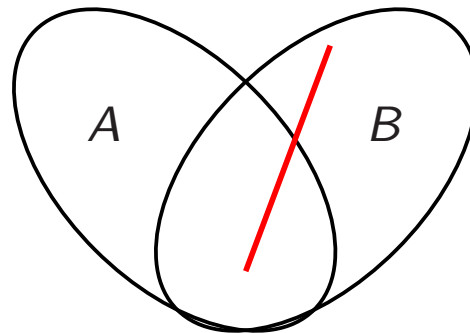


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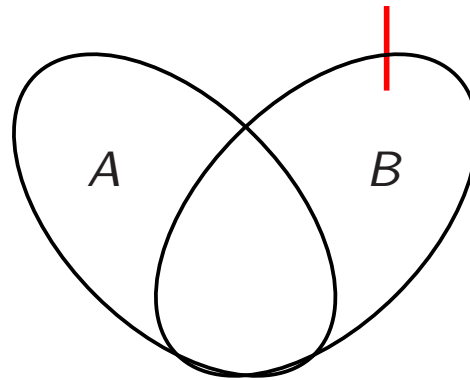


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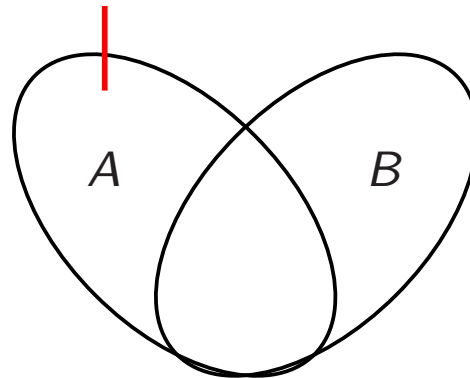


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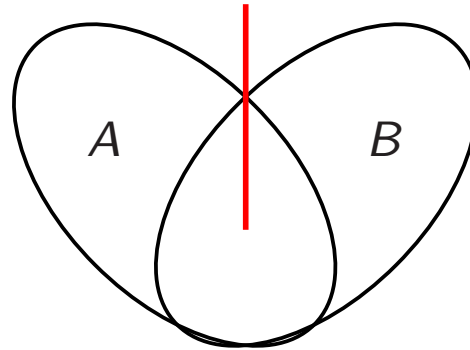


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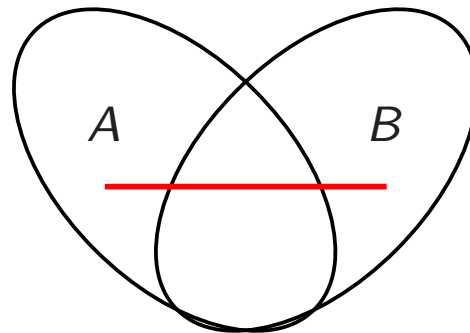


Submodularity

Fact: The function δ is **submodular**: for arbitrary sets A, B ,

$$\underbrace{|\delta(A)|}_1 + \underbrace{|\delta(B)|}_1 \geq \underbrace{|\delta(A \cap B)|}_0 + \underbrace{|\delta(A \cup B)|}_0$$

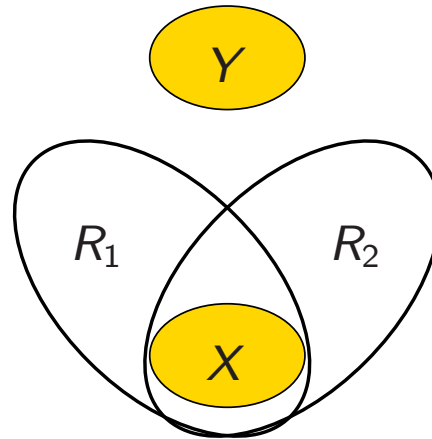
Proof: Determine separately the contribution of the different types of edges.



Submodularity

Consequence: There is a unique maximal $R_{\max} \supseteq X$ such that $\delta(R_{\max})$ is an (X, Y) -separator of size $\lambda(X, Y)$.

Proof: Let $R_1, R_2 \supseteq X$ be two sets such that $\delta(R_1), \delta(R_2)$ are (X, Y) -separators of size $\lambda := \lambda(X, Y)$.



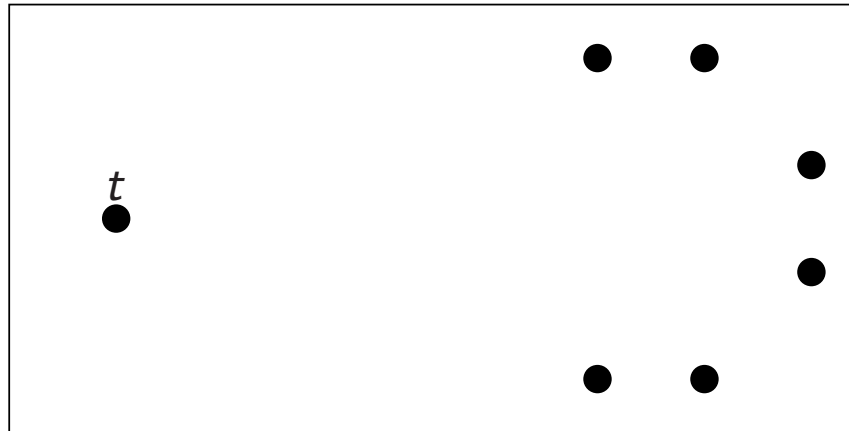
$$\begin{array}{ccccccc} |\delta(R_1)| & + & |\delta(R_2)| & \geq & |\delta(R_1 \cap R_2)| & + & |\delta(R_1 \cup R_2)| \\ \lambda & & \lambda & & \geq \lambda & & \end{array}$$

$$\Rightarrow |\delta(R_1 \cup R_2)| \leq \lambda$$

Note: Analogous result holds for a unique minimal R_{\min} .

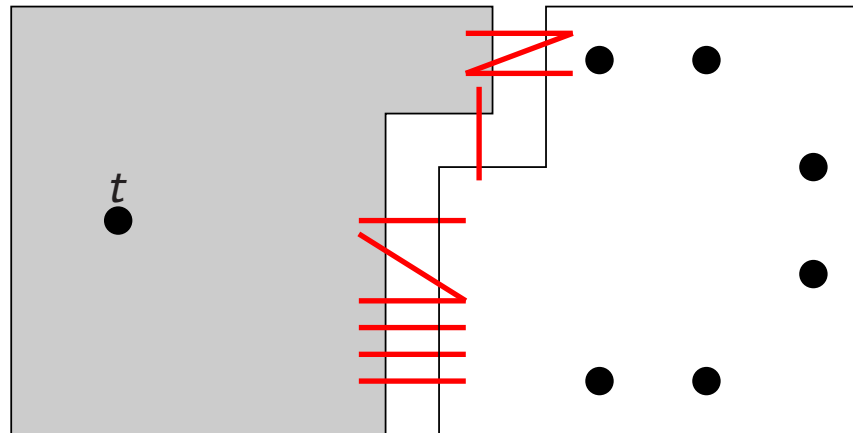
MULTIWAY CUT

Intuition: Consider a $t \in T$. A subset of the solution separates t and $T \setminus t$.



MULTIWAY CUT

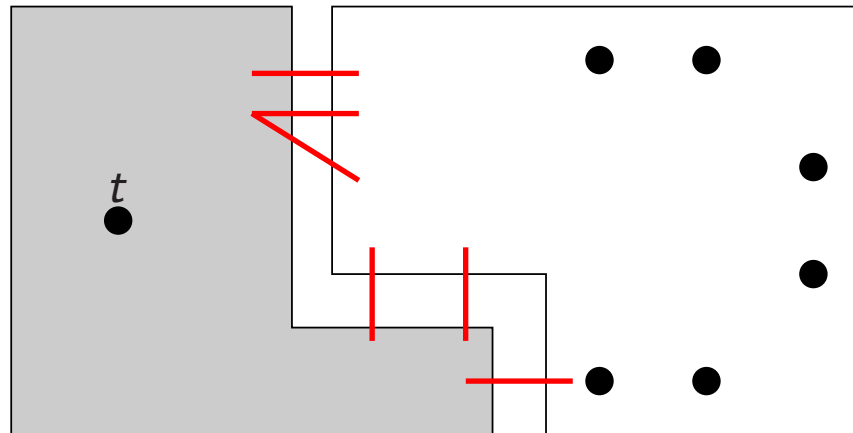
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There are many such separators.

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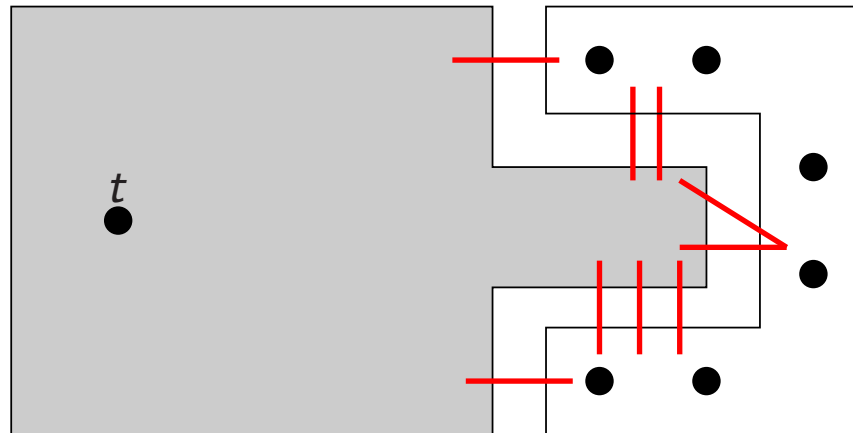
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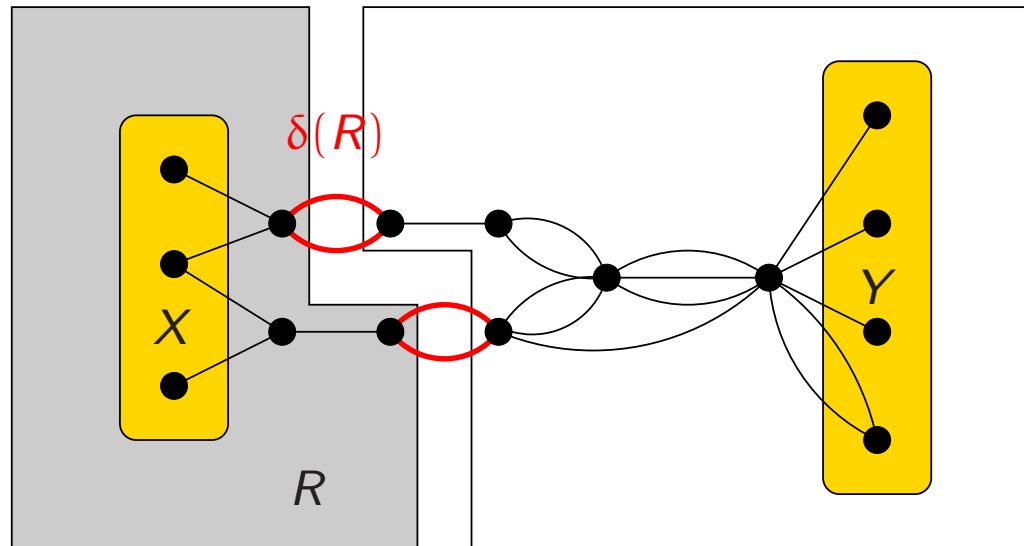


There are many such separators.

But a separator farther from t and closer to $T \setminus t$ seems to be more useful.

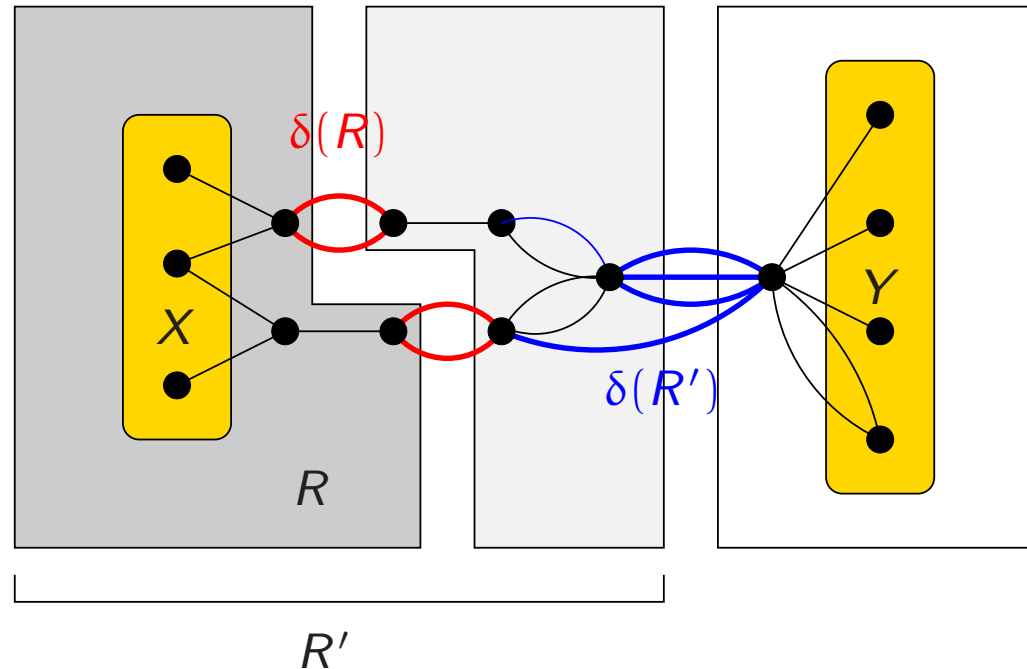
Important separators

Definition: An (X, Y) -separator $\delta(R)$ is **important** if there is no (X, Y) -separator $\delta(R')$ with $R \subset R'$ and $|\delta(R')| \leq |\delta(R)|$.



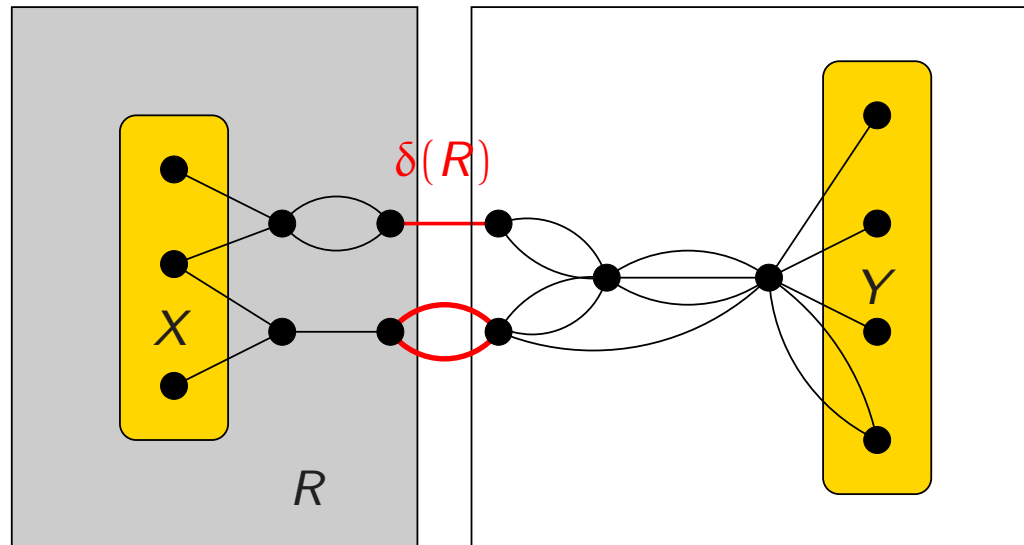
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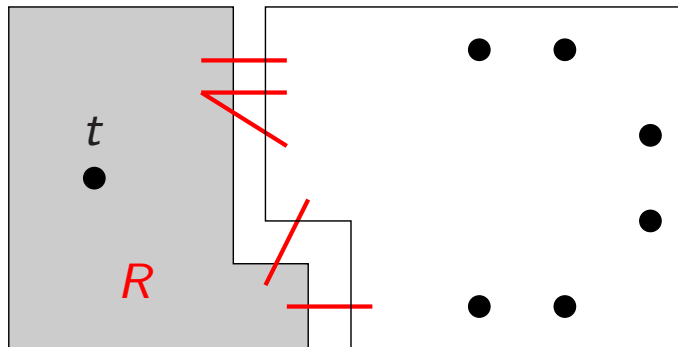
MULTIWAY CUT *and important separators*

Lemma: Let $t \in T$. The MULTIWAY CUT problem has a solution S that contains an important $(t, T \setminus t)$ -separator.

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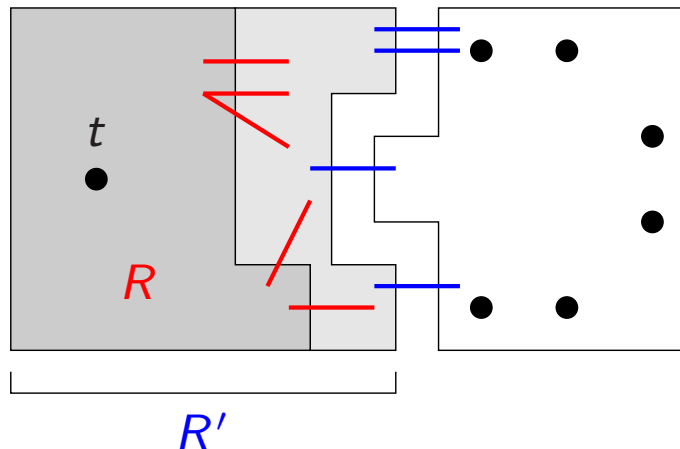
Proof: Let R be the vertices reachable from t in $G \setminus S$ for a solution S .



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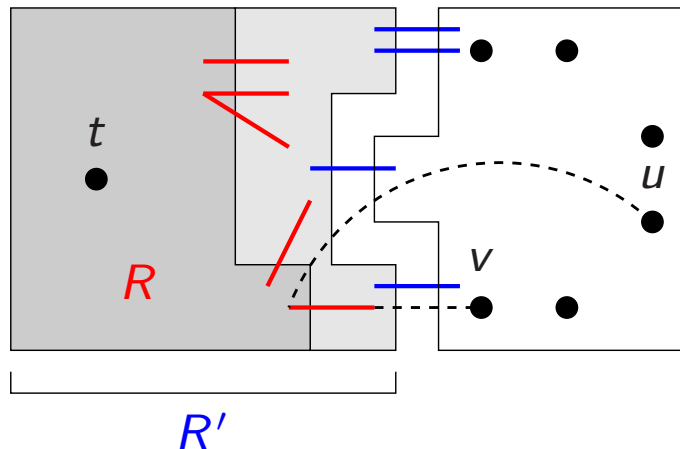


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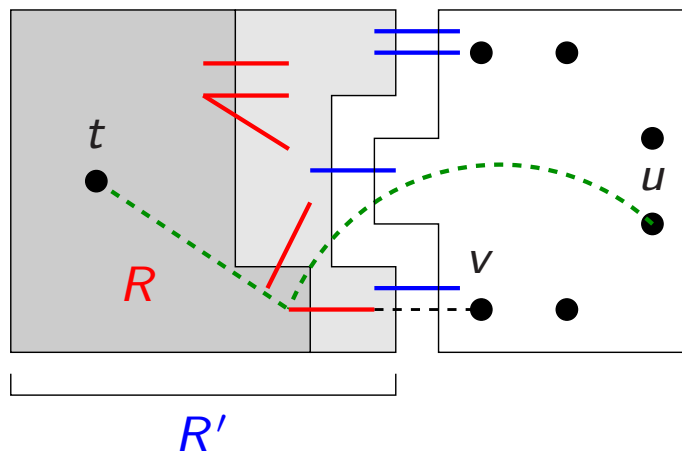
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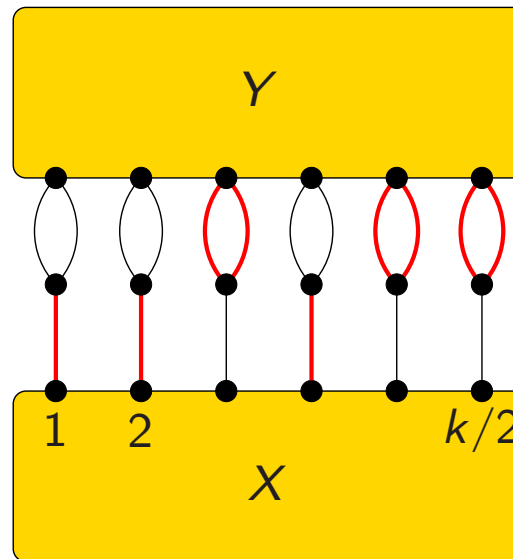
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Important separators

Lemma: There are at most 4^k important (X, Y) -separators of size at most k .

Example:



There are exactly $2^{k/2}$ important (X, Y) -separators of size at most k in this graph.

Important separators

Lemma: There are at most 4^k important (X, Y) -separators of size at most k .

Proof: First we show that $R_{\max} \subseteq R$ for every important separator $\delta(R)$.

$$\begin{array}{ccccccc} |\delta(R_{\max})| & + & |\delta(R)| & \geq & |\delta(R_{\max} \cap R)| & + & |\delta(R_{\max} \cup R)| \\ \lambda & & & & \geq \lambda & & \end{array}$$

\Downarrow

$$|\delta(R_{\max} \cup R)| \leq |\delta(R)|$$

\Downarrow

If $R \neq R_{\max} \cup R$, then $\delta(R)$ is not important.

Thus the important (X, Y) - and (R_{\max}, Y) -separators are the same.

\Rightarrow We can assume $X = R_{\max}$.

Important separators

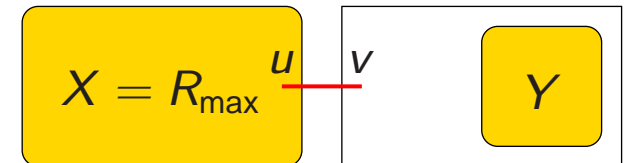
Lemma: There are at most 4^k important (X, Y) -separators of size at most k .

Search tree algorithm for finding all these separators:

An (arbitrary) edge uv leaving $X = R_{\max}$ is either in the separator or not.

Branch 1: If $uv \in S$, then $S \setminus uv$ is an important (X, Y) -separator of size at most $k - 1$ in $G \setminus uv$.

Branch 2: If $uv \notin S$, then S is an important $(X \cup v, Y)$ -separator of size at most k in G .



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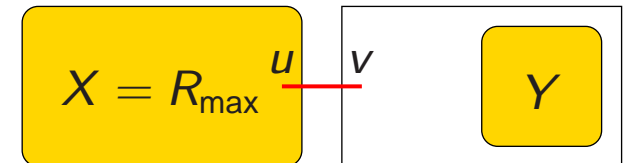
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$\Rightarrow k$ decreases by one, λ decreases by at most 1.

Branch 2: If $uv \notin S$, then S is an important $(X \cup v, Y)$ -separator of size at most k in G .

$\Rightarrow k$ remains the same, λ increases by 1.



The measure $2k - \lambda$ decreases in each step.

\Rightarrow Height of the search tree $\leq 2k \Rightarrow \leq 2^{2k} = 4^k$ important separators.

Algorithm for MULTIWAY CUT

1. If every vertex of T is in a different component, then we are done.
2. Let $t \in T$ be a vertex with that is not separated from every $T \setminus t$.
3. Branch on a choice of an important $(t, T \setminus t)$ separator S of size at most k .
4. Set $G := G \setminus S$ and $k := k - |S|$.
5. Go to step 1.

We branch into at most 4^k directions at most k times.

Algorithm for MULTIWAY CUT

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Better estimate of the search tree size:

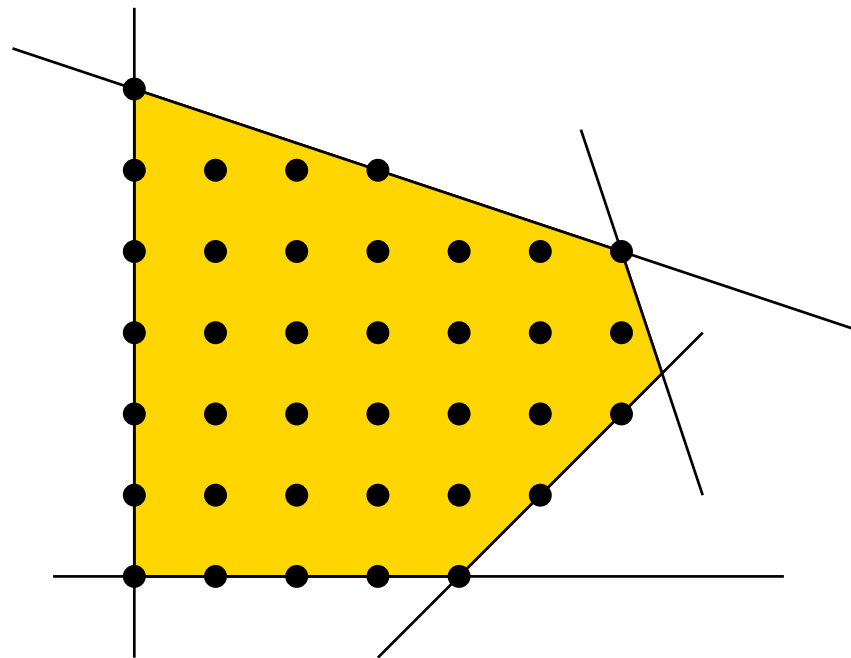
- ⑥ When choosing the important separator, $2k - \lambda$ decreases at each branching, until λ reaches 0.
- ⑥ When choosing the next vertex t , λ changes from 0 to positive, thus $2k - \lambda$ does not increase.

Size of the search tree is at most $2^{2k} = 4^k$.

Other separation problems

- ⑥ Some other variants:
 - △ $|T|$ as a parameter
 - △ MULTITERMINAL CUT: pairs $(s_1, t_1), \dots, (s_\ell, t_\ell)$ have to be separated.
 - △ Directed graphs
 - △ Planar graphs
- ⑥ Useful for deletion-type problems such as DIRECTED FEEDBACK VERTEX SET (via iterative compression).
- ⑥ Important separators: is it relevant for a given problem?

Integer Linear Programming



Integer Linear Programming

Linear Programming (LP): important tool in (continuous) combinatorial optimization. Sometimes very useful for discrete problems as well.

$$\max c_1x_1 + c_2x_2 + c_3x_3$$

s.t.

$$x_1 + 5x_2 - x_3 \leq 8$$

$$2x_1 - x_3 \leq 0$$

$$3x_2 + 10x_3 \leq 10$$

$$x_1, x_2, x_3 \in \mathbb{R}$$

Fact: It can be decided if there is a solution (feasibility) and an optimum solution can be found in polynomial time.

Integer Linear Programming

Integer Linear Programming (ILP): Same as LP, but we require that every x_i is integer.

Very powerful, able to model many NP-hard problems. (Of course, no polynomial-time algorithm is known.)

Theorem: ILP with p variables can be solved in time $p^{O(p)} \cdot n^{O(1)}$.

CLOSEST STRING

Task: Given strings s_1, \dots, s_k of length L over alphabet Σ , and an integer d , find a string s (of length L) such that $d(s, s_i) \leq d$ for every $1 \leq i \leq k$.

Note: $d(s, s_i)$ is the Hamming distance.

Theorem: CLOSEST STRING parameterized by k is FPT.

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Theorem: CLOSEST STRING is NP-hard for $\Sigma = \{0, 1\}$.

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CLOSEST STRING

An instance with $k = 5$ and a solution for $d = 4$:

s_1	CBDCCACBB
s_2	ABDBCABDB
s_3	CDDBACCB
s_4	DDABACCB
s_5	ACDBDDCBC
<hr/>	
	ADDBCACBD

Each column can be described by a partition \mathcal{P} of $[k]$.

The instance can be described by an integer $c_{\mathcal{P}}$ for each partition \mathcal{P} : the number of columns with this type.

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s_1	C B D C C A C B B
s_2	A B D B C A B D B
s_3	C D D B A C C B D
s_4	D D A B A C C B D
s_5	A C D B D D C B C
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Describing a solution: If C is a class of \mathcal{P} , let $x_{\mathcal{P},C}$ be the number of type \mathcal{P} columns where the solution agrees with class C .

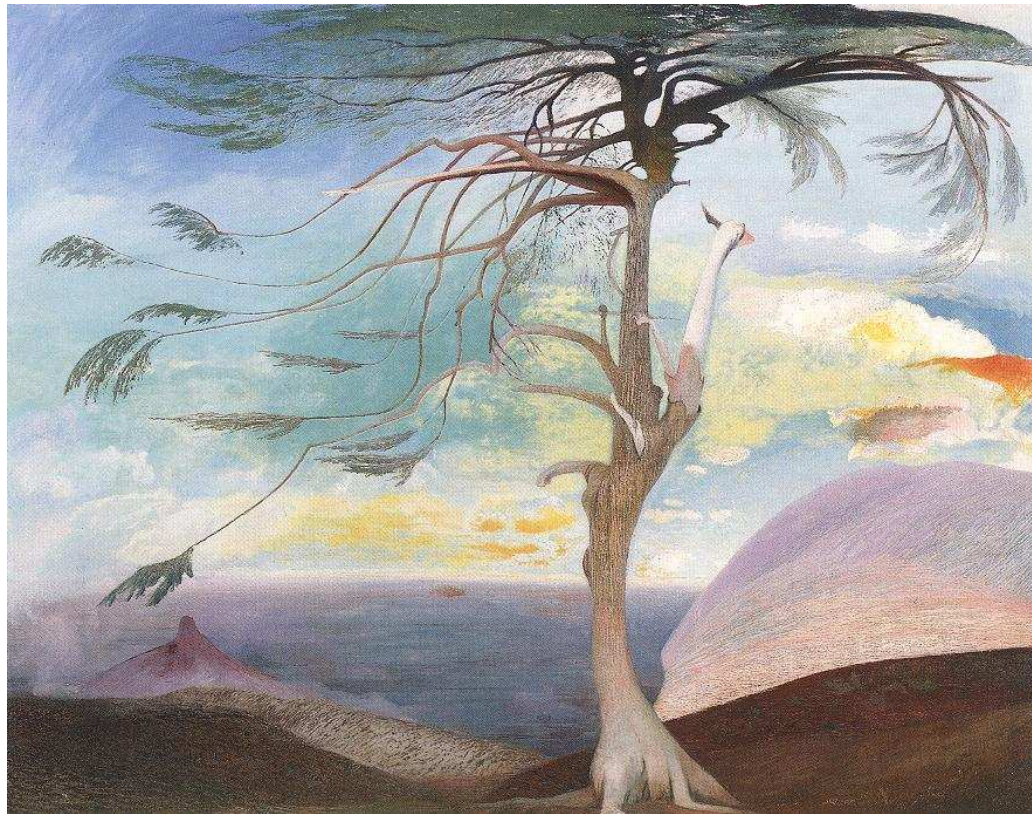
There is a solution iff the following ILP has a feasible solution:

$$\begin{aligned} \sum_{C \in \mathcal{P}} x_{\mathcal{P},C} &\leq c_{\mathcal{P}} && \forall \text{partition } \mathcal{P} \\ \sum_{i \notin C, C \in \mathcal{P}} x_{\mathcal{P},C} &\leq d && \forall 1 \leq i \leq k \\ x_{\mathcal{P},C} &\geq 0 && \forall \mathcal{P}, C \end{aligned}$$

Number of variables is $\leq B(k) \cdot k$, where $B(k)$ is the no. of partitions of $[k]$

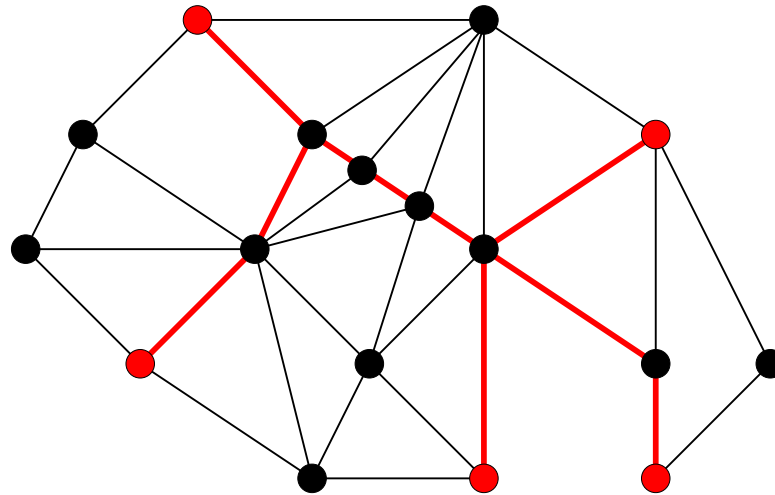
\Rightarrow The ILP algorithm solves the problem in time $f(k) \cdot n^{O(1)}$.

STEINER TREE



STEINER TREE

Task: Given a graph G with weighted edges and a set S of k vertices, find a tree T of minimum weight that contains S .



Known to be NP-hard. For fixed k , we can solve it in polynomial time: we can guess the Steiner points and the way they are connected.

Theorem: STEINER TREE is FPT parameterized by $k = |S|$.

STEINER TREE

Solution by dynamic programming. For $v \in V(G)$ and $X \subseteq S$,

$c(v, X) :=$ minimum cost of a Steiner tree of X that contains v

$d(u, v) :=$ distance of u and v

Recurrence relation:

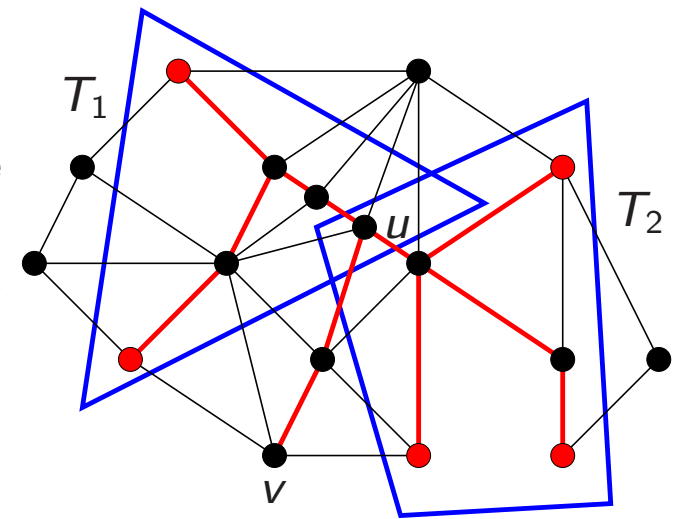
$$c(v, X) = \min_{\substack{u \in V(G) \\ \emptyset \subset X' \subset X}} c(u, X' \setminus u) + c(u, (X \setminus X') \setminus u) + d(u, v)$$

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- ⑥ \leq : A tree T_1 realizing $c(u, X' \setminus u)$, a tree T_2 realizing $c(u, (X \setminus X') \setminus u)$, and the path uv gives a (superset of a) Steiner tree of X containing v .

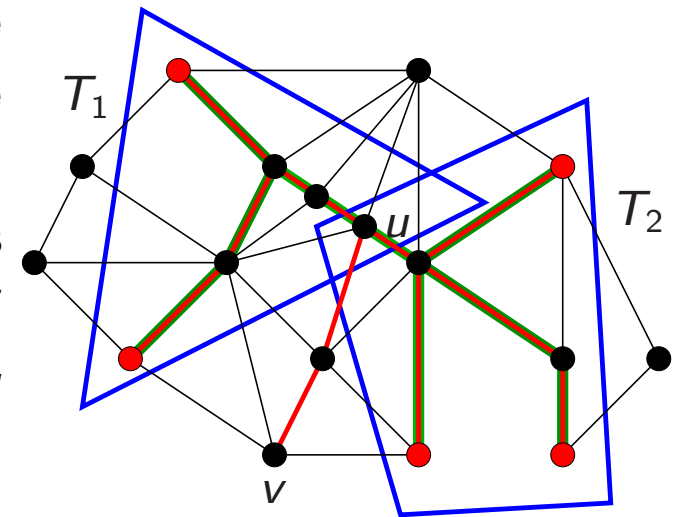


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- ⑥ \geq : Suppose T realizes $c(v, X)$, let T' be the minimum subtree containing X . Let u be a vertex of T' closest to v . If $|X| > 1$, then there is a component C of $T \setminus u$ that contains a subset $\emptyset \subset X' \subset X$ of terminals. Thus T is the disjoint union of a tree containing $X' \setminus u$ and u , a tree containing $(X \setminus X') \setminus u$ and u , and the path uv .



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Running time:

$2^k |V(G)|$ variables $c(v, X)$, determine them in increasing order of $|X|$. Variable $c(v, X)$ can be determined by considering $2^{|X|}$ cases. Total number of cases to consider:

$$\sum_{X \subseteq T} 2^{|X|} = \sum_{i=1}^k \binom{k}{i} 2^i \leq (1 + 2)^k = 3^k.$$

Running time is $O^*(3^k)$.

Note: Running time can be reduced to $O^*(2^k)$ with clever techniques.

Conclusions

- ⑥ Many nice techniques invented so far — and probably many more to come.
- ⑥ A single technique might provide the key for several problems.
- ⑥ How to find new techniques? By attacking the open problems!
- ⑥ Next (January):
 - △ Treewidth
 - △ Hardness theory