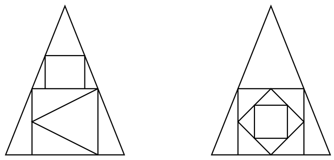
Introduction to the Theory of Computing 2.

Exercise-set 5.

1. If it is possible, draw the figures below with one line, without lifting the pen.



2. a) Let the vertices of the graph G be all the 2-element subsets of the 8-element set $S = \{a, b, c, d, e, f, g, h\}$, and two vertices be adjacent if and only if the corresponding subsets are disjoint. Does this graph G contain an Euler circuit?
b) (MT+'19) Let the vertices of the graph G be all the 3-element subsets of the 6-element set $S = \{a, b, c, d, e, f\}$, and two vertices be adjacent if and only if the corresponding subsets have at most one element in common. Does this graph G contain an Euler circuit?

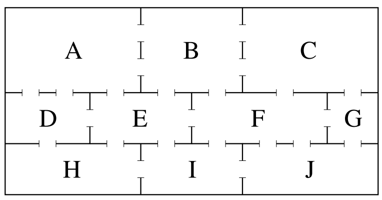
3. Let the vertices of the graph G be all the 0-1 sequences of length 4, and two vertices be adjacent if and only if the corresponding sequences differ in exactly two digits. Does the graph G contain an Euler circuit?

4. (MT'11) Show that there exists an integer n (in decimal form), in which the sum of the adjacent digits is never 9, but no matter how we choose two different integers between 0 and 9 whose sum is not 9, the two chosen numbers occur exactly once in n as adjacent digits (in some order).

5. (MT'12) In an imaginary language there are 10 vowels and 21 consonants. In this language there are no double letters (i.e. no letter can stand next to itself) and two different consonants cannot stand next to each other (either). But except for these anything else is possible, that is, any two different letters can stand next to each other if at least one of them is a vowel. What is the length of the longest sequence of letters in this language under the conditions that every letter can be used several times, but any two different letters can stand next to each other at most once in it?

6. (MT++'16) 11 children play a game. They stand in a circle, and one of them starts passing a ball to somebody else, who in turn passes it on to a third child, etc. The rules are the following: nobody can throw the ball to somebody he/she has thrown it before, also nobody can throw the ball to somebody who has thrown it to him/her before, and nobody can throw the ball to either of the two children standing next to him/her in the circle. At most how many passes are possible in the game under these rules?

7. (MT'16) The picture below is the layout of a royal palace. Each morning the king starts a walk from his apartment denoted by A. He decides that he wants to pass through each door exactly once. If he succeeds in it, then he would assign the last room of the walk as the throne-room. But he never succeeds in it, so the court sage tells him to wall up one of the doors. Is there a door in the palace, such that after walling it up there is a walk satisfying the king's wish? If yes, which room can become the throne-room?

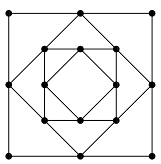
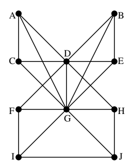


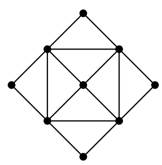
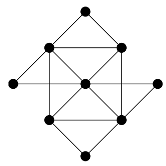
8. (MT+'17) In a simple graph on 20 vertices the degree of each vertex is 6. Prove that we can add one new edge to the graph in such a way that the resulting graph is still simple and contains an Euler trail.

9. (MT'19) For which of the values $r = 1, 2, \ldots, 9$ does it hold that every simple r-regular graph on 10 vertices contains an Euler circuit? (A graph is r-regular if the degree of each of its vertices is r.)

10. (MT++'19) In a simple graph with two components exactly four vertices have odd degrees. Moreover we know that neither of the components comtain an Euler circuit. Is it always true that we can add two edges to the graph in such a way that the graph obtained is still simple and contains an Euler circuit?

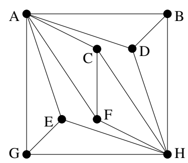
11. (MT+'10, MT++'16, MT'17, MT+'17) Do the following graphs contain a Hamilton cycle? And a Hamilton path?

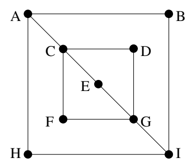



12. (MT'13) Let the vertices of the graph G be the squares of a 5×5 chessboard, and two vertices be adjacent if and only if the corresponding squares have a common edge. The graph G_1 is obtained from G by deleting a vertex corresponding to one of the corners of the chessboard from it (so G_1 has 24 vertices). The graph G_2 is obtained from G by deleting two vertices corresponding to opposite corners of the chessboard from it (so G_2 has 23 vertices).
a) Does G_1 contain a Hamilton cycle? And a Hamilton path?
b) Does G_2 contain a Hamilton cycle? And a Hamilton path?

13. (MT++'11) At least how many edges must be added to the graphs below so that the graphs obtained contain a Hamilton cycle?





14. (MT'15) The graph G is a star on 101 vertices (i.e. G has one vertex of degree 100 and hundred vertices of degree 1). At least how many edges must be added to G so that the graph obtained contains a Hamilton cycle?

15. a) Show that it is impossible to visit each square of a 4×4 chessboard (exactly once) with a horse.
b) Show that it is impossible to visit each square of a 5×5 chessboard (exactly once) with a horse such that in the 25th move we arrive back to the starting square.
c) $(*)$(MT+'19) Can we visit each square of a 3×5 chessboard exactly once with a horse?

16. In a company of 12 everybody knows at least 6 others (acquaintances are mutual). Show that this company can be seated around a round table in such a way that everybody knows his/her neighbors.

17. (MT'03) The simple graph G has $2k+1$ vertices. One of its vertices has degree k, and all the other vertices have degree at least $k+1$. Prove that G contains a Hamilton cycle.

18. In a company of 20 everybody knows the same number of people (acquaintances are mutual). Show that this company can be seated around a round table in such a way that either everybody knows his/her neighbors or nobody knows his or her neighbors.

19. a) Show that for each $n \geq 5$ there is a graph G on n vertices such that both G and its complement contain a Hamilton cycle.
b) Give a simple, connected graph G on 8 vertices, whose complement is also connected, and neither G nor its complement contain a Hamilton cycle.

20. In the simple graph G on $2k+1$ vertices each vertex has degree at least k. Prove that G contains a Hamilton path.

21. (MT'16) In the simple graph G on 201 vertices the degree of each vertex, except for v, is at least 101. About v we only know that it is not an isolated vertex. Show that G contains a Hamilton path.

22. (MT++'08) Show that if G is a simple 9-regular graph on 16 vertices, then we can delete 8 edges of G in such a way that the remaining graph contains an Euler circuit.

23. (MT'17) In a simple graph on 20 vertices the degree of each vertex is 8. Prove that we can add 20 new edges to the graph in such a way that the resulting graph is still simple and contains an Euler circuit.

24. (MT+'18) $(*)$ Can we list all the four-element subsets of the set $\{1, 2, \ldots, 10\}$ in one sequence in such a way that the subsets which are next to each other in the sequence have at least two elements in common (and each subset appears exactly once in the sequence)?

25. (MT++'19) In a simple bipartite graph on 20 vertices the degree of each vertex is at least 9. Show that this doesn't imply that the graph contains a Hamilton path.