Exercise-set 7.
Solutions

1. a) Vertices of the same color are independent.
   b) It is a good coloring, if each vertex in the complement of a maximum independent set of vertices gets a different color.
2. No (counterexample).
3. a) It is an interval graph ($\exists$ a representation of the vertices with intervals).
   b) It is not an interval graph, since $\chi(G) \neq \omega(G)$.
   c) It is not an interval graph, since there is no representation of the vertices with intervals.
   d) It is an interval graph.
   e) It is not an interval graph (the 5-cycle cannot be represented with intervals).
   f) It is an interval graph.
4. It is an interval graph ($\exists$ a representation of the vertices with intervals).
5. $\chi(G) = \omega(G) = 11$ (=max. # of intervals though a point)
6. $\omega = 10$ in the original graph. We can delete at most 2 vertices from a clique $\Longrightarrow$ $\chi = \omega \geq 8$.
7. $\chi(G) = 3,\ \nu(G) = 9,\ \tau(G) = 12,\ \alpha(G) = 6,\ \rho(G) = 9.$
8. a) $\nu(G) = 4,\ \tau(G) = 4,\ \alpha(G) = 6,\ \rho(G) = 6.$
   b) $\nu(G) = 5,\ \tau(G) = 5,\ \alpha(G) = 7,\ \rho(G) = 7.$
   a) $\nu(G) = 4,\ \tau(G) = 4,\ \alpha(G) = 6,\ \rho(G) = 6.$
9. $G = K_{668} \cup K_{668,669} \implies \chi(G) = 668,\ \nu(G) = 334+668 = 1002,\ \tau(G) = 667+668 = 1335,\ \alpha(G) = 1 + 669 = 670,\ \rho(G) = 334 + 669 = 1003.$
10. $\nu(G) = 20 = \tau(G).$
11. $\alpha(G) = 86,\ \tau(G) = 14,\ \nu(G) = 14,\ \rho(G) = 86.$
12. $\nu(G) = 25,\ \alpha(G) = 75.$
13. a) $\{b, c, g, h\}$ ($\nu(G) = 4$).
    b) $\{b, d, f, h\}$ ($\nu(G) = 4$).
14. $\nu(G) = 4 = \tau(G).$
15. a) By contradiction: otherwise the matching would not be maximum.
    b) Follows from a).
    c) Follows from b) and Gallai's theorem.
16. a) True.
    b) False.
    c) No.
17. G contains a Hamilton cycle $\Longrightarrow$ $\nu(G) \geq \lfloor 2k + 1/2 \rfloor = k$, and $\nu(G) \leq (2k + 1)/2 = k$.
18. If we add the edge $\{u, v\}$ to G then it contains a Hamilton cycle.
19. If we add two new vertices (connected to all the old ones) to G then the new graph contains a Hamilton cycle.
20. $\det M \neq 0 \implies \exists$ a nonzero elementary product, corresponding to a perfect matching.
21. $|E(G)| \leq \binom{20}{2} + 20 \cdot 80 = 1790$, and this is possible (example).
22. $\nu(G) = 10 = \tau(G).$