Introduction to the theory of computing 2
Midterm grading guide
May.09, 2024

General principles.

The aim of the scoring guide is for the correctors to evaluate the papers in a uniform manner. Therefore, the guide provides the main ideas of (at least one possible) solution to each task and the sub-scores assigned to them. The guide is not intended to be a detailed description of the full solution of the tasks; the described steps can be considered as an outline of a solution that will achieve a maximum score.
The partial marks indicated in the guide are awarded to the solver only if the related idea is included in the thesis as a step of a clear, clearly described and justified solution. Thus, for example, the mere description of the knowledge, definitions, and theorems included in the material is not worth points without their application (even if, by the way, one of the described facts actually plays a role in the solution). Considering whether the score indicated in the guide is due to the solver (in part or in whole) taking into account the above, is entirely the competence of the corrector.
A partial score is awarded for any idea or partial solution from which a perfect solution to the task could have been obtained by suitable addition of the thought process described in the thesis. If a solver starts several significantly different solutions to the same task, then score can be given to only one of them. If each described solution or solution part is correct or can be supplemented to be correct, then the solution initiative with the most partial points is evaluated. However, if among several solution attempts there is one that is correct and one that contains (significant) errors, and the thesis does not reveal which one the solver considered to be correct, then the solution attempt with fewer points is evaluated (even if this score is 0).
The sub-scores in the guide can be divided further if necessary. A good solution different from the one described in the guide is of course worth maximum points, but without proof you can only refer to the theorems and statements in the lectures.

Question 1.

First solution. Let P_6 and P_8 denote the 6- and 8-vertex paths used to construct G, respectively. **(0 points)**
Then the degree of the endpoints of P_6 in G is 9, and the degree of the other vertices of P_6 is 10 (since the vertices of P_6 are adjacent to all 8 vertices of P_8 in addition to their neighbors within P_6). Similarly, the degree of the endpoints of P_8 is 7, and the degree of the other vertices of P_8 is 8. **(1 point)**
The number of edges of G is 60 (of which 5 + 7 = 12 edges belong to P_6 and P_8, respectively, and the other $6 \cdot 8 = 48$ edges connect the vertices of the two paths). **(1 point)**
Since it has more than 2 vertices of odd degree, according to the learned theorem, there is no Euler trail in G. **(1 point)**
This means that there is no 60-edge edge sequence in G that contains all edges of G; that is, the length of the shortest sequence of such edges is at least 61. **(2 points)**
Let u be an endpoint of the path P_8 , and v be an endpoint of the path P_6. Add a new edge $\{u, v\}$ to G. (Therefore, two parallel edges run between u and v.) **(1 point)**
In the resulting graph G', the degrees of u and v increased by 1, so they became even. In other words, the degrees of all vertices in G' are even, with two exceptions. **(1 point)**
It is also true that G' is a connected graph. Indeed, if two vertices in G' are not adjacent, then they must belong to the same one of P_6 and P_8, so there is obviously a path between them (either along P_6 or P_8 or through an arbitrary vertex of the other path). **(1 point)**
Thus, according to the learned theorem, G' contains an Euler trail. **(1 point)**
It follows that G contains a sequence of edges of length 61 that contains all edges of G: for this, the Euler trail in G' only needs to be modified to the extent that both of its edges between u and v must be the unique edge $\{u, v\}$ of G, repeated twice. **(1 point)**
Thus, the length of the shortest such sequence of edges is 61.

Second solution. The justification that the length of the required edge sequence is at least 61 is the same as that written in the first solution. **(5 points)**
Let u be an endpoint of the path P_8 , and v be an endpoint of the path P_6. Delete the edge $\{u, v\}$ from G. **(1 point)**
In the resulting graph G', the degrees of u and v decreased by 1, thus changing to even. In other words, the degrees of all vertices in G' are even, with two exceptions. **(1 point)**

It is also true that G' is a connected graph. Indeed, a path connects u and v, for example, through their neighbors in P_6 and P_8, which are adjacent to each other. Otherwise, if two vertices are not adjacent in G', then they must belong to the same one of P_6 and P_8, so there is obviously a path between them (for example, along P_6 or P_8). **(1 point)**

Thus, according to the learned theorem, G' contains an Euler trail. **(1 point)**

It follows that Gcontains a sequence of edges of length 61 that contains all edges of G: for this, the Euler trail in G' only needs to be modified to the extent that when the trail reaches (say) u, we insert twice in a row the edge $\{u, v\}$ - that is, we first go from u to v and then back to u; then we continue with the Euler trail. **(1 point)**

Thus, the length of the shortest such sequence of edges is 61.

Question 2.

a) $Parent(g)=f, parent(f)=c, parent(c)=a$, we get back to a by walking along the series, **(1 point)** so following the operation of the BFS algorithm, the path touching a, c, f, g is a shortest path from a to g. Therefore $distance(g)=3$. **(1 point)**

b) Since the procedure starts from a, it reaches all neighbors of a from a. **(1 point)**
Thus, the degree is the number of vertices v for which $parent(v)=a$, that is $deg(a)=2$. **(1 point)**

c) f is a neighbor of g, since $parent(g)=f$. **(1 point)**
Since $parent(g)$ received a value only when f became active, **(1 point)**
and vertices a, b, c, d, e were all active earlier, **(1 point)**
therefore g cannot be adjacent to any of the vertices a, b, c, d, e. **(1 point)**
(Indeed, if v were the first vertex among a, b, c, d, e, to which g is adjacent, then when v becomes active, the algorithm would also examine the edge $\{v, g\}$ and thus we would have $parent(g)=v$.) (0 points)
It follows that the degree of g is 1. **(2 points)**

If a solver draws the BFS tree associated with the traversal and reads the degree of a and g from it without any further justification, they can only get the first 1 point in the scoring of tasks (a) and (c) out of the total of 8 points for these tasks (even though their results are correct), since the task did not ask for the degree of the vertices in the BFS tree, but rather the degrees in the graph G itself. An additional partial score can only be awarded if the solution (at least partially) reveals that a and g cannot have additional neighbors in G.

Question 3.

The vertices of G can be colored with 6 colors. Indeed, for each of the cycles with 5 and 7 vertices, we can color one of the vertices with one color, and then, starting from this vertex and walking along the cycle, we use two additional colors alternately. If there is no color in common between the colors used for the vertices of the two cycles, then we have indeed obtained a proper coloring with 6 colors. **(2 points)**

We show that the vertices of G cannot be colored with 5 colors. **(0 points)**

Since all vertices of one cycle are adjacent to all vertices of the other cycle, if one of the vertices of one cycle is colored with, say, a red color, then none of the vertices of the other cycle can be red. In other words, the set of colors used to color the two cycles is disjoint. **(2 points)**

At least three colors are needed to color the two cycles separately. This follows, on the one hand, from the learned theorem that graphs containing an odd cycle are not bipartite, so their chromatic number is at least 3; but it can also be seen directly: if we tried to color the vertices of any cycle with two colors, these two colors would have to be used alternately while moving along the cycle, but thus, when we reach the last vertex, its two neighbors would already have a different color, so this vertex could not get a color. **(1 point)**

With this, we realized that G cannot be colored with 5 colors, only with 6. Thus $\chi(G) = 6$. **(1 point)**

If two vertices adjacent along the cycle are selected in both circles, then together they form a 4-vertex clique in G. **(1 point)**

We show that there is no 5-vertex clique in G. **(0 points)**

Let us choose any 5 of the vertices of G. Then at least 3 of the selected vertices will belong to the same cycle (of the 5 and 7 vertex cycles used to make G). **(1 point)**

Among these at least 3 vertices, however, there must be two that are not adjacent, because obviously there is no clique with three vertices within either the 5-vertex or the 7-vertex cycle. **(1 point)**
Thus, the selected 5 vertices do not form a clique.
With this we have shown that ω(G) = 4. **(1 point)**

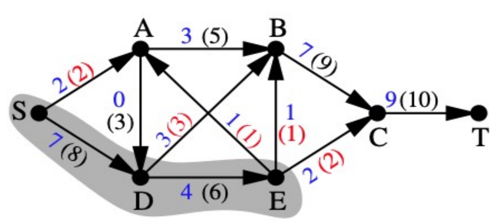
Question 4.

The degree of every vertex in G is 3, so the maximum degree in G is also $\Delta(G)=3$. **(1 point)**
Thus, according to what was learned, $\chi_e(G)\geq 3$. **(2 points)**
(after all, G does not contain a loop). **(0 points)**
We show that the edges of G can be colored with 3 colors. **(0 points)**
Let's color the edges of one cycle with three colors, for example by painting one edge of the cycle red and the rest alternately blue and green. Now we also color the edges of the other cycle with the same three colors so that the color of each edge is the same as the respective edge of the other cycle. Finally, each edge between the two cycles also receives a color from the same three: the color which is missing from the colors at its endpoints. Since the same two colors are missing at the two endpoints of these edges, the resulting coloring is proper. **(6 points)**
Since the edges of G can be colored with 3 colors, but not with less, $\chi_e(G)=3$. **(1 point)**

Of course, the edge coloring of G with three colors can also be specified with a drawing instead of a text similar to the one above. This can be appreciated if, on the one hand, the drawing really represents the graph in the task, on the other hand, the color of each edge is clearly and unambiguously revealed (either from the coloring of the drawing or from the mark written next to the edge), and, third, the depicted edge coloring is correct. However, the 6 points awarded for correct edge coloring cannot be divided further: if any of the listed conditions are violated, no partial score can be given from these 6 points. In addition, partial points cannot be given for the application of the Vizing theorem (that is, for the statement $\chi_e(G)\leq 4$ and its justification), because this has no added value for the solution of the task.

Question 5.

a) The capacity of the cut with $X=\{S,D,E\}$ is 9 (the sum of the capacities marked in red in the figure below). **(2 points)**
The value of the flow shown in the figure (the sum of the flow values on the edges leaving S) is also 9. **(2 points)**
Since the value of any flow is at most the capacity of any cut, a flow of value 9 proves that a cut of capacity 9 is minimal. **(2 points)**



b) The solution to part **(a)** works without change in all cases $p\geq 9$: the flow given there also proves that the 9-capacity cut with $X=\{S,D,E\}$ is minimal. **(1 point)**
We therefore examine the case $0\leq p<9$. Then the capacity of the cut with $X=\{S,A,B,C,D,E\}$ is p (because only the capacity of the edge (C,T) contributes), which is therefore less than 9. **(1 point)**
Let X be an arbitrary cut. If $C\in X$, then the edge (C,T) leaves X, so the capacity of X is $c(X)\geq p$. If, on the other hand, $C\notin X$, then the edge (C,T) does not leave X, so the value of $c(X)$ does not depend on p. Then, from the solution of part **(a)**, we know that $c(X)\geq 9>p$. **(1 point)**
It follows that in the case $0\leq p<9$, the cut with $X=\{S,A,B,C,D,E\}$ has minimal capacity. **(1 point)**

In the scoring of part (a), the last 2 points are awarded to the person who gives a substantial reason that the specified cut has a minimum capacity. This can be done in a different way than in the above solution: for example, by running the augmenting

path algorithm, referring to the fact that the set of vertices reachable from S after the procedure is stopped is, according to what has been learned, a minimum capacity cut; however, empty phrases (such as "because of the Ford-Fulkerson theorem") are not worth points.
In part (b) in the case of $0 \leq p < 9$, the minimal capacity of the cut with $X = \{S, A, B, C, D, E\}$ can be justified as follows: multiply the value of the flow given in part (a) taken at each edge by $\frac{p}{9}$. The values obtained in this way also form a flow, because neither the capacity conditions nor the flow conservation conditions are violated by the multiplication (due to $\frac{p}{9} < 1$ in the case of the former). The value of the flow thus obtained is p, which thus proves the minimality of the cut with $X = \{S, A, B, C, D, E\}$ with capacity p in a manner analogous to what was written in problem (a).

Question 6.

First solution. According to Gallai's theorem, $\nu(G) + \rho(G) = n$, where n is the number of vertices of the graph. **(0 points)**
Thus, the proposition to be proved is equivalent to this: $\chi(\overline{G}) \leq n - \nu(G)$. **(2 points)**
Let M be a maximal matching in G (that is, $¿M¿ = \nu(G)$) and create the following coloring of the vertices of $\overline{G}$: paint the endpoints of the edges in M with the same color, but for each edge in M use a different color. Finally, for each of the vertices not covered by M give an independent color (that is, their color should be different from the color of all other vertices). **(2 points)**
This gives us a proper coloring of $\overline{G}$. Indeed, only the endpoints of the edges in M were given the same color, but since the edges of M are missing from $E(\overline{G})$, adjacent vertices in $\overline{G}$ were not given the same color. **(2 points)**
On the other hand, the number of colors used for coloring is $n - \nu(G)$, because we used a $\nu(G)$ colors for the endpoints of the edges of M, and one color for each of the remaining $n - 2 \cdot \nu(G)$ vertices, and this is really $(n - 2 \cdot \nu(G)) + \nu(G) = n - \nu(G)$ colors. **(2 points)**
Since the vertices of $\overline{G}$ were colored with $n - \nu(G)$ colors, it follows that $\chi(\overline{G}) \leq n - \nu(G) = \rho(G)$. **(2 points)**

Second solution. Let Z be a minimal edge-covering set in G (that is, $¿Z¿ = \rho(G)$) and create the following coloring of the vertices of $\overline{G}$. Let us number the edges of Z from 1 to $\rho(G)$ arbitrarily, then for each vertex v choose an arbitrary edge in Z that is incident to it and color v in the same color as the number of the chosen edge. **(3 points)**
Since Z is a set of covering edges, we assigned a color to each vertex (because there is an edge in Z incident to it). **(1 point)**
If now the (arbitrary) vertices u and v have the same color, let's say the jth one, then they are connected by the jth edge in Z. **(2 points)**
Since the edge e_j is missing from $E(\overline{G})$, u and v are not adjacent in $\overline{G}$. With this we have shown that the specified coloring is a proper coloring of the vertices of $\overline{G}$. **(2 points)**
Since we have colored the vertices of $\overline{G}$ with $\rho(G)$ colors, it follows that $\chi(\overline{G}) \leq \rho(G)$. **(2 points)**