Introduction to the theory of computing I.
Retake of 2nd Midterm — Grading guide
December 11, 2023.

General Rules are as before.

1. The augmented matrix of the system is: $\left(\begin{array}{cccc|c} 1 & 3 & 2 & 4 & 2 \\ 2 & 7 & 5 & 10 & 6 \\ -1 & -2 & -1 & 5 & 7 \end{array}\right)$ (0 points)

   We use Gaussian Elimination:

$$\left(\begin{array}{cccc|c} 1 & 3 & 2 & 4 & 2 \\ 0 & 1 & 1 & 2 & 2 \\ 0 & 1 & 1 & 9 & 9 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 3 & 2 & 4 & 2 \\ 0 & 1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 7 & 7 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 3 & 2 & 4 & 2 \\ 0 & 1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right) \to$$

$$\left(\begin{array}{cccc|c} 1 & 3 & 2 & 0 & -2 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 0 & -1 & 0 & -2 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right).$$

   (4 points)

   Based on the reduced row echelon form, the solutions of the system of equations are $x_3=\alpha \in R, x_1=\alpha-2, x_2=-\alpha, x_4=1$. (3 points)

   Substituting the obtained values into the equation $x_1=3(x_4)^3-2x_4+1$ (1 point) yields $\alpha-2=3-2+1 \Rightarrow \alpha=4$, i.e. the equation is satisfied with $\alpha=4$. (1 point)

   Since α can take any real value, $\alpha=4$ is also possible, so the system of equations has a solution that meets the condition. (1 point)

2. We will use Gaussian elimination and track the changes happening to the determinant according to the elementary row operations we perform.

$$\begin{vmatrix} 1 & 1 & 2 & 5 \\ 2 & p & 3 & 5 \\ 3 & 4 & 5 & 2 \\ 1 & 1 & 3 & 2 \end{vmatrix} = -\begin{vmatrix} 1 & 1 & 2 & 5 \\ 1 & 1 & 3 & 2 \\ 3 & 4 & 5 & 2 \\ 2 & p & 3 & 5 \end{vmatrix} = -\begin{vmatrix} 1 & 1 & 2 & 5 \\ 0 & 0 & 1 & -3 \\ 0 & 1 & -1 & -13 \\ 0 & p-2 & -1 & -5 \end{vmatrix} =$$

$$= \begin{vmatrix} 1 & 1 & 2 & 5 \\ 0 & 1 & -1 & -13 \\ 0 & 0 & 1 & -3 \\ 0 & p-2 & -1 & -5 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 2 & 5 \\ 0 & 1 & -1 & -13 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & p-3 & 13p-31 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 2 & 5 \\ 0 & 1 & -1 & -13 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 16p-40 \end{vmatrix} =$$

$$= 16p-40.$$

   (10 pont)

   Of course other methods can be used like the Laplace expansion fixing a row or a column. One who only states the expansion but doesn't perform it to find all the minor determinants and reach to a formula depending on p, receives 2 points.

   Solutions that resemble Gaussian elimination, but do not actually follow it, can only be of full value if the effects of each step are discussed. Deduct 1 point for each piece for missing them, as well as for calculation errors. For errors of principle (e.g. failure to multiply by (−1) when swapping lines, use of the wrong sign during the expansion theorem, failure to indicate the sign), on the other hand, 3 points are deducted per piece.

3. **(a)** The statement is not true. (0 points)

   Let for example both A and B be 2×2 identity matrices, then C has two identical rows, so its determinant is 0, while $det\, A=det\, B=1$, so the equality is not fulfilled. (3 points)

Of course, there are plenty of other counterexamples. Partial points are awarded for an incomplete solution if the determinants in question are calculated by the solver or if this is not done, then there is some convincing argument that the equality really will not be fulfilled. If the counterexample is good, but the solver makes a calculation error during the calculation of the determinants, then 1 point should be deducted for such errors. In the case of a wrong counterexample, no points are awarded for calculating the determinants.

(b) This statement is true (0 points).

Let $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B=\begin{bmatrix} e & f \\ g & h \end{bmatrix}$, then $D=\begin{bmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & e & f \\ 0 & 0 & g & h \end{bmatrix}$. (2 points)

Then we have $det\, A=ad-bc$ and $det\, B=eh-gf$. (1 point)

Using the expansion theorem, fixing the first row, the determinant of D is:

$$det\, D=a\begin{vmatrix} d & 0 & 0 \\ 0 & e & f \\ 0 & g & h \end{vmatrix}-b\begin{vmatrix} c & 0 & 0 \\ 0 & e & f \\ 0 & g & h \end{vmatrix} \qquad \text{(2 points)}$$

Again, expressing the two 3x3 determinants according to the first rows:

$$det\, D=ad\begin{vmatrix} e & f \\ g & h \end{vmatrix}-bc\begin{vmatrix} e & f \\ g & h \end{vmatrix}=(ad-bc)\begin{vmatrix} e & f \\ g & h \end{vmatrix}=det\, A\cdot det\, B \qquad \text{(2 points)}$$

A student who gives an example to show the truth of the statement of part (b) receives only 1 point.

4. Let A be the matrix in the problem, we are looking for a matrix B for which $B^{-1}=A$. (0 points) Then $I=BB^{-1}=BA$ and $I=B^{-1}B=AB$ are also satisfied, so B must be the inverse of A. (2 points)
With Gaussian elimination:

$$\left(\begin{array}{cccc|cccc} 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 2 & 3 & 4 & 5 & 0 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{cccc|cccc} 2 & 3 & 4 & 5 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \end{array}\right) \to$$

$$\left(\begin{array}{cccc|cccc} 1 & 3/2 & 2 & 5/2 & 0 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \end{array}\right) \to \left(\begin{array}{cccc|cccc} 1 & 3/2 & 2 & 5/2 & 0 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{array}\right) \to$$

$$\left(\begin{array}{cccc|cccc} 1 & 3/2 & 2 & 0 & 0 & 0 & -5/2 & 1/2 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{array}\right) \to \left(\begin{array}{cccc|cccc} 1 & 3/2 & 0 & 0 & -2 & 0 & -5/2 & 1/2 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{array}\right) \to$$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -2 & -3/2 & -5/2 & 1/2 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{array}\right). \qquad \text{(4 pont)}$$

Since we got the identity matrix on the left side, A has an inverse (3 points)
and it is the matrix shown on the right, so this will be the B we are looking for. (1 point)

The points for showing the existence of the inverse can of course be obtained in other ways, e.g.: the inverse exists (exactly) if the determinant of A is not 0. The statement is worth 1 point, and the calculation of the determinant of A is worth another 2 points.

5. Since A has four independent rows, its rank is at least 4 (by the definition of row rank). (3 points)

The first four columns of A are dependent, so all the columns of A are also dependent. (2 points)
Indeed: if the columns of A were independent, then (according to what was learned in the lecture) any subset of the columns, including the first four columns, would be independent. (1 point)

Since the columns of A are dependent, due to the definition of the column rank, the rank of A is less than 5, so it can be at most 4, (3 points)

The rank is therefore exactly 4. (1 point)

Strictly speaking, it should still be justified that the rank can be 4 (it is easy to give an example), but since we can conclude from the text of the task that such a matrix exists and its rank can only be 4, no points should be deducted for the lack of this. If, on the other hand, someone addresses this and gives a good example, they can get an extra point (of course, only if their solution would not be worth maximum points otherwise).

6. **(a)** If the rank of the new matrix after the change were 5, then its columns would form an independent system (according to the definition of column rank). (1 point)
In this case, however, the four columns of the original matrix that do not contain the changed entry would also be independent, (1 point) since they form a subset of an independent system. (1 point)
A contradiction, since the rank of the original matrix is 3.

(b) Let B be a 3x3 submatrix of A whose determinant is not 0. Such a submatrix exists because of the definition of the determinant rank. (1 point)
Let i be a row and j be a column that have no entry in B. Let C be a 4x4 submatrix of A that is formed by adding row i and column j to the rows and columns of B. Then, we must have $det\, C=0$, by the definition of determinant rank. (1 point)
We claim that by changing the a_{ij} entry in A arbitrarily, we get a 4-rank matrix. (1 point)
To prove this, let's make the change in C as well and call the resulting matrix C'. The determinant of C' (in A) expressed through row i, has a value different from the determinant of C, (1 point)
Since fixing row i and expressing the determinant of C and C', they are the same everywhere except for the entry a_{ij}, whose cofactor is $(-1)^{i+j} det\, B$, which is not 0. Thus, $det\, C' \neq det\, C = 0$. (2 points)
Since the matrix after the change has a 4×4 submatrix whose determinant is not 0, its rank is at least 4. (1 point)
In subtask (a), we realized that the rank cannot be more than this, so it must be exactly 4. (0 points)