1

REGULAR LANGUAGES

The theory of computation begins with a question: What is a computer? It is perhaps a silly question, as even my four-year-old daughter knows that this thing I type on is a computer. But these real computers are quite complicated—too much so to allow us to set up a manageable mathematical theory of them directly. Instead we use an idealized computer called a ***computational model.*** As with any model in science, a computational model may be accurate in some ways but perhaps not in others. Thus we will use several different computational models, depending on the features we want to focus on. We begin with the simplest model, called the ***finite state machine*** or ***finite automaton.***

1.1

FINITE AUTOMATA

Finite automata are good models for computers with an extremely limited amount of memory. What can a computer do with such a small memory? Many useful things! In fact, we interact with such computers all the time, as they lie at the heart of various electromechanical devices.

As shown in the following figures, the controller for an automatic door is one example of such a device. Often found at supermarket entrances and exits, automatic doors swing open when sensing that a person is approaching. An automatic door has a pad in front to detect the presence of a person about to walk through the doorway. Another pad is located to the rear of the doorway so that the controller can hold the door open long enough for the person to pass all the way through and also so that the door does not strike someone standing behind it as it opens.

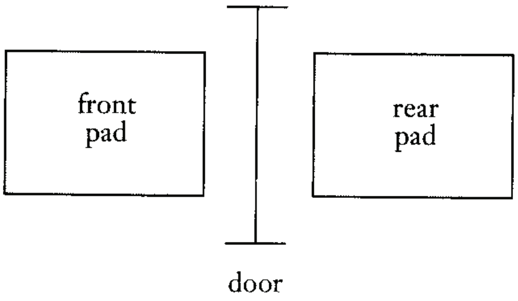


FIGURE 1.1
Top view of an automatic door

The controller is in either of two states: "OPEN" or "CLOSED," representing the corresponding condition of the door. As shown in the following figures, there are four possible input conditions: "FRONT" (meaning that a person is standing on the pad in front of the doorway), "REAR" (meaning that a person is standing on the pad to the rear of the doorway), "BOTH" (meaning that people are standing on both pads), and "NEITHER" (meaning that no one is standing on either pad).

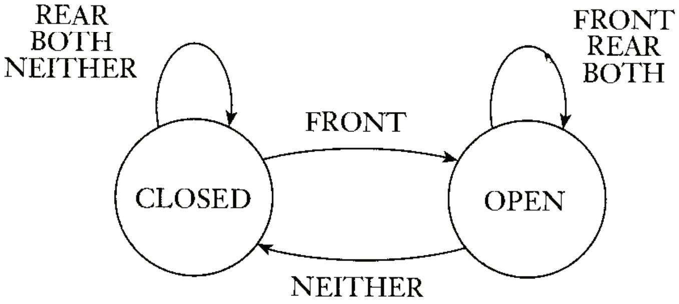


FIGURE 1.2
State diagram for automatic door controller

		input signal			
		NEITHER	FRONT	REAR	BOTH
state	CLOSED	CLOSED	OPEN	CLOSED	CLOSED
	OPEN	CLOSED	OPEN	OPEN	OPEN

FIGURE 1.3
State transition table for automatic door controller

The controller moves from state to state, depending on the input it receives. When in the CLOSED state and receiving input NEITHER or REAR, it remains in the CLOSED state. In addition, if the input BOTH is received, it stays CLOSED because opening the door risks knocking someone over on the rear pad. But if the input FRONT arrives, it moves to the OPEN state. In the OPEN state, if input FRONT, REAR, or BOTH is received, it remains in OPEN. If input NEITHER arrives, it returns to CLOSED.

For example, a controller might start in state CLOSED and receive the following series of input signals: FRONT, REAR, NEITHER, FRONT, BOTH, NEITHER, REAR, NEITHER. It then would go through the series of states: CLOSED (starting), OPEN, OPEN, CLOSED, OPEN, OPEN, CLOSED, CLOSED, CLOSED.

Thinking of an automatic door controller as a finite automaton is useful because that suggests standard ways of representation as in Figures 1.2 and 1.3. This controller is a computer that has just a single bit of memory, capable of recording which of the two states the controller is in. Other common devices have controllers with somewhat larger memories. In an elevator controller a state may represent the floor the elevator is on and the inputs might be the signals received from the buttons. This computer might need several bits to keep track of this information. Controllers for various household appliances such as dishwashers and electronic thermostats, as well as parts of digital watches and calculators, are additional examples of computers with limited memories. The design of such devices requires keeping the methodology and terminology of finite automata in mind.

Finite automata and their probabilistic counterpart ***Markov chains*** are useful tools when we are attempting to recognize patterns in data. These devices are used in speech processing and in optical character recognition. Markov chains have even been used to model and predict price changes in financial markets.

We will now take a close look at finite automata from a mathematical perspective. We will develop a precise definition of a finite automaton, terminology for describing and manipulating finite automata, and theoretical results that describe their power and limitations. Besides giving us a clearer understanding of what finite automata are and what they can and cannot do, the theoretical development allows us to practice and become more comfortable with mathematical definitions, theorems, and proofs in a relatively simple setting.

In beginning to describe the mathematical theory of finite automata, we do so in the abstract, without reference to any particular application. The following figure depicts a finite automaton called M_1.

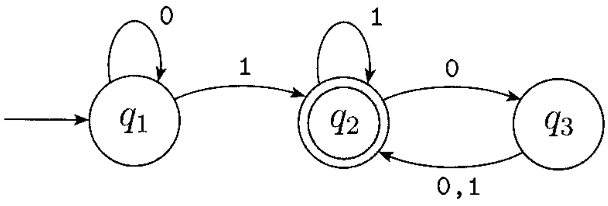


FIGURE 1.4
A finite automaton called M_1 that has three states

Figure 1.4 is called the ***state diagram*** of M_1. It has three ***states***, labeled q_1, q_2, and q_3. The ***start state***, q_1, is indicated by the arrow pointing at it from nowhere. The ***accept state***, q_2, is the one with a double circle. The arrows going from one state to another are called ***transitions***.

When this automaton receives an input string such as 1101, it processes that string and produces an output. The output is either ***accept*** or ***reject***. We will consider only this yes/no type of output for now to keep things simple. The processing begins in M_1's start state. The automaton receives the symbols from the input string one by one from left to right. After reading each symbol, M_1 moves from one state to another along the transition that has that symbol as its label. When it reads the last symbol, M_1 produces its output. The output is *accept* if M_1 is now in an accept state and *reject* if it is not.

For example, when we feed the input string 1101 to the machine M_1 in Figure 1.4, the processing proceeds as follows.

1. start in state q_1;
2. read 1, follow transition from q_1 to q_2;
3. read 1, follow transition from q_2 to q_2;
4. read 0, follow transition from q_2 to q_3;
5. read 1, follow transition from q_3 to q_2;
6. *accept* because M_1 is in an accept state q_2 at the end of the input.

Experimenting with this machine on a variety of input strings reveals that it accepts the strings 1, 01, 11, and 0101010101. In fact, M_1 accepts any string that ends with a 1, as it goes to its accept state q_2 whenever it reads the symbol 1. In addition, it accepts strings 100, 0100, 110000, and 0101000000, and any string that ends with an even number of 0s following the last 1. It rejects other strings, such as 0, 10, 101000. Can you describe the language consisting of all strings that M_1 accepts? We will do so shortly.

FORMAL DEFINITION OF A FINITE AUTOMATON

In the preceding section we used state diagrams to introduce finite automata. Now we define finite automata formally. Though state diagrams are easier to grasp intuitively, we need the formal definition, too, for two specific reasons.

First, a formal definition is precise. It resolves any uncertainties about what is allowed in a finite automaton. If you were uncertain about whether finite automata were allowed to have 0 accept states or whether they must have exactly one transition exiting every state for each possible input symbol, you could consult the formal definition and verify that the answer is yes in both cases. Second, a formal definition provides notation. Good notation helps you think and express your thoughts clearly.

The language of a formal definition is somewhat arcane, having some similarity to the language of a legal document. Both need to be precise, and every detail must be spelled out.

A finite automaton has several parts. It has a set of states and rules for going from one state to another, depending on the input symbol. It has an input alphabet that indicates the allowed input symbols. It has a start state and a set of accept states. The formal definition says that a finite automaton is a list of those five objects: set of states, input alphabet, rules for moving, start state, and accept states. In mathematical language a list of five elements is often called a 5-tuple. Hence we define a finite automaton to be a 5-tuple consisting of these five parts.

We use something called a ***transition function***, frequently denoted δ, to define the rules for moving. If the finite automaton has an arrow from a state x to a state y labeled with the input symbol 1, that means that, if the automaton is in state x when it reads a 1, it then moves to state y. We can indicate the same thing with the transition function by saying that $\delta(x, 1) = y$. This notation is a kind of mathematical shorthand. Putting it all together we arrive at the formal definition of finite automata.

DEFINITION 1.1

A ***finite automaton*** is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$, where

1. Q is a finite set called the ***states***,
2. Σ is a finite set called the ***alphabet***,
3. $\delta\colon Q \times \Sigma \longrightarrow Q$ is the ***transition function***,[1]
4. $q_0 \in Q$ is the ***start state***, and
5. $F \subseteq Q$ is the ***set of accept states***.[2]

The formal definition precisely describes what we mean by a finite automaton. For example, returning to the earlier question of whether 0 accept states is

[1] Refer back to page 7 if you are uncertain about the meaning of $\delta\colon Q \times \Sigma \longrightarrow Q$.

[2] Accept states sometimes are called ***final states***.

allowable, you can see that setting F to be the empty set $\emptyset$ yields 0 accept states, which is allowable. Furthermore the transition function δ specifies exactly one next state for each possible combination of a state and an input symbol. That answers our other question affirmatively, showing that exactly one transition arrow exits every state for each possible input symbol.

We can use the notation of the formal definition to describe individual finite automata by specifying each of the five parts listed in Definition 1.1. For example, let's return to the finite automaton M_1 depicted in Figure 1.4.

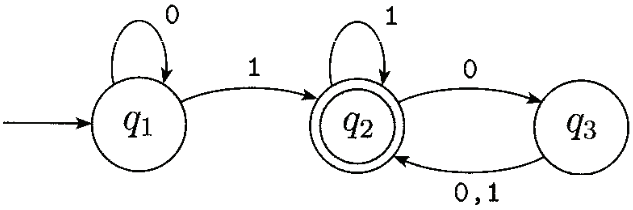


FIGURE 1.5
The finite automaton M_1

We can describe M_1 formally by writing $M_1 = (Q, \Sigma, \delta, q_1, F)$, where

1. $Q = \{q_1, q_2, q_3\}$,
2. $\Sigma = \{0,1\}$,
3. δ is described as

	0	1
q_1	q_1	q_2
q_2	q_3	q_2
q_3	q_2	q_2

,

4. q_1 is the start state, and
5. $F = \{q_2\}$.

If A is the set of all strings that machine M accepts, we say that A is the ***language of machine M*** and write $L(M) = A$. We say that ***M recognizes A*** or that ***M accepts A***. Because the term *accept* has different meanings when we refer to machines accepting strings and machines accepting languages, we prefer the term *recognize* for languages in order to avoid confusion.

A machine may accept several strings, but it always recognizes only one language. If the machine accepts no strings, it still recognizes one language, namely, the empty language $\emptyset$.

In our example, let

$$A = \{w \mid w \text{ contains at least one } 1 \text{ and}$$
$$\text{an even number of 0s follow the last } 1\}.$$

Then $L(M_1) = A$, or equivalently, M_1 recognizes A.

EXAMPLES OF FINITE AUTOMATA

EXAMPLE 1.2

The following is the state diagram of finite automaton M_2.

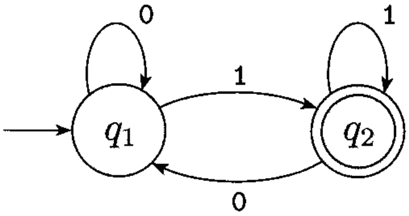


FIGURE 1.6
State diagram of the two-state finite automaton M_2

In the formal description $M_2 = (\{q_1, q_2\}, \{0,1\}, \delta, q_1, \{q_2\})$. The transition function δ is

	0	1
q_1	q_1	q_2
q_2	q_1	q_2

Remember that the state diagram of M_2 and the formal description of M_2 contain the same information, only in different form. You can always go from one to the other if necessary.

A good way to begin understanding any machine is to try it on some sample input strings. When you do these "experiments" to see how the machine is working, its method of functioning often becomes apparent. On the sample string 1101 the machine M_2 starts in its start state q_1 and proceeds first to state q_2 after reading the first 1, and then to states q_2, q_1, and q_2 after reading 1, 0, and 1. The string is accepted because the state q_2 is an accept state. But string 110 leaves M_2 in state q_1, so it is rejected. After trying a few more examples, you would see that M_2 accepts all strings that end in a 1. Thus $L(M_2) = \{w \mid w \text{ ends in a } 1\}$. ▪

EXAMPLE 1.3

Consider the finite automaton M_3.

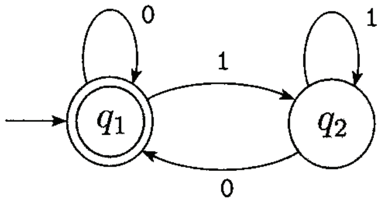


FIGURE 1.7
State diagram of the two-state finite automaton M_3

Machine M_3 is similar to M_2 except for the location of the accept state. As usual, the machine accepts all strings that leave it in an accept state when it has finished reading. Note that, because the start state is also an accept state, M_3 accepts the empty string ε. As soon as a machine begins reading the empty string it is at the end, so if the start state is an accept state, ε is accepted. In addition to the empty string, this machine accepts any string ending with a 0. Here,

$$L(M_3) = \{w \mid w \text{ is the empty string } \varepsilon \text{ or ends in a } 0\}.$$

EXAMPLE 1.4

The following figure shows a five-state machine M_4

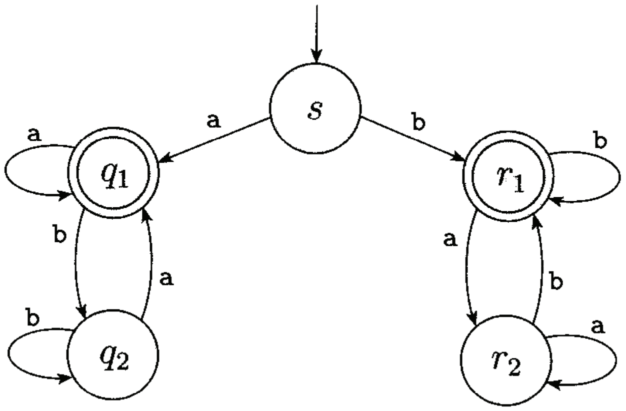


FIGURE 1.8
Finite automaton M_4

M_4 has two accept states, q_1 and r_1 and operates over the alphabet $\Sigma = \{\texttt{a}, \texttt{b}\}$. Some experimentation shows that it accepts strings a, b, aa, bb, and bab, but not strings ab, ba, or bbba. This machine begins in state s, and after it reads the first symbol in the input, it either goes left into the q states or right into the r states. In either case it can never return to the start state (in contrast to the previous examples), as it has no way to get from any other state back to s. If the first symbol in the input string is a, then it goes left and accepts when the string ends with an a. Similarly, if the first symbol is a b, the machine goes right, and accepts when the string ends in b. So M_4 accepts all strings that start and end with a, or that start and end with b. In other words, M_4 accepts strings that start and end with the same symbol.

EXAMPLE 1.5

The following diagram shows machine M_5, which has a four-symbol input alphabet, $\Sigma = \{\langle\text{RESET}\rangle, 0, 1, 2\}$. We treat $\langle\text{RESET}\rangle$ as a single symbol.

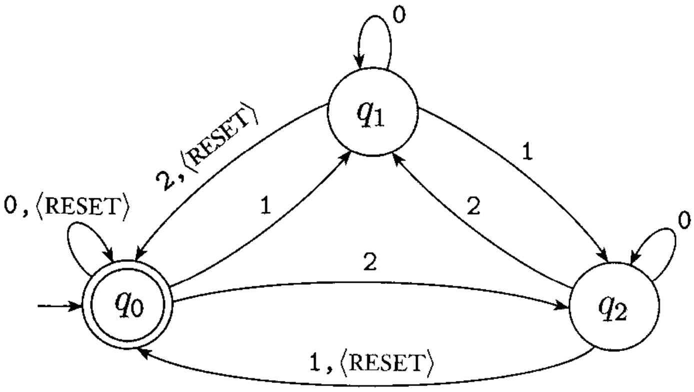


FIGURE 1.9
Finite automaton M_5

M_5 keeps a running count of the sum of the numerical input symbols it reads, modulo 3. Every time it receives the $\langle\text{RESET}\rangle$ symbol it resets the count to 0. It accepts if the sum is 0, modulo 3, or in other words, if the sum is a multiple of 3.

EXAMPLE 1.6

Describing a finite automaton by state diagram is not possible in some cases. That may occur when the diagram would be too big to draw or if, as in this example, the description depends on some unspecified parameter. In these cases we resort to a formal description to specify the machine.

Consider a generalization of Example 1.5 using the same four symbol alphabet Σ. For each $i \geq 1$ let A_i be the language of all strings where the sum of the numbers is a multiple of i, except that the sum is reset to 0 whenever the symbol $\langle\text{RESET}\rangle$ appears. For each A_i we give a finite automaton B_i recognizing A_i. We describe the machine B_i formally as follows: $B_i = (Q_i, \Sigma, \delta_i, q_0, \{q_0\})$, where Q_i is the set of i states $\{q_0, q_1, q_2, \ldots, q_{i-1}\}$, and we design the transition function δ_i so that for each j, if B_i is in q_j, the running sum is j, modulo i. For each q_j let

$$\begin{aligned}
&\delta_i(q_j, 0) = q_j,\\
&\delta_i(q_j, 1) = q_k \text{ where } k = j + 1 \text{ modulo } i,\\
&\delta_i(q_j, 2) = q_k \text{ where } k = j + 2 \text{ modulo } i, \text{ and}\\
&\delta_i(q_j, \langle\text{RESET}\rangle) = q_0.
\end{aligned}$$

FORMAL DEFINITION OF COMPUTATION

So far we have described finite automata informally, using state diagrams, and with a formal definition, as a 5-tuple. The informal description is easier to grasp at first, but the formal definition is useful for making the notion totally precise, resolving any ambiguities that may have occurred in the informal description. Next we do the same for a finite automaton's computation. We already have an informal idea of the way it computes, and we now formalize it mathematically.

Let $M = (Q, \Sigma, \delta, q_0, F)$ be a finite automaton and $w = w_1 w_2 \cdots w_n$ be a string over the alphabet Σ. Then M ***accepts*** w if a sequence of states $r_0, r_1, \ldots, r_n$ exists in Q with the following three conditions:

1. $r_0 = q_0$,
2. $\delta(r_i, w_{i+1}) = r_{i+1}$ for $i = 0, \ldots, n-1$, and
3. $r_n \in F$.

Condition 1 says that the machine starts in the start state. Condition 2 says that the machine goes from state to state according to the transition function. Condition 3 says that the machine accepts its input if it ends up in an accept state. We say that M ***recognizes language*** A if $A = \{w \mid M \text{ accepts } w\}$.

DEFINITION 1.7

A language is called a ***regular language*** if some finite automaton recognizes it.

EXAMPLE 1.8

Take machine M_5 from Example 1.5. Let w be the string

$$\texttt{10}\langle\text{RESET}\rangle\texttt{22}\langle\text{RESET}\rangle\texttt{012}$$

Then M_5 accepts w according to the formal definition of computation because the sequence of states it enters when computing on w is

$$q_0, q_1, q_1, q_0, q_2, q_1, q_0, q_0, q_1, q_0,$$

which satisfies the three conditions. The language of M_5 is

$$L(M_5) = \{w \mid \text{the sum of the symbols in } w \text{ is } 0 \text{ modulo } 3, \\ \text{except that } \langle\text{RESET}\rangle \text{ resets the count to } 0\}.$$

As M_5 recognizes this language, it is a regular language.

DESIGNING FINITE AUTOMATA

Whether it be of automaton or artwork, design is a creative process. As such it cannot be reduced to a simple recipe or formula. However, you might find a particular approach helpful when designing various types of automata. That is, put *yourself* in the place of the machine you are trying to design and then see how you would go about performing the machine's task. Pretending that you are the machine is a psychological trick that helps engage your whole mind in the design process.

Let's design a finite automaton using the "reader as automaton" method just described. Suppose that you are given some language and want to design a finite automaton that recognizes it. Pretending to be the automaton, you receive an input string and must determine whether it is a member of the language the automaton is supposed to recognize. You get to see the symbols in the string one by one. After each symbol you must decide whether the string seen so far is in the language. The reason is that you, like the machine, don't know when the end of the string is coming, so you must always be ready with the answer.

First, in order to make these decisions, you have to figure out what you need to remember about the string as you are reading it. Why not simply remember all you have seen? Bear in mind that you are pretending to be a finite automaton and that this type of machine has only a finite number of states, which means a finite memory. Imagine that the input is extremely long, say, from here to the moon, so that you could not possibly remember the entire thing. You have a finite memory, say, a single sheet of paper, which has a limited storage capacity. Fortunately, for many languages you don't need to remember the entire input. You only need to remember certain crucial information. Exactly which information is crucial depends on the particular language considered.

For example, suppose that the alphabet is {0,1} and that the language consists of all strings with an odd number of 1s. You want to construct a finite automaton E_1 to recognize this language. Pretending to be the automaton, you start getting an input string of 0s and 1s symbol by symbol. Do you need to remember the entire string seen so far in order to determine whether the number of 1s is odd? Of course not. Simply remember whether the number of 1s seen so far is even or odd and keep track of this information as you read new symbols. If you read a 1, flip the answer, but if you read a 0, leave the answer as is.

But how does this help you design E_1? Once you have determined the necessary information to remember about the string as it is being read, you represent this information as a finite list of possibilities. In this instance, the possibilities would be

1. even so far, and
2. odd so far.

Then you assign a state to each of the possibilities. These are the states of E_1, as shown in the following figure.

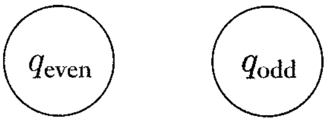


FIGURE 1.10
The two states q_{even} and q_{odd}

Next, you assign the transitions by seeing how to go from one possibility to another upon reading a symbol. So, if state q_{even} represents the even possibility and state q_{odd} represents the odd possibility, you would set the transitions to flip state on a 1 and stay put on a 0, as shown in the following following figure.

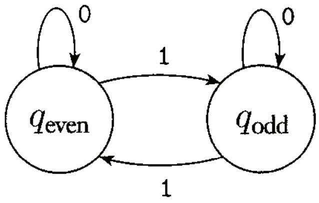


FIGURE 1.11
Transitions telling how the possibilities rearrange

Next, you set the start state to be the state corresponding to the possibility associated with having seen 0 symbols so far (the empty string ε). In this case the start state corresponds to state q_{even} because 0 is an even number. Last, set the accept states to be those corresponding to possibilities where you want to accept the input string. Set q_{odd} to be an accept state because you want to accept when you have seen an odd number of 1s. These additions are shown in the following figure.

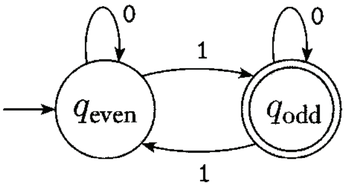


FIGURE 1.12
Adding the start and accept states

EXAMPLE 1.9

This example shows how to design a finite automaton E_2 to recognize the regular language of all strings that contain the string 001 as a substring. For example, 0010, 1001, 001, and 11111110011111 are all in the language, but 11 and 0000 are not. How would you recognize this language if you were pretending to be E_2? As symbols come in, you would initially skip over all 1s. If you come to a 0, then you note that you may have just seen the first of the three symbols in the pattern 001 you are seeking. If at this point you see a 1, there were too few 0s, so you go back to skipping over 1s. But if you see a 0 at that point, you should remember that you have just seen two symbols of the pattern. Now you simply need to continue scanning until you see a 1. If you find it, remember that you succeeded in finding the pattern and continue reading the input string until you get to the end.

So there are four possibilities: You

1. haven't just seen any symbols of the pattern,
2. have just seen a 0,
3. have just seen 00, or
4. have seen the entire pattern 001.

Assign the states q, q_0, q_{00}, and q_{001} to these possibilities. You can assign the transitions by observing that from q reading a 1 you stay in q, but reading a 0 you move to q_0. In q_0 reading a 1 you return to q, but reading a 0 you move to q_{00}. In q_{00}, reading a 1 you move to q_{001}, but reading a 0 leaves you in q_{00}. Finally, in q_{001} reading a 0 or a 1 leaves you in q_{001}. The start state is q, and the only accept state is q_{001}, as shown in the following figure.

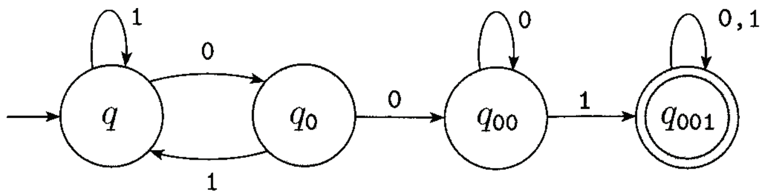


FIGURE 1.13
Accepts strings containing 001

THE REGULAR OPERATIONS

In the preceding two sections we introduced and defined finite automata and regular languages. We now begin to investigate their properties. Doing so will help develop a toolbox of techniques to use when you design automata to recognize particular languages. The toolbox also will include ways of proving that certain other languages are nonregular (i.e., beyond the capability of finite automata).

In arithmetic, the basic objects are numbers and the tools are operations for manipulating them, such as + and ×. In the theory of computation the objects are languages and the tools include operations specifically designed for manipulating them. We define three operations on languages, called the ***regular operations***, and use them to study properties of the regular languages.

DEFINITION 1.10

Let A and B be languages. We define the regular operations ***union***, ***concatenation***, and ***star*** as follows.

- **Union**: $A \cup B = \{x|\, x \in A \text{ or } x \in B\}$.
- **Concatenation**: $A \circ B = \{xy|\, x \in A \text{ and } y \in B\}$.
- **Star**: $A^* = \{x_1x_2 \ldots x_k|\, k \geq 0 \text{ and each } x_i \in A\}$.

You are already familiar with the union operation. It simply takes all the strings in both A and B and lumps them together into one language.

The concatenation operation is a little trickier. It attaches a string from A in front of a string from B in all possible ways to get the strings in the new language.

The star operation is a bit different from the other two because it applies to a single language rather than two. That is, the star operation is a ***unary operation*** instead of a ***binary operation***. It works by attaching any number of strings in A together to get a string in the new language. Because "any number" includes 0 as a possibility, the empty string ε is always a member of A^*, no matter what A is.

EXAMPLE 1.11

Let the alphabet Σ be the standard 26 letters $\{\texttt{a}, \texttt{b}, \ldots, \texttt{z}\}$. If $A = \{\texttt{good}, \texttt{bad}\}$ and $B = \{\texttt{boy}, \texttt{girl}\}$, then

$$A \cup B = \{\texttt{good}, \texttt{bad}, \texttt{boy}, \texttt{girl}\},$$

$$A \circ B = \{\texttt{goodboy}, \texttt{goodgirl}, \texttt{badboy}, \texttt{badgirl}\}, \text{ and}$$

$$A^* = \{\varepsilon, \texttt{good}, \texttt{bad}, \texttt{goodgood}, \texttt{goodbad}, \texttt{badgood}, \texttt{badbad}, \texttt{goodgoodgood}, \texttt{goodgoodbad}, \texttt{goodbadgood}, \texttt{goodbadbad}, \ldots\}.$$

Let $\mathcal{N} = \{1, 2, 3, \ldots\}$ be the set of natural numbers. When we say that $\mathcal{N}$ is *closed under multiplication* we mean that, for any x and y in $\mathcal{N}$, the product $x \times y$ also is in $\mathcal{N}$. In contrast $\mathcal{N}$ is not closed under division, as 1 and 2 are in $\mathcal{N}$ but $1/2$ is not. Generally speaking, a collection of objects is ***closed*** under some operation if applying that operation to members of the collection returns an object still in the collection. We show that the collection of regular languages is closed under all three of the regular operations. In Section 1.3 we show that these are useful tools for manipulating regular languages and understanding the power of finite automata. We begin with the union operation.

THEOREM 1.12

The class of regular languages is closed under the union operation.

In other words, if A_1 and A_2 are regular languages, so is $A_1 \cup A_2$.

PROOF IDEA We have regular languages A_1 and A_2 and want to show that $A_1 \cup A_2$ also is regular. Because A_1 and A_2 are regular, we know that some finite automaton M_1 recognizes A_1 and some finite automaton M_2 recognizes A_2. To prove that $A_1 \cup A_2$ is regular we demonstrate a finite automaton, call it M, that recognizes $A_1 \cup A_2$.

This is a proof by construction. We construct M from M_1 and M_2. Machine M must accept its input exactly when either M_1 or M_2 would accept it in order to recognize the union language. It works by *simulating* both M_1 and M_2 and accepting if either of the simulations accept.

How can we make machine M simulate M_1 and M_2? Perhaps it first simulates M_1 on the input and then simulates M_2 on the input. But we must be careful here! Once the symbols of the input are read and used to simulate M_1, we cannot "rewind the input tape" to try the simulation on M_2. We need another approach.

Pretend that you are M. As the input symbols arrive one by one, you simulate both M_1 and M_2 simultaneously. That way only one pass through the input is necessary. But can you keep track of both simulations with finite memory? All you need to remember is the state that each machine would be in if it had read up to this point in the input. Therefore you need to remember a pair of states. How many possible pairs are there? If M_1 has k_1 states and M_2 has k_2 states, the number of pairs of states, one from M_1 and the other from M_2, is the product $k_1 \times k_2$. This product will be the number of states in M, one for each pair. The transitions of M go from pair to pair, updating the current state for both M_1 and M_2. The accept states of M are those pairs wherein either M_1 or M_2 is in an accept state.

PROOF

Let M_1 recognize A_1, where $M_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$, and
M_2 recognize A_2, where $M_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$.

Construct M to recognize $A_1 \cup A_2$, where $M = (Q, \Sigma, \delta, q_0, F)$.

1. $Q = \{(r_1, r_2) | \, r_1 \in Q_1 \text{ and } r_2 \in Q_2\}$.
   This set is the ***Cartesian product*** of sets Q_1 and Q_2 and is written $Q_1 \times Q_2$. It is the set of all pairs of states, the first from Q_1 and the second from Q_2.
2. Σ, the alphabet, is the same as in M_1 and M_2. In this theorem and in all subsequent similar theorems, we assume for simplicity that both M_1 and M_2 have the same input alphabet Σ. The theorem remains true if they have different alphabets, Σ_1 and Σ_2. We would then modify the proof to let $\Sigma = \Sigma_1 \cup \Sigma_2$.
3. δ, the transition function, is defined as follows. For each $(r_1, r_2) \in Q$ and each $a \in \Sigma$, let

   $$\delta\big((r_1, r_2), a\big) = \big(\delta_1(r_1, a), \delta_2(r_2, a)\big).$$

   Hence δ gets a state of M (which actually is a pair of states from M_1 and M_2), together with an input symbol, and returns M's next state.
4. q_0 is the pair (q_1, q_2).
5. F is the set of pairs in which either member is an accept state of M_1 or M_2. We can write it as

   $$F = \{(r_1, r_2) | \, r_1 \in F_1 \text{ or } r_2 \in F_2\}.$$

   This expression is the same as $F = (F_1 \times Q_2) \cup (Q_1 \times F_2)$. (Note that it is *not* the same as $F = F_1 \times F_2$. What would that give us instead?[3])

This concludes the construction of the finite automaton M that recognizes the union of A_1 and A_2. This construction is fairly simple, and thus its correctness is evident from the strategy that is described in the proof idea. More complicated constructions require additional discussion to prove correctness. A formal

[3] This expression would define M's accept states to be those for which *both* members of the pair are accept states. In this case M would accept a string only if both M_1 *and* M_2 accept it, so the resulting language would be the *intersection* and not the union. In fact, this result proves that the class of regular languages is closed under intersection.

correctness proof for a construction of this type usually proceeds by induction. For an example of a construction proved correct, see the proof of Theorem 1.28. Most of the constructions that you will encounter in this course are fairly simple and so do not require a formal correctness proof.

We have just shown that the union of two regular languages is regular, thereby proving that the class of regular languages is closed under the union operation. We now turn to the concatenation operation and attempt to show that the class of regular languages is closed under that operation, too.

THEOREM 1.13

The class of regular languages is closed under the concatenation operation.

In other words, if A_1 and A_2 are regular languages then so is $A_1 \circ A_2$.

To prove this theorem let's try something along the lines of the proof of the union case. As before, we can start with finite automata M_1 and M_2 recognizing the regular languages A_1 and A_2. But now, instead of constructing automaton M to accept its input if either M_1 or M_2 accept, it must accept if its input can be broken into two pieces, where M_1 accepts the first piece and M_2 accepts the second piece. The problem is that M doesn't know where to break its input (i.e., where the first part ends and the second begins). To solve this problem we introduce a new technique called nondeterminism.

1.2 NONDETERMINISM

Nondeterminism is a useful concept that has had great impact on the theory of computation. So far in our discussion, every step of a computation follows in a unique way from the preceding step. When the machine is in a given state and reads the next input symbol, we know what the next state will be—it is determined. We call this ***deterministic*** computation. In a ***nondeterministic*** machine, several choices may exist for the next state at any point.

Nondeterminism is a generalization of determinism, so every deterministic finite automaton is automatically a nondeterministic finite automaton. As the following figure shows, nondeterministic finite automata may have additional features.

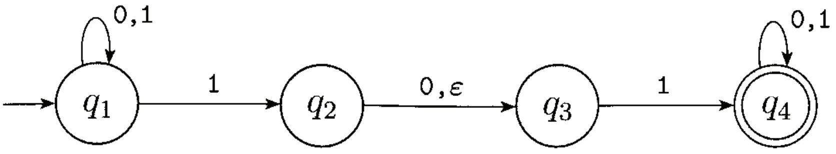


FIGURE 1.14
The nondeterministic finite automaton N_1

The difference between a deterministic finite automaton, abbreviated DFA, and a nondeterministic finite automaton, abbreviated NFA, is immediately apparent. First, every state of a DFA always has exactly one exiting transition arrow for each symbol in the alphabet. The nondeterministic automaton shown in Figure 1.14 violates that rule. State q_1 has one exiting arrow for 0, but it has two for 1; q_2 has one arrow for 0, but it has none for 1. In an NFA a state may have zero, one, or many exiting arrows for each alphabet symbol.

Second, in a DFA, labels on the transition arrows are symbols from the alphabet. This NFA has an arrow with the label ε. In general, an NFA may have arrows labeled with members of the alphabet or ε. Zero, one, or many arrows may exit from each state with the label ε.

How does an NFA compute? Suppose that we are running an NFA on an input string and come to a state with multiple ways to proceed. For example, say that we are in state q_1 in NFA N_1 and that the next input symbol is a 1. After reading that symbol, the machine splits into multiple copies of itself and follows *all* the possibilities in parallel. Each copy of the machine takes one of the possible ways to proceed and continues as before. If there are subsequent choices, the machine splits again. If the next input symbol doesn't appear on any of the arrows exiting the state occupied by a copy of the machine, that copy of the machine dies, along with the branch of the computation associated with it. Finally, if *any one* of these copies of the machine is in an accept state at the end of the input, the NFA accepts the input string.

If a state with an ε symbol on an exiting arrow is encountered, something similar happens. Without reading any input, the machine splits into multiple copies, one following each of the exiting ε-labeled arrows and one staying at the current state. Then the machine proceeds nondeterministically as before.

Nondeterminism may be viewed as a kind of parallel computation wherein several "processes" can be running concurrently. When the NFA splits to follow several choices, that corresponds to a process "forking" into several children, each proceeding separately. If at least one of these processes accepts then the entire computation accepts.

Another way to think of a nondeterministic computation is as a tree of possibilities. The root of the tree corresponds to the start of the computation. Every branching point in the tree corresponds to a point in the computation at which the machine has multiple choices. The machine accepts if at least one of the computation branches ends in an accept state, as shown in the following figure.

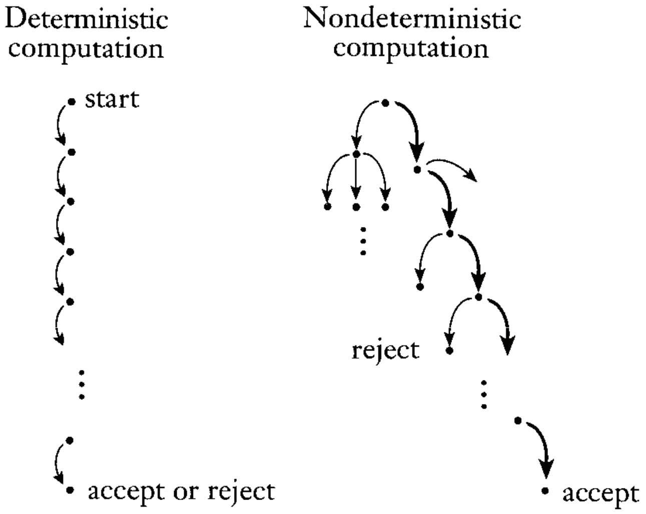


FIGURE 1.15
Deterministic and nondeterministic computations with an accepting branch

Let's consider some sample runs of the NFA N_1 shown in Figure 1.14. On input 010110 start in the start state q_1 and read the first symbol 0. From q_1 there is only one place to go on a 0, namely, back to q_1, so remain there.

Next read the second symbol 1. In q_1 on a 1 there are two choices: either stay in q_1 or move to q_2. Nondeterministically, the machine splits in two to follow each choice. Keep track of the possibilities by placing a finger on each state where a machine could be. So you now have fingers on states q_1 and q_2. An ε arrow exits state q_2 so the machine splits again; keep one finger on q_2, and move the other to q_3. You now have fingers on q_1, q_2, and q_3.

When the third symbol 0 is read, take each finger in turn. Keep the finger on q_1 in place, move the finger on q_2 to q_3, and remove the finger that has been on q_3. That last finger had no 0 arrow to follow and corresponds to a process that simply "dies." At this point you have fingers on states q_1 and q_3.

When the fourth symbol 1 is read, split the finger on q_1 into fingers on states q_1 and q_2, then further split the finger on q_2 to follow the ε arrow to q_3, and move the finger that was on q_3 to q_4. You now have a finger on each of the four states.

When the fifth symbol 1 is read, the fingers on q_1 and q_3 result in fingers on states q_1, q_2, q_3, and q_4, as you saw with the fourth symbol. The finger on state q_2 is removed. The finger that was on q_4 stays on q_4. Now you have two fingers on q_4, so remove one, because you only need to remember that q_4 is a possible state at this point, not that it is possible for multiple reasons.

When the sixth and final symbol 0 is read, keep the finger on q_1 in place, move the one on q_2 to q_3, remove the one that was on q_3, and leave the one on q_4 in place. You are now at the end of the string, and you accept if some finger is on an accept state. You have fingers on states q_1, q_3, and q_4, and as q_4 is an accept state, N_1 accepts this string. The computation of N_1 on input 010110 is depicted in Figure 1.16.

What does N_1 do on input 010? Start with a finger on q_1. After reading the 0 you still have a finger only on q_1, but after the 1 there are fingers on q_1, q_2, and q_3 (don't forget the ε arrow). After the third symbol 0, remove the finger on q_3, move the finger on q_2 to q_3, and leave the finger on q_1 where it is. At this point you are at the end of the input, and as no finger is on an accept state, N_1 rejects this input.

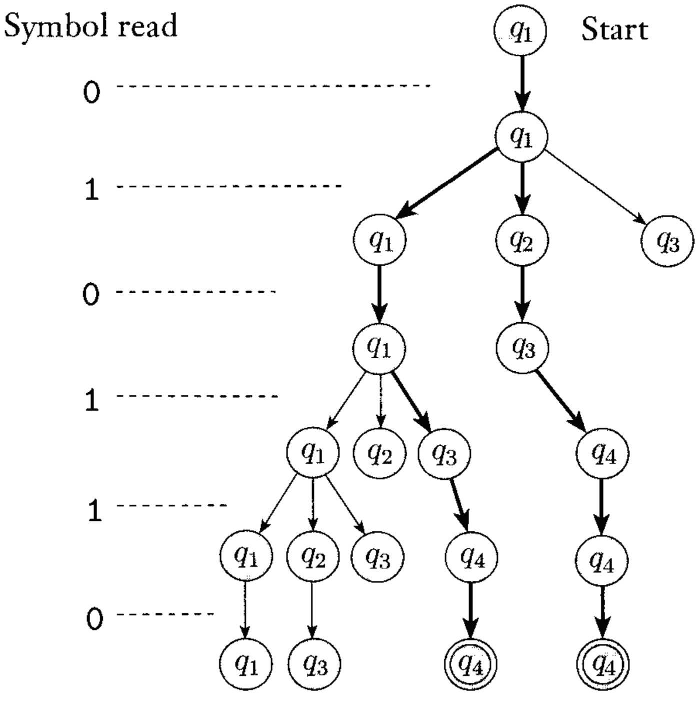


FIGURE 1.16
The computation of N_1 on input 010110

By continuing to experiment in this way, you will see that N_1 accepts all strings that contain either 101 or 11 as a substring.

Nondeterministic finite automata are useful in several respects. As we will show, every NFA can be converted into an equivalent DFA, and constructing NFAs is sometimes easier than directly constructing DFAs. An NFA may be much smaller than its deterministic counterpart, or its functioning may be easier to understand. Nondeterminism in finite automata is also a good introduction to nondeterminism in more powerful computational models because finite automata are especially easy to understand. Now we turn to several examples of NFAs.

EXAMPLE 1.14

Let A be the language consisting of all strings over $\{0,1\}$ containing a 1 in the third position from the end (e.g., 000100 is in A but 0011 is not). The following four-state NFA N_2 recognizes A.

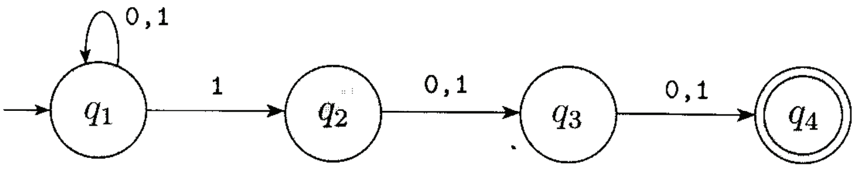


FIGURE 1.17
The NFA N_2 recognizing A

One good way to view the computation of this NFA is to say that it stays in the start state q_1 until it "guesses" that it is three places from the end. At that point, if the input symbol is a 1, it branches to state q_2 and uses q_3 and q_4 to "check" on whether its guess was correct.

As mentioned, every NFA can be converted into an equivalent DFA, but sometimes that DFA may have many more states. The smallest DFA for A contains eight states. Furthermore, understanding the functioning of the NFA is much easier, as you may see by examining the following figure for the DFA.

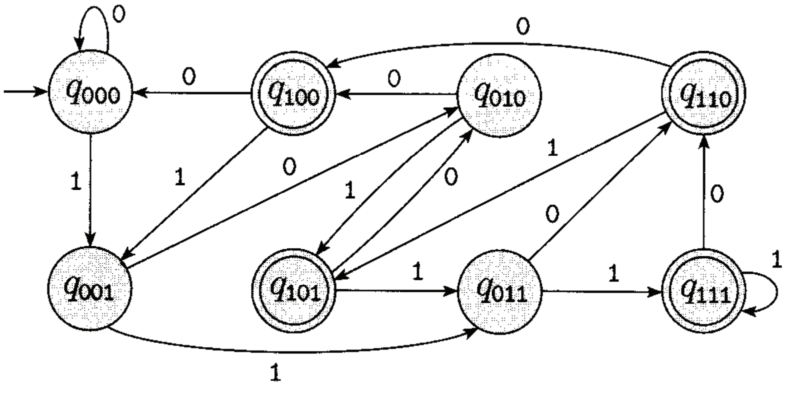


FIGURE 1.18
A DFA recognizing A

Suppose that we added ε to the labels on the arrows going from q_2 to q_3 and from q_3 to q_4 in machine N_2 in Figure 1.17. In other words, both arrows would then have the label 0, 1, ε instead of just 0, 1. What language would N_2 recognize with this modification? Try modifying the DFA in Figure 1.18 to recognize that language.

EXAMPLE 1.15

Consider the following NFA N_3 that has an input alphabet $\{0\}$ consisting of a single symbol. An alphabet containing only one symbol is called a ***unary*** alphabet.

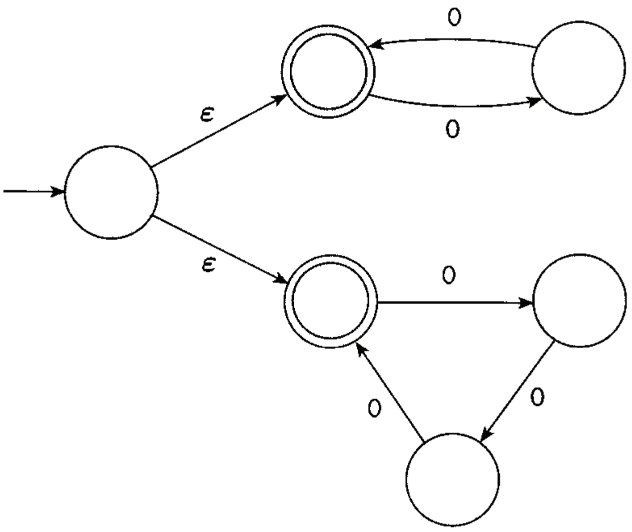


FIGURE **1.19**
The NFA N_3

This machine demonstrates the convenience of having ε arrows. It accepts all strings of the form 0^k where k is a multiple of 2 or 3. (Remember that the superscript denotes repetition, not numerical exponentiation.) For example, N_3 accepts the strings ε, 00, 000, 0000, and 000000, but not 0 or 00000.

Think of the machine operating by initially guessing whether to test for a multiple of 2 or a multiple of 3 by branching into either the top loop or the bottom loop and then checking whether its guess was correct. Of course, we could replace this machine by one that doesn't have ε arrows or even any nondeterminism at all, but the machine shown is the easiest one to understand for this language.

EXAMPLE 1.16

We give another example of an NFA in the following figure. Practice with it to satisfy yourself that it accepts the strings ε, a, baba, and baa, but that it doesn't accept the strings b, bb, and babba. Later we use this machine to illustrate the procedure for converting NFAs to DFAs.

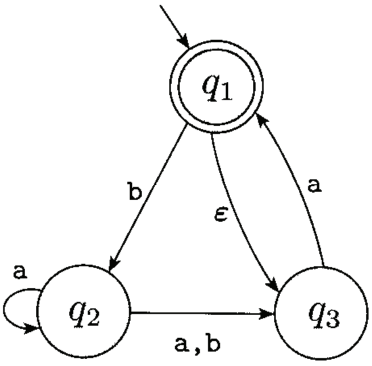


FIGURE 1.20
The NFA N_4

FORMAL DEFINITION OF A NONDETERMINISTIC FINITE AUTOMATON

The formal definition of a nondeterministic finite automaton is similar to that of a deterministic finite automaton. Both have states, an input alphabet, a transition function, a start state, and a collection of accept states. However, they differ in one essential way: in the type of transition function. In a DFA the transition function takes a state and an input symbol and produces the next state. In an NFA the transition function takes a state and an input symbol *or the empty string* and produces *the set of possible next states.* In order to write the formal definition, we need to set up some additional notation. For any set Q we write $\mathcal{P}(Q)$ to be the collection of all subsets of Q. Here $\mathcal{P}(Q)$ is called the ***power set*** of Q. For any alphabet Σ we write Σ_ε to be $\Sigma \cup \{\varepsilon\}$. Now we can easily write the formal description of the type of the transition function in an NFA. It is $\delta\colon Q \times \Sigma_\varepsilon \longrightarrow \mathcal{P}(Q)$, and we are ready to give the formal definition.

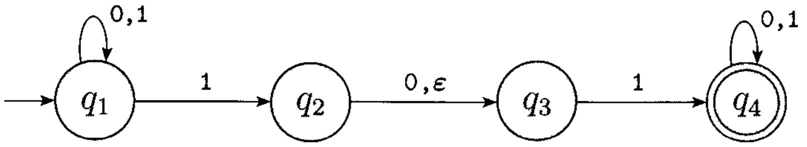
DEFINITION 1.17

A ***nondeterministic finite automaton*** is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$, where

1. Q is a finite set of states,
2. Σ is a finite alphabet,
3. $\delta\colon Q \times \Sigma_\varepsilon \longrightarrow \mathcal{P}(Q)$ is the transition function,
4. $q_0 \in Q$ is the start state, and
5. $F \subseteq Q$ is the set of accept states.

EXAMPLE 1.18

Recall the NFA N_1:



The formal description of N_1 is $(Q, \Sigma, \delta, q_1, F)$, where

1. $Q = \{q_1, q_2, q_3, q_4\}$,
2. $\Sigma = \{0,1\}$,
3. δ is given as

	0	1	ε
q_1	$\{q_1\}$	$\{q_1, q_2\}$	$\emptyset$
q_2	$\{q_3\}$	$\emptyset$	$\{q_3\}$
q_3	$\emptyset$	$\{q_4\}$	$\emptyset$
q_4	$\{q_4\}$	$\{q_4\}$	$\emptyset$

,

4. q_1 is the start state, and
5. $F = \{q_4\}$.

The formal definition of computation for an NFA also is similar to that for a DFA. Let $N = (Q, \Sigma, \delta, q_0, F)$ be an NFA and w a string over the alphabet Σ. Then we say that N ***accepts*** w if we can write w as $w = y_1 y_2 \cdots y_m$, where each y_i is a member of Σ_ε and a sequence of states $r_0, r_1, \ldots, r_m$ exists in Q with the following three conditions:

1. $r_0 = q_0$,
2. $r_{i+1} \in \delta(r_i, y_{i+1})$, for $i = 0, \ldots, m-1$, and
3. $r_m \in F$.

Condition 1 says that the machine starts out in the start state. Condition 2 says that state r_{i+1} is one of the allowable next states when N is in state r_i and reading y_{i+1}. Observe that $\delta(r_i, y_{i+1})$ is the *set* of allowable next states and so we say that r_{i+1} is a member of that set. Finally, Condition 3 says that the machine accepts its input if the last state is an accept state.

EQUIVALENCE OF NFAS AND DFAS

Deterministic and nondeterministic finite automata recognize the same class of languages. Such equivalence is both surprising and useful. It is surprising because NFAs appear to have more power than DFAs, so we might expect that NFAs recognize more languages. It is useful because describing an NFA for a given language sometimes is much easier than describing a DFA for that language.

Say that two machines are ***equivalent*** if they recognize the same language.

THEOREM 1.19

Every nondeterministic finite automaton has an equivalent deterministic finite automaton.

PROOF IDEA If a language is recognized by an NFA, then we must show the existence of a DFA that also recognizes it. The idea is to convert the NFA into an equivalent DFA that simulates the NFA.

Recall the "reader as automaton" strategy for designing finite automata. How would you simulate the NFA if you were pretending to be a DFA? What do you need to keep track of as the input string is processed? In the examples of NFAs you kept track of the various branches of the computation by placing a finger on each state that could be active at given points in the input. You updated the fingers by moving, adding, and removing them according to the way the NFA operates. All you needed to keep track of was the set of states with fingers.

If k is the number of states of the NFA, it has 2^k subsets of states. Each subset corresponds to one of the possibilities that DFA must remember, so the DFA simulating the NFA will have 2^k states. Now we need to figure out which will be the start state and accept states of the DFA, and what will be its transition function. We can discuss this more easily after setting up some formal notation.

PROOF Let $N = (Q, \Sigma, \delta, q_0, F)$ be the NFA recognizing some language A. We construct a DFA M recognizing A. Before doing the full construction, let's first consider the easier case wherein N has no ε arrows. Later we take the ε arrows into account.

Construct $M = (Q', \Sigma, \delta', q_0', F')$.

1. $Q' = \mathcal{P}(Q)$.
   Every state of M is a set of states of N. Recall that $\mathcal{P}(Q)$ is the set of subsets of Q.
2. For $R \in Q'$ and $a \in \Sigma$ let $\delta'(R, a) = \{q \in Q \mid q \in \delta(r, a) \text{ for some } r \in R\}$. If R is a state of M, it is also a set of states of N. When M reads a symbol a in state R, it shows where a takes each state in R. Because each state may go to a set of states, we take the union of all these sets. Another way to write this expression is
   $$\delta'(R, a) = \bigcup_{r \in R} \delta(r, a).^4$$
3. $q_0' = \{q_0\}$.
   M starts in the state corresponding to the collection containing just the start state of N.
4. $F' = \{R \in Q' \mid R \text{ contains an accept state of } N\}$.
   The machine M accepts if one of the possible states that N could be in at this point is an accept state.

[4] The notation $\bigcup_{r \in R} \delta(r, a)$ means: the union of the sets $\delta(r, a)$ for each possible r in R.

Now we need to consider the ε arrows. To do so we set up an extra bit of notation. For any state R of M we define $E(R)$ to be the collection of states that can be reached from R by going only along ε arrows, including the members of R themselves. Formally, for $R \subseteq Q$ let

$$E(R) = \{q|\ q \text{ can be reached from } R \text{ by traveling along 0 or more } \varepsilon \text{ arrows}\}.$$

Then we modify the transition function of M to place additional fingers on all states that can be reached by going along ε arrows after every step. Replacing $\delta(r,a)$ by $E(\delta(r,a))$ achieves this effect. Thus

$$\delta'(R,a) = \{q \in Q|\ q \in E(\delta(r,a)) \text{ for some } r \in R\}.$$

Additionally we need to modify the start state of M to move the fingers initially to all possible states that can be reached from the start state of N along the ε arrows. Changing $q_0{}'$ to be $E(\{q_0\})$ achieves this effect. We have now completed the construction of the DFA M that simulates the NFA N.

The construction of M obviously works correctly. At every step in the computation of M on an input, it clearly enters a state that corresponds to the subset of states that N could be in at that point. Thus our proof is complete.

If the construction used in the preceding proof were more complex we would need to prove that it works as claimed. Usually such proofs proceed by induction on the number of steps of the computation. Most of the constructions that we use in this book are straightforward and so do not require such a correctness proof. To see an example of a more complex construction that we do prove correct turn to the proof of Theorem 1.28.

Theorem 1.19 states that every NFA can be converted into an equivalent DFA. Thus nondeterministic finite automata give an alternative way of characterizing the regular languages. We state this fact as a corollary of Theorem 1.19.

COROLLARY 1.20

A language is regular if and only if some nondeterministic finite automaton recognizes it.

One direction of the "if and only if" states that a language is regular if some NFA recognizes it. Theorem 1.19 shows that any NFA can be converted into an equivalent DFA, so if an NFA recognizes some language, so does some DFA, and hence the language is regular. The other direction states that a language is regular only if some NFA recognizes it. That is, if a language is regular, some NFA must be recognizing it. Obviously, this condition is true because a regular language has a DFA recognizing it and any DFA is also an NFA.

EXAMPLE **1.21**

Let's illustrate the procedure of converting an NFA to a DFA using the machine N_4 that was given in Example 1.16. For clarity, we have relabeled the states of N_4 to be $\{1, 2, 3\}$. Thus in the formal description of $N_4 = (Q, \{\mathtt{a},\mathtt{b}\}, \delta, 1, \{1\})$, the set of states Q is $\{1, 2, 3\}$ as shown in the following figure.

To construct a DFA D that is equivalent to N_4, we first determine D's states. N_4 has three states, $\{1, 2, 3\}$, so we construct D with eight states, one for each subset of N_4's states. We label each of D's states with the corresponding subset. Thus D's state set is

$$\{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}.$$

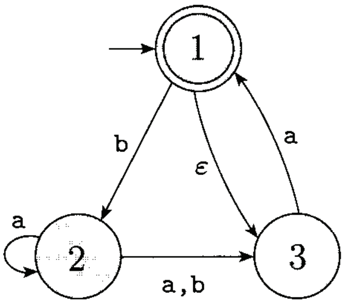


FIGURE **1.21**
The NFA N_4

Next, we determine the start and accept states of D. The start state is $E(\{1\})$, the set of states that are reachable from 1 by traveling along ε arrows, plus 1 itself. An ε arrow goes from 1 to 3, so $E(\{1\}) = \{1, 3\}$. The new accept states are those containing N_4's accept state; thus $\{\{1\}, \{1,2\}, \{1,3\}, \{1,2,3\}\}$.

Finally, we determine D's transition function. Each of D's states goes to one place on input a, and one place on input b. We illustrate the process of determining the placement of D's transition arrows with a few examples.

In D, state $\{2\}$ goes to $\{2,3\}$ on input a, because in N_4, state 2 goes to both 2 and 3 on input a and we can't go farther from 2 or 3 along ε arrows. State $\{2\}$ goes to state $\{3\}$ on input b, because in N_4, state 2 goes only to state 3 on input b and we can't go farther from 3 along ε arrows.

State $\{1\}$ goes to $\emptyset$ on a, because no a arrows exit it. It goes to $\{2\}$ on b.

State $\{3\}$ goes to $\{1,3\}$ on a, because in N_4, state 3 goes to 1 on a and 1 in turn goes to 3 with an ε arrow. State $\{3\}$ on b goes to $\emptyset$.

State $\{1,2\}$ on a goes to $\{2,3\}$ because 1 points at no states with a arrows and 2 points at both 2 and 3 with a arrows and neither point anywhere with ε arrows. State $\{1,2\}$ on b goes to $\{2,3\}$. Continuing in this way we obtain the following diagram for D.

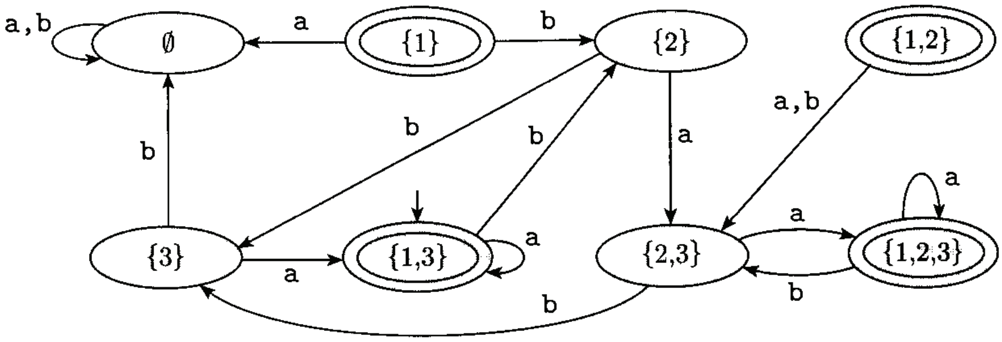


FIGURE 1.22
A DFA D that is equivalent to the NFA N_4

We may simplify this machine by observing that no arrows point at states $\{1\}$ and $\{1, 2\}$, so they may be removed without affecting the performance of the machine. Doing so yields the following figure.

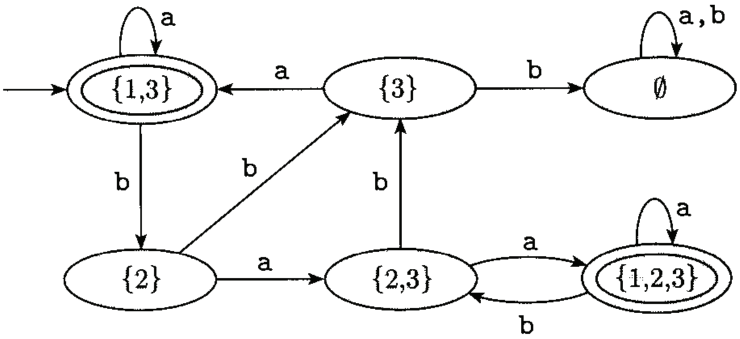


FIGURE 1.23
DFA D after removing unnecessary states

CLOSURE UNDER THE REGULAR OPERATIONS

Now we return to the closure of the class of regular languages under the regular operations that we began in Section 1.1. Our aim is to prove that the union, concatenation, and star of regular languages are still regular. We abandoned the original attempt to do so when dealing with the concatenation operation was too complicated. The use of nondeterminism makes the proofs much easier.

First, let's consider again closure under union. Earlier we proved closure under union by simulating deterministically both machines simultaneously via a