Fine-Grained Complexity and Algorithm Design Boot Camp

Lower Bounds Based on ETH

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Exponential Time Hypothesis (ETH)

Hypothesis introduced by Impagliazzo, Paturi, and Zane:

**Exponential Time Hypothesis (ETH) [consequence of]**

There is no $2^{o(n)}$-time algorithm for $n$-variable 3SAT.

**Note:** current best algorithm is $1.30704^n$ [Hertli 2011].

**Note:** an $n$-variable 3SAT formula can have $\Omega(n^3)$ clauses.
Exponential Time Hypothesis (ETH)

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**Sparsification Lemma [Impagliazzo, Paturi, Zane 2001]**

There is a $2^{o(n)}$-time algorithm for $n$-variable 3SAT.

$\Leftrightarrow$

There is a $2^{o(m)}$-time algorithm for $m$-clause 3SAT.
Lower bounds for exact and parameterized problems
Lower bounds based on ETH

Exponential Time Hypothesis (ETH)

There is no $2^{o(m)}$-time algorithm for $m$-clause 3SAT.

The textbook reduction from 3SAT to 3-Coloring:

\[
\begin{align*}
\text{3SAT formula } &\phi \\
n \text{ variables} &\\
m \text{ clauses} &
\end{align*} \quad \Rightarrow \quad \begin{align*}
\text{Graph } &G \\
O(n + m) \text{ vertices} &\\
O(n + m) \text{ edges} &
\end{align*}
\]

Corollary

Assuming ETH, there is no $2^{o(n)}$ algorithm for 3-Coloring on an $n$-vertex graph $G$. 
Lower bounds based on ETH

Exponential Time Hypothesis (ETH)

There is no $2^{o(m)}$-time algorithm for $m$-clause 3SAT.

The textbook reduction from 3SAT to 3-Coloring:

<table>
<thead>
<tr>
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Corollary

Assuming ETH, there is no $2^{o(n)}$ algorithm for 3-Coloring on an $n$-vertex graph $G$. 
Transfering bounds

There are polynomial-time reductions from, say, \textbf{3-Coloring} to many other problems such that the reduction increases the number of vertices by at most a constant factor.

\textbf{Consequence:} Assuming ETH, there is no $2^{o(n)}$ time algorithm on $n$-vertex graphs for

- \textbf{Independent Set}
- \textbf{Clique}
- \textbf{Dominating Set}
- \textbf{Vertex Cover}
- \textbf{Hamiltonian Path}
- \textbf{Feedback Vertex Set}
- \ldots
Transfering bounds

There are polynomial-time reductions from, say, 3-Coloring to many other problems such that the reduction increases the number of vertices by at most a constant factor.

Consequence: Assuming ETH, there is no $2^{o(k)} \cdot n^{O(1)}$ time algorithm for

- $k$-Independent Set
- $k$-Clique
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SPEED LIMIT $2^{O(k)}$
Lower bounds based on ETH

What about \textbf{3-Coloring} on planar graphs?

The textbook reduction from \textbf{3-Coloring} to \textbf{Planar 3-Coloring} uses a “crossover gadget” with 4 external connectors:

- In every 3-coloring of the gadget, opposite external connectors have the same color.
- Every coloring of the external connectors where the opposite vertices have the same color can be extended to the whole gadget.
- If two edges cross, replace them with a crossover gadget.
Lower bounds based on ETH

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- Every coloring of the external connectors where the opposite vertices have the same color can be extended to the whole gadget.
- If two edges cross, replace them with a crossover gadget.
Lower bounds based on ETH

- The reduction from \textbf{3-Coloring} to \textbf{Planar 3-Coloring} introduces $O(1)$ new edges/vertices for each crossing.
- A graph with $m$ edges can be drawn with $O(m^2)$ crossings.

\[
\begin{array}{|c|c|c|}
\hline
\text{3SAT formula } \phi & \text{Graph } G & \text{Planar graph } G' \\
\text{n variables} & O(m) \text{ vertices} & O(m^2) \text{ vertices} \\
\text{m clauses} & O(m) \text{ edges} & O(m^2) \text{ edges} \\
\hline
\end{array}
\]

**Corollary**

Assuming ETH, there is no $2^{o(\sqrt{n})}$ algorithm for \textbf{3-Coloring} on an $n$-vertex planar graph $G$.

(Essentially observed by [Cai and Juedes 2001])
Lower bounds for planar problems

**Consequence:** Assuming ETH, there is no $2^{o(\sqrt{n})}$ time algorithm on $n$-vertex **planar graphs** for

- **Independent Set**
- **Dominating Set**
- **Vertex Cover**
- **Hamiltonian Path**
- **Feedback Vertex Set**
- ...
Lower bounds for planar problems

Consequence: Assuming ETH, there is no $2^{o(\sqrt{k})} \cdot n^{O(1)}$ time algorithm on planar graphs for

- $k$-Independent Set
- $k$-Dominating Set
- $k$-Vertex Cover
- $k$-Path
- $k$-Feedback Vertex Set
- ...
Lower bounds for planar problems

**Consequence:** Assuming ETH, there is no \(2^{o(\sqrt{k})} \cdot n^{O(1)}\) time algorithm on planar graphs for

- **k-Independent Set**
- **k-Dominating Set**
- **k-Vertex Cover**
- **k-Path**
- **k-Feedback Vertex Set**
- ... 

**Note:** Reduction to planar graphs does not work for **Clique** (why?).
Recall from Tuesday:
FPT algorithms parameterized by treewidth.
Treewidth

Given a tree decomposition of width $w$, FPT algorithms with running time $2^{O(w)} \cdot n^{O(1)}$ for

- Independent Set
- Dominating Set
- 3-Coloring
- Hamiltonian Cycle
- ...

Observation: A $2^{O(w)} \cdot n^{O(1)}$ algorithm implies a $2^{O(n)} \cdot n^{O(1)}$ algorithm.

$\Rightarrow$ Assuming ETH, no $2^{O(w)} \cdot n^{O(1)}$ algorithms for these problems!
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⇒ Assuming ETH, no $2^{o(w)} \cdot n^{O(1)}$ algorithms for these problems!
The following problems have $w^{O(w)} \cdot n^{O(1)} = 2^{O(w \log w)} \cdot n^{O(1)}$ algorithms:

- **Vertex Coloring**
- **Cycle Packing**
- **Vertex Disjoint Paths**
Treewidth

The following problems have $w^{O(w)} \cdot n^{O(1)} = 2^{O(w \log w)} \cdot n^{O(1)}$ algorithms:

- **Vertex Coloring**
- **Cycle Packing**
- **Vertex Disjoint Paths**

... and assuming ETH, they do not have $2^{o(w \log w)} \cdot n^{O(1)}$ algorithms.

**Proof:** Reduce an instance of a graph problem on $N$ vertices to an instance with treewidth $O(N/\log N)$. 
**Edge Clique Cover**

**Edge Clique Cover**: Given a graph $G$ and an integer $k$, cover the edges of $G$ with at most $k$ cliques.

(the cliques need not be edge disjoint)

**Equivalently**: can $G$ be represented as an intersection graph over a $k$ element universe?
**Edge Clique Cover**

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[Diagram of a graph with 6 cliques]
**Edge Clique Cover**

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(the cliques need not be edge disjoint)

**Equivalently**: can $G$ be represented as an intersection graph over a $k$ element universe?

5 cliques
**Edge Clique Cover**

**Edge Clique Cover**: Given a graph $G$ and an integer $k$, cover the edges of $G$ with at most $k$ cliques. (the cliques need not be edge disjoint)

**Simple algorithm (sketch)**

- If two adjacent vertices have the same neighborhood (“twins”), then remove one of them.
- If there are no twins and $|V(G)| > 2^k$, then there is no solution.
- Use brute force.

Running time: $2^{2^{O(k)}} \cdot n^{O(1)}$ — double exponential dependence on $k$!
**Edge Clique Cover**

**Edge Clique Cover**: Given a graph $G$ and an integer $k$, cover the edges of $G$ with at most $k$ cliques.

(the cliques need not be edge disjoint)

Double-exponential dependence on $k$ cannot be avoided!

**Theorem [Cygan, Pilipczuk, Pilipczuk 2013]**

Assuming ETH, there is no $2^{o(k)} \cdot n^{O(1)}$ time algorithm for **Edge Clique Cover**.

**Proof**: Reduce an $n$-variable 3SAT instance into and instance of **Edge Clique Cover** with $k = O(\log n)$. 
Lower bounds for $\text{W}[1]$-hard problems
Exponential Time Hypothesis

Engineers’ Hypothesis

\( k\text{-CLIQUE} \) cannot be solved in time \( f(k) \cdot n^{O(1)} \).

Theorists’ Hypothesis

\( k\text{-STEP HALTING PROBLEM} \) (is there a path of the given NTM that stops in \( k \) steps?) cannot be solved in time \( f(k) \cdot n^{O(1)} \).

Exponential Time Hypothesis (ETH)

\( n\text{-variable 3SAT} \) cannot be solved in time \( 2^{o(n)} \).

What do we have to show to prove that ETH implies Engineers’ Hypothesis?
### Exponential Time Hypothesis

#### Engineers’ Hypothesis

$k$-Clique cannot be solved in time $f(k) \cdot n^{O(1)}$.

#### Theorists’ Hypothesis

$k$-Step Halting Problem (is there a path of the given NTM that stops in $k$ steps?) cannot be solved in time $f(k) \cdot n^{O(1)}$.

#### Exponential Time Hypothesis (ETH)

$n$-variable 3SAT cannot be solved in time $2^{o(n)}$.

What do we have to show to prove that ETH implies Engineers’ Hypothesis?

We have to show that an $f(k) \cdot n^{O(1)}$ algorithm implies that there is a $2^{o(n)}$ time algorithm for $n$-variable 3SAT.
Exponential Time Hypothesis

Engineers’ Hypothesis

\( k\text{-CLIQUE} \) cannot be solved in time \( f(k) \cdot n^{O(1)} \).

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Exponential Time Hypothesis (ETH)

\( n\)-variable 3SAT cannot be solved in time \( 2^{o(n)} \).

We actually show something much stronger and more interesting:

Theorem [Chen et al. 2004]

Assuming ETH, there is no \( f(k) \cdot n^{o(k)} \) algorithm for \( k\text{-CLIQUE} \) for any computable function \( f \).
Lower bound on the exponent

Theorem [Chen et al. 2004]

Assuming ETH, there is no $f(k) \cdot n^{o(k)}$ algorithm for $k$-CLIQUE for any computable function $f$.

Suppose that $k$-CLIQUE can be solved in time $f(k) \cdot n^{k/s(k)}$, where $s(k)$ is a monotone increasing unbounded function. We use this algorithm to solve $3$-$\text{COLORING}$ on an $n$-vertex graph $G$ in time $2^{o(n)}$. 
Lower bound on the exponent

**Theorem [Chen et al. 2004]**

Assuming ETH, there is no \( f(k) \cdot n^{o(k)} \) algorithm for \( k\text{-\textsc{Clique}} \) for any computable function \( f \).

Suppose that \( k\text{-\textsc{Clique}} \) can be solved in time \( f(k) \cdot n^{k/s(k)} \), where \( s(k) \) is a monotone increasing unbounded function. We use this algorithm to solve \( 3\text{-\textsc{Coloring}} \) on an \( n \)-vertex graph \( G \) in time \( 2^{o(n)} \).

Let \( k \) be the largest integer such that \( f(k) \leq n \) and \( k^{k/s(k)} \leq n \). Function \( k := k(n) \) is monotone increasing and unbounded.

Split the vertices of \( G \) into \( k \) groups. Let us build a graph \( H \) where each vertex corresponds to a proper 3-coloring of one of the groups. Connect two vertices if they are not conflicting.
Lower bound on the exponent

**Theorem [Chen et al. 2004]**

Assuming ETH, there is no \( f(k) \cdot n^{o(k)} \) algorithm for \( k\text{-CLIQUE} \) for any computable function \( f \).

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Split the vertices of \( G \) into \( k \) groups. Let us build a graph \( H \) where each vertex corresponds to a proper \( 3 \)-coloring of one of the groups. Connect two vertices if they are not conflicting.

Every \( k \)-clique of \( H \) corresponds to a proper \( 3 \)-coloring of \( G \).

\[ \Rightarrow \text{A } 3\text{-coloring of } G \text{ can be found in time } f(k) \cdot |V(H)|^{k/s(k)} \leq n \cdot (k^{3^n/k})^{k/s(k)} = n \cdot k^{k/s(k)} \cdot 3^{n/s(k)} = 2^{o(n)}. \]
Tight bounds

Theorem [Chen et al. 2004]
Assuming ETH, there is no $f(k) \cdot n^{o(k)}$ algorithm for $k$-CLIQUE for any computable function $f$.

Transfering to other problems:

$k$-CLIQUE $(x, k)$ \quad \Rightarrow \quad \text{Problem A} (x', O(k))

$f(k) \cdot n^{o(k)}$ algorithm \quad \Leftarrow \quad f(k) \cdot n^{o(k)}$ algorithm
Tight bounds

**Theorem [Chen et al. 2004]**

Assuming ETH, there is no \( f(k) \cdot n^{o(k)} \) algorithm for \( k\text{-CLIQUE} \) for any computable function \( f \).

**Transfering to other problems:**

\[
k\text{-CLIQUE} (x, k) \quad \Rightarrow \quad \text{Problem } A (x', g(k))
\]

\[
f(k) \cdot n^{o(k)} \quad \text{algorithm}
\]

\[
\leftarrow \quad f(k) \cdot n^{o(g^{-1}(k))} \quad \text{algorithm}
\]
Tight bounds

Theorem [Chen et al. 2004]

Assuming ETH, there is no $f(k) \cdot n^{o(k)}$ algorithm for $k$-CLIQUE for any computable function $f$.

Transfering to other problems:

$$k$$-CLIQUE $(x, k) \Rightarrow \text{Problem } A (x', k^2)$$

$f(k) \cdot n^{o(k)}$ algorithm $\iff$ $f(k) \cdot n^{o(\sqrt{k})}$ algorithm
**Tight bounds**

**Theorem [Chen et al. 2004]**

Assuming ETH, there is no $f(k) \cdot n^{o(k)}$ algorithm for $k$-CLIQUE for any computable function $f$.

**Transfering to other problems:**

\[
\begin{align*}
\text{k-CLIQUE} & \quad \Rightarrow \quad \text{Problem A} \\
(x, k) & \quad \Rightarrow \quad (x', k^2) \\
f(k) \cdot n^{o(k)} \quad \text{algorithm} & \quad \iff \quad f(k) \cdot n^{o(\sqrt{k})} \quad \text{algorithm}
\end{align*}
\]

**Bottom line:**

- To rule out $f(k) \cdot n^{o(k)}$ algorithms, we need a parameterized reduction that blows up the parameter at most *linearly*.
- To rule out $f(k) \cdot n^{o(\sqrt{k})}$ algorithms, we need a parameterized reduction that blows up the parameter at most *quadratically*. 
Tight bounds

Assuming ETH, there is no $f(k)n^{o(k)}$ time algorithms for

- Set Cover
- Hitting Set
- Connected Dominating Set
- Independent Dominating Set
- Partial Vertex Cover
- Dominating Set in bipartite graphs
- ...
The odd case of **Odd Set**

**Odd Set**: Given a set system \( \mathcal{F} \) over a universe \( U \) and an integer \( k \), find a set \( S \) of at most \( k \) elements such that \( |S \cap F| \) is odd for every \( F \in \mathcal{F} \).

We have seen:

**Theorem**

**Odd Set** is \( W[1] \)-hard parameterized by \( k \).

New parameter: \( k' := k + \binom{k}{2} = O(k^2) \).
The odd case of **ODD SET**

**ODD SET**: Given a set system $\mathcal{F}$ over a universe $U$ and an integer $k$, find a set $S$ of at most $k$ elements such that $|S \cap F|$ is odd for every $F \in \mathcal{F}$.

We have seen:

**Theorem**

**ODD SET** is $W[1]$-hard parameterized by $k$.

We immediately get:

**Corollary**

Assuming ETH, there is no $f(k)n^{o(\sqrt{k})}$ time algorithm for **ODD SET**.

But this does not seem to be tight...

**Problem**: **k-CLIQUE** is a very densely constrained problem, which makes the reduction very expensive.
**Subgraph Isomorphism**

**Subgraph Isomorphism**: Given two graphs $H$ and $G$, decide if $H$ is isomorphic to a subgraph of $G$.

Trivial reduction from $k$-Clique:

**Corollary (parameterized by no. of vertices of $H$)**

Assuming ETH, **Subgraph Isomorphism** parameterized by $k := |V(H)|$ has no $f(k)n^{o(k)}$ time algorithm.
**Subgraph Isomorphism**

Given two graphs $H$ and $G$, decide if $H$ is isomorphic to a subgraph of $G$.

Trivial reduction from $k$-Clique:

**Corollary (parameterized by no. of edges of $H$)**

Assuming ETH, **Subgraph Isomorphism** parameterized by $k := |E(H)|$ has no $f(k)n^{o(\sqrt{k})}$ time algorithm.

Is this tight?
**Subgraph Isomorphism**

**Subgraph Isomorphism:** Given two graphs $H$ and $G$, decide if $H$ is isomorphic to a subgraph of $G$.

Trivial reduction from $k$-Clique:

**Corollary** (parameterized by no. of edges of $H$)

Assuming ETH, **Subgraph Isomorphism** parameterized by $k := |E(H)|$ has no $f(k)n^{o(\sqrt{k})}$ time algorithm.

Is this tight?

An almost tight result:

**Theorem [M. 2010]**

Assuming ETH, **Subgraph Isomorphism** parameterized by $k := |E(H)|$ has no $f(k)n^{o(k/\log k)}$ time algorithm.

Open question: can we remove the $\log k$ from this lower bound?
**Odd Set**

Reduction from \texttt{k-Clique} to \texttt{Odd Set}:

New parameter: \( k' := k + \binom{k}{2} = O(k^2) \).
**Odd Set**

Reduction from **Subgraph Isomorphism** to **Odd Set**:

New parameter: \( k' := |V(H)| + |E(H)| = O(k) \).

(Where \( k := |E(H)| \))
Odd Set

Reduction from Subgraph Isomorphism to Odd Set:

New parameter: $k' := |V(H)| + |E(H)| = O(k)$.
(Where $k := |E(H)|$)

Theorem

Assuming ETH, there is no $f(k)n^{o(k/\log k)}$ time algorithm for Odd Set.
Tight bounds

Assuming ETH, there is no $f(k)n^{o(k)}$ time algorithms for

- **Set Cover**
- **Hitting Set**
- **Connected Dominating Set**
- **Independent Dominating Set**
- **Partial Vertex Cover**
- **Dominating Set** in bipartite graphs
- ...
Tight bounds

Assuming ETH, there is no $f(k)n^{o(k)}$ time algorithms for

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What about planar problems?

- More problems are FPT, more difficult to prove $W[1]$-hardness.
- The problem **Grid Tiling** is the key to many of these results.
Grid Tiling

**GRID TILING**

Input: A $k \times k$ matrix and a set of pairs $S_{i,j} \subseteq [D] \times [D]$ for each cell.

A pair $s_{i,j} \in S_{i,j}$ for each cell such that

- Vertical neighbors agree in the 1st coordinate.
- Horizontal neighbors agree in the 2nd coordinate.

<table>
<thead>
<tr>
<th></th>
<th>(1,1)</th>
<th>(5,1)</th>
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<td>(3,1)</td>
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$k = 3, \ D = 5$
**Grid Tiling**

**Input:** A \( k \times k \) matrix and a set of pairs \( S_{i,j} \subseteq [D] \times [D] \) for each cell.

**Find:** A pair \( s_{i,j} \in S_{i,j} \) for each cell such that
- Vertical neighbors agree in the 1st coordinate.
- Horizontal neighbors agree in the 2nd coordinate.

\[
\begin{array}{ccc}
(1,1) & (5,1) & (1,1) \\
(3,1) & (1,4) & (2,4) \\
(2,4) & (5,3) & (3,3)
\end{array}
\begin{array}{c}
(2,2) \\
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\( k = 3, \ D = 5 \)
# Grid Tiling

## Grid Tiling

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A pair \( s_{i,j} \in S_{i,j} \) for each cell such that

- Vertical neighbors agree in the 1st coordinate.
- Horizontal neighbors agree in the 2nd coordinate.

**Find:**

- Vertical neighbors agree in the 1st coordinate.
- Horizontal neighbors agree in the 2nd coordinate.

**Simple proof:**

**Fact**

There is a parameterized reduction from \( k\)-Clique to \( k \times k \) Grid Tiling.
Grid Tiling is W[1]-hard

**Reduction from** \(k\text{-CLIQUE}\)**

**Definition of the sets:**

- For \(i = j\): \((x, y) \in S_{i,j} \iff x = y\)
- For \(i \neq j\): \((x, y) \in S_{i,j} \iff x \text{ and } y \text{ are adjacent.}\)

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\((v_i, v_i)\)

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Each diagonal cell defines a value \(v_i\) ...
Grid Tiling is $W[1]$-hard

**Reduction from $k$-CLIQUE**

**Definition of the sets:**

- For $i = j$: $(x, y) \in S_{i,j} \iff x = y$
- For $i \neq j$: $(x, y) \in S_{i,j} \iff x$ and $y$ are adjacent.

\[
\begin{array}{cccc}
(\cdot, v_i) & (v_i, \cdot) & (\cdot, v_i) & (\cdot, v_i) \\
(\cdot, v_i) & (v_i, v_i) & (\cdot, v_i) & (\cdot, v_i) \\
(v_i, \cdot) & & & \\
(v_i, \cdot) & & & \\
(v_i, \cdot) & & & \\
(v_i, \cdot) & & & \\
\end{array}
\]

... which appears on a “cross”
Grid Tiling is W[1]-hard

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$v_i$ and $v_j$ are adjacent for every $1 \leq i < j \leq k$. 

<table>
<thead>
<tr>
<th></th>
<th>$(v_i,)$</th>
<th>$(v_i, v_i)$</th>
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**Theorem**

$k \times k$ Grid Tiling is $W[1]$-hard and, assuming ETH, cannot be solved in time $f(k)n^{o(k)}$ for any function $f$.

This lower bound is the key for proving hardness results for planar graphs.

**Examples:**

- **Multiway Cut** on planar graphs with $k$ terminals
- **Independent Set** for unit disks
A classical problem

$s – t$ Cut

Input: A graph $G$, an integer $p$, vertices $s$ and $t$
Output: A set $S$ of at most $p$ edges such that removing $S$ separates $s$ and $t$.

Theorem [Ford and Fulkerson 1956]
A minimum $s – t$ cut can be found in polynomial time.

What about separating more than two terminals?
More than two terminals

$k$-Terminal Cut (aka Multiway Cut)

| Input:       | A graph $G$, an integer $p$, and a set $T$ of $k$ terminals |
| Output:      | A set $S$ of at most $p$ edges such that removing $S$ separates any two vertices of $T$ |

Theorem [Dalhaus et al. 1994]

NP-hard already for $k = 3$. 
More than two terminals

**k-Terminal Cut (aka Multiway Cut)**

<table>
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<th>Input:</th>
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**Theorem** [Dalhaus et al. 1994] [Hartvigsen 1998] [Bentz 2012]

**Planar $k$-Terminal Cut** can be solved in time $n^{O(k)}$.

**Theorem** [Klein and M. 2012]

**Planar $k$-Terminal Cut** can be solved in time $2^{O(k)} \cdot n^{O(\sqrt{k})}$.
## Lower bounds

<table>
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<th>Theorem [Klein and M. 2012]</th>
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| **Planar** $k$-**Terminal Cut** can be solved in time $2^{O(k)} \cdot n^{O(\sqrt{k})}$.

Natural questions:

- Is there an $f(k) \cdot n^{o(\sqrt{k})}$ time algorithm?
- Is there an $f(k) \cdot n^{O(1)}$ time algorithm (i.e., is it fixed-parameter tractable)?
Lower bounds

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**Lower bounds:**

**Theorem [M. 2012]**

**Planar $k$-Terminal Cut** is $W[1]$-hard and has no $f(k) \cdot n^{o(\sqrt{k})}$ time algorithm (assuming ETH).
Reduction from $k \times k$ Grid Tiling to Planar $k^2$-Terminal Cut

For every set $S_{i,j}$, we construct a gadget with 4 terminals such that
- for every $(x, y) \in S_{i,j}$, there is a minimum multiway cut that represents $(x, y)$.
- every minimum multiway cut represents some $(x, y) \in S_{i,j}$.

Main part of the proof: constructing these gadgets.

The gadget.
Reduction from $k \times k$ Grid Tiling to Planar $k^2$-Terminal Cut

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Main part of the proof: constructing these gadgets.

A cut representing $(4, 2)$.
Reduction from $k \times k$ Grid Tiling to Planar $k^2$-Terminal Cut

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Main part of the proof: constructing these gadgets.

A cut not representing any pair.
Putting together the gadgets
Putting together the gadgets

Oops!
Putting together the gadgets
Grid Tiling with \( \leq \)

**Grid Tiling with \( \leq \)**

**Input:** A \( k \times k \) matrix and a set of pairs \( S_{i,j} \subseteq [D] \times [D] \) for each cell.

A pair \( s_{i,j} \in S_{i,j} \) for each cell such that

- 1st coordinate of \( s_{i,j} \leq 1 \)st coordinate of \( s_{i+1,j} \).
- 2nd coordinate of \( s_{i,j} \leq 2 \)nd coordinate of \( s_{i,j+1} \).

**Find:**

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<th>(5,1)</th>
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\( k = 3, \ D = 5 \)
## Grid Tiling with \( \leq \)

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Variant of the previous proof:

### Theorem

There is a parameterized reduction from \( k \times k \)-Grid Tiling to \( O(k) \times O(k) \) Grid Tiling with \( \leq \).

Very useful starting point for geometric problems!
**Theorem**

Given a set of $n$ unit disks in the plane, we can find $k$ independent disks in time $n^{O(\sqrt{k})}$. 
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Given a set of \( n \) unit disks in the plane, we can find \( k \) independent disks in time \( n^{O(\sqrt{k})} \).

Matching lower bound:

**Theorem**

There is a reduction from \( k \times k \) Grid Tiling with \( \leq \) to \( k^2\)-Independent Set for unit disks. Consequently, Independent Set for unit disks is

- is \( W[1] \)-hard, and
- cannot be solved in time \( f(k)n^{o(\sqrt{k})} \) for any function \( f \).
Reduction to unit disks

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Every pair is represented by a unit disk in the plane.

\( \leq \) relation between coordinates \( \iff \) disks do not intersect.
Reduction to unit disks

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\( \leq \) relation between coordinates \( \iff \) disks do not intersect.
Reduction to unit disks

Every pair is represented by a unit disk in the plane.

≤ relation between coordinates ⇐⇒ disks do not intersect.
Center-pivot irrigation
Higher dimensions

Bidimensionality for planar graphs:

- $2^{O(\sqrt{n})}$, $2^{O(\sqrt{k})} \cdot n^{O(1)}$, $n^{O(\sqrt{k})}$ time algorithms.
- There is no tridimensionality!
Higher dimensions

Bidimensionality for 2-dimensional geometric problems:

- $2^{O(\sqrt{n})}$, $2^{O(\sqrt{k})} \cdot n^{O(1)}$, $n^{O(\sqrt{k})}$ time algorithms.
- What about higher dimensions?
Higher dimensions

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- What about higher dimensions?

“Limited blessing of low dimensionality:”

Theorem

**Independent Set** for unit spheres in $d$ dimensions can be solved in time $n^{O(k^{1-1/d})}$.

Matching lower bound:

Theorem [M. and Sidiropoulos 2014]

Assuming ETH, **Independent Set** for unit spheres in $d$ dimensions cannot be solved in time $n^{o(k^{1-1/d})}$.
Higher dimensions

Bidimensionality for 2-dimensional geometric problems:

- $2^{O(\sqrt{n})}, 2^{O(\sqrt{k})} \cdot n^{O(1)}, n^{O(\sqrt{k})}$ time algorithms.
- What about higher dimensions?

“Limited blessing of low dimensionality:”

**Theorem** [Smith and Wormald 1998]

**Euclidean TSP** in $d$ dimensions can be solved in time $2^{O(n^{1-1/d+\epsilon})}$.

Matching lower bound:

**Theorem** [M. and Sidiropoulos 2014]

Assuming ETH, **Euclidean TSP** in $d$ dimension cannot be solved in time $2^{O(n^{1-1/d-\epsilon})}$ for any $\epsilon > 0$. 

Summary

We used ETH to rule out

1. $2^{o(n)}$ time algorithms for, say, **Independent Set**.
2. $2^{o(\sqrt{n})}$ time algorithms for, say, **Independent Set** on planar graphs.
3. $2^{o(k)} \cdot n^{O(1)}$ time algorithms for, say, **Vertex Cover**.
4. $2^{o(\sqrt{k})} \cdot n^{O(1)}$ time algorithms for, say, **Vertex Cover** on planar graphs.
5. $f(k)n^{o(k)}$ time algorithms for **Clique**.
6. $f(k)n^{o(\sqrt{k})}$ time algorithms for planar problems such as **$k$-Terminal Cut** and **Independent Set** for unit disks.

Other tight lower bounds on $f(k)$ having the form $2^{o(k \log k)}$, $2^{o(k^2)}$, or $2^{2^{o(k)}}$ exist.