Exercise-set 5.
Solutions

1. Not possible; possible.
2. a) $|V(G)| = \binom{8}{2} = 28$, $\deg(v) = \binom{6}{2} = 15\ \forall v \in V(G) \implies$ no Euler-circuit.
   b) $|V(G)| = \binom{6}{3} = 20$, $\deg(v) = 1 + 3 \cdot 3 = 10\ \forall v \in V(G)$, and G is connected $\implies \exists$ Euler-circuit.
3. $|V(G)| = 2^4 = 16$, $\deg(v) = \binom{4}{2} = 6\ \forall v \in V(G)$, but G is not connected $\implies$ no Euler-circuit.
4. Construct a graph G: $V(G) =$ digits $= \{0, 1, \dots, 9\}$, and u and v are adjacent $\iff u + v \neq 9$. This graph contains an Euler-circuit ($\deg(v) = 8\ \forall v \in V(G)$, connected) $\iff \exists n$.
5. Construct a graph G: $V(G) =$ letters, and u and v are adjacent $\iff$ can stand next to each other. This graph contains an Euler-circuit ($\deg(v) = 30$ for vowels and $\deg(v) = 10$ for consonants, connected). Length of the longest sequence of letters = length of an Euler-circuit + 1 = $|E(G)| + 1 = \binom{10}{2} + 10 \cdot 21 + 1 = 256$.
6. Construct a graph G: $V(G) =$ children, and u and v are adjacent $\iff$ not next to each other in the circle. This graph contains an Euler-circuit ($\deg(v) = 8\ \forall v \in V(G)$, connected). Most number of passes = length of an Euler-circuit = $|E(G)| = 40$.
7. Wall up the door between F and G. The throne-room is H.
8. There can be at most 2 components $\implies$ adding one edge can make it connected, and the degrees will be OK.
9. r = 1,2,3,4,5,7,9 NO; r = 6,8 YES (each degree is even + connected).
10. YES: both components contain 2 vertices of odd degree.
11. a) There is no Hamilton cycle: if we delete 4 vertices we get 5 components.
    There is a Hamilton path: draw.
    b) There is no Hamilton cycle: if we delete 2 vertices we get 3 components.
    There is a Hamilton path: draw.
    c), d) There is no Hamilton cycle: if we delete 4 vertices we get 5 components.
    There is a Hamilton path: draw.
12. a) Yes (draw); yes.
    b) No (delete 11 vertices); yes (draw).
13. a) If we delete 2 vertices we get 3 components $\implies$ need to add at least 1 edge. That is enough (draw).
    b) If we delete 2 vertices we get 4 components $\implies$ need to add at least 2 edges. That is enough (draw).
14. If we delete 1 vertex (the center) we get 100 components $\implies$ need to add at least 99 edges. That is enough (path).
15. a) Construct a graph G: $V(G) =$ squares, and the edges are the possible moves of the horse. This graph contains no Hamilton path: if we delete the 4 middle vertices we get 6 components.
    b) Construct a graph G: $V(G) =$ squares, and the edges are the possible moves of the horse. This graph contains no Hamilton cycle: if we delete the 12 vertices we get 13 components.
16. Construct a graph G: $V(G) =$ people, and the edges are the acquaintances. Then $\deg(v) \geq 6 = 12/2$ $\implies$ by Dirac's theorem $\exists$ a Hamilton cycle.
17. The condition in Ore's theorem holds for $G \implies \exists$ a Hamilton cycle.
18. Construct a graph G: $V(G) =$ people, and the edges are the acquaintances. G is k-regular for some k. If $k \geq 10 \implies G$ contains a Hamilton cycle, if $k \leq 9 \implies \overline{G}$ contains a Hamilton cycle.
19. a) A cycle on n vertices is like that (check).
    b) E.g. K_7 with the edge $\{u, v\}$ missing and the 8th vertex is connected to u.
20. Add a new vertex to G, and connect it to all the old vertices. Then the new graph contains a Hamilton cycle from which we can get a Hamilton path of G.

21. Delete v from G. Then the new graph contains a Hamilton cycle from which we can get a Hamilton path of G.

22. The 8 edges must have pairwise no common endpoints (i.e. be independent). Every second edge of a Hamilton cycle will do (which exists because $\deg(v) \geq n/2 \ \forall v$).

23. We can add the edges of a Hamilton cycle of $\overline{G}$.

24. Construct a graph G: $V(G)$ = four-element subsets, and they are adjacent, if they have at least two elements in common. Then G contains a Hamilton cycle.

25. Eg. if the graph is $K_{9,11}$.