Introduction to the Theory of Computing 1.
First Retake of the Second Midterm Test
December 11, 2023

1. Does the system of linear equations below have a solution for which $x_1 = 3(x_4)^3 - 2x_4 + 1$?

$$\begin{aligned} x_1 + 3x_2 + 2x_3 + 4x_4 &= 2 \\ 2x_1 + 7x_2 + 5x_3 + 10x_4 &= 6 \\ -x_1 - 2x_2 - x_3 + 5x_4 &= 7 \end{aligned}$$

2. Evaluate the determinant of the matrix below depending on the parameter p.

$$\begin{pmatrix} 1 & 1 & 2 & 5 \\ 2 & p & 3 & 5 \\ 3 & 4 & 5 & 2 \\ 1 & 1 & 3 & 2 \end{pmatrix}$$

3. Let A and B be 2×2 matrices. Let C be the 4×4 matrix whose upper left and upper right 2×2 submatrices are A, and lower left and lower right 2×2 submatrices are B. Furthermore, let D be the 4×4 matrix whose upper left 2×2 submatrix is A, lower right 2×2 submatrix is B, and the other eight entries of it are 0.
   a) Does it always hold that $\det(C) = \det(A) \cdot \det(B)$?
   a) Does it always hold that $\det(D) = \det(A) \cdot \det(B)$?

4. Is there such a matrix whose inverse is the matrix A below? If yes, then determine it.

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 3 & 4 & 5 \end{pmatrix}$$

5. The first four rows of a 5×5 matrix A are linearly independent, and the first four columns of A are linearly dependent. Determine all the possible values of the rank of A.

6. * The rank of a 5×5 matrix A is 3.
   a) Prove that no matter how we change one of the entries of A the rank of the matrix obtained is at most 4.
   b) Prove that A has such an entry which when changed appropriately the rank of the matrix obtained will be (exactly) 4.

Please work on stapled sheets only, and submit all of them at the end of the midterm, including drafts.

Write your name on every sheet you work on, and write your Neptun code and the number of the group you are registered to in Neptun (A1, A2 or A3) on the first page.

You have 90 minutes to work on the problems. Each of them is worth 10 points. To obtain a signature you have to achieve at least 24 points on each of the two midterm tests.

The details of the solutions must be explained; giving the result only is not worth any points. Notes, calculators or any additional devices cannot be used. The problem marked with an * is supposed to be more difficult.