**Introduction to the theory of computing 1.**
**2nd Midterm — December 1, 2023.**
**Grading guide**

General principles.

The aim of the scoring guide is for the correctors to evaluate the papers in a uniform manner. The aim of the guide is not to provide a detailed description of the complete solution of the tasks; the described steps can be considered as an outline of a solution that will achieve a maximum score.

The sub-points indicated in the guide are awarded to the solver only if the related idea is included in the thesis as a step of a clear, clearly described and justified solution. Thus, for example, the mere description of the knowledge, definitions, and theorems included in the material is not worth points without their application (even if, by the way, one of the described facts actually plays a role in the solution). Considering whether the score indicated in the guide is due to the solver (in part or in whole) taking into account the above, is entirely the responsibility of the corrector.

Partial points are awarded for all ideas and thoughts that can have a meaningful role in a solution and from which, with a suitable addition to the thought process described in the thesis, a flawless solution to the task could be obtained. If a solver starts several, significantly different solutions to the same task, then the maximum score that can be given is one. If each described solution or solution part is correct or can be supplemented to be correct, then the solution initiative with the most partial points is evaluated. However, if among several solution attempts there is both a correct one and one containing a (significant) error, and the thesis does not reveal which one the solver considered correct, then the solution attempt with fewer points is evaluated (even if this score is 0).

The sub-scores in the guide can be divided further if necessary. A good solution different from the one described in the guide is of course worth maximum points, but without proof you can only refer to the theorems and statements in the lecture.

1. **First solution.**
   The system of linear equations in the problem definitely has a solution for all values of p and q: $x_1=x_2=x_3=x_4=0$. (2 points)
   If a system of linear equations has a unique solution, then (according to the learned theorem) the number of equations in this system is at least the same as the number of unknowns. Thus, the system of equations in the problem cannot have a unique solution (since it consists of three equations and four unknowns). (3 points)
   It is true for every system of linear equations (according to what we have learned) that either it has no solution, or a unique solution, or it has an infinite number of solutions, (2 points)
   and according to the above, the first two cases are excluded, so the system of equations has an infinite number of solutions for all values of p and q. (3 points)

   **Second solution.**
   Applying Gaussian elimination, we get:
   $$\left(\begin{array}{cccc|c} -1 & -4 & 3 & 5 & 0 \\ 2 & 5 & -24 & 3p-1 & 0 \\ 3 & 17 & 22 & q-5p & 0 \end{array}\right)\left(\begin{array}{cccc|c} 1 & 4 & -3 & -5 & 0 \\ 0 & -3 & -18 & 3p+9 & 0 \\ 0 & 5 & 31 & q-5p+15 & 0 \end{array}\right)\left(\begin{array}{cccc|c} 1 & 4 & -3 & -5 & 0 \\ 0 & 1 & 6 & -p-3 & 0 \\ 0 & 0 & 1 & q+30 & 0 \end{array}\right) \quad \text{(2 points)}$$

   With this, we have reached the row echelon form, and then we continue the execution of the algorithm with the second phase until the reduced row echelon form: (0 points)

$$\begin{pmatrix} 1 & 4 & 0 & 3q+85 & 0 \\ 0 & 1 & 0 & -p-6q-183 & 0 \\ 0 & 0 & 1 & q+30 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 4p+27q+817 & 0 \\ 0 & 1 & 0 & -p-6q-183 & 0 \\ 0 & 0 & 1 & q+30 & 0 \end{pmatrix}$$ (2 points)

According to the obtained reduced row echelon form, $x_4=\alpha \in R$ is a free parameter and $x_1=(-4p-27q-817)\alpha$, $x_2=(p+6q+183)\alpha$ and $x_3=(-q-30)\cdot\alpha$. (3 points)

Thus, for every value of $p$and $q$, the system of equations has an infinite number of solutions. (3 points)

If a solver stops calculating either after reaching the row echelon form or the reduced row echelon form and argues that since no forbidden row was created at the end of the first phase (2 points), there is no leading 1 in the fourth column and thus $x_4$ is a free parameter (2 points), so there is definitely an infinite number of solutions for every value of $p$and $q$ (1+1 points), it is of course a full-valued solution and therefore deserves maximum points according to the above distribution. Calculation errors (in a good solution) mean a deduction of 1 point per piece. However, the incorrect interpretation of the reduced row echelon form is considered an essential (error, in which case the last 3+3 points according to the scoring are lost. If a solver performs (even correct) calculations (for example, expresses an unknown, substitutes, etc.), but these are not purposeful, do not show the direction of a correct solution, only very few points (maximum 1-3) can be awarded depending on the usefulness of the work performed.

2. According to the definition, among the products listed to be added up, those that contain a factor of 0 do not have to be taken into account, because their value is 0, so they contribute 0 to the value of the determinant. (1 point)
   In a product that does not contain 0, only the 1 in the 6th position from the 3rd row can be chosen, because due to $\pi_3=6$, the first 5 elements of the row are 0. (1 point)
   Therefore, only the 1 in the 5th position from the 1st row can be chosen, because due to $\pi_1=5$, the first 4 elements of the row are 0 and we have already chosen an element from the 6th column. (1 point)
   Similarly: from the 5th row, only the 1 in the 4th place can be chosen, because due to $\pi_5=4$, the first 3 elements of the row are 0 and we have already chosen an element from the last two columns. Continuing in this way, we finally get that the only non-zero product is the product corresponding to the given permutation $\pi$. (3 points)
   The value of this product is obviously $1^6=1$, (1 point)
   and for its sign, we determine the inversion number of π: $I(\pi)=4+1+3+0+1+0=9$ (where the number of elements to the right of it, smaller than it, is added to the elements in a row). (or you can clearly list those pairs that are inversions and count them) (1 point)
   Since $I(\pi)$ is odd, the product of π has a negative sign. (1 point)
   Thus, by definition, $det\,A=-1$ (because one of the 6! = 720 terms to be added is (−1), the others are all 0). (1 points)

   Every calculation error or inattention that does not significantly influence the solution results in a deduction of 1 point. However, errors of principle indicating a (even partial) lack of knowledge of the definition of the determinant mean a deduction of 3 points per piece.

3. **First solution.**
   Subtract the third column from the fourth and fifth. These steps (according to what was learned) do not change the value of the determinant. (3+1 points)
   This made the fourth column $(0,2,0,0,0)^T$ , the fifth $(0,6,0,0,0)^T$ . (1 point)
   Now subtract three times the fourth column from the fifth column. This step does not change the value of the determinant either. (2 points)
   This changed the fifth column to all zeros. (1 point)

Due to the all-zero column (as learned), the value of the determinant is 0 (and thus the same is true for the original determinant). (2 points)

**Second solution.**
Subtract the first row from the third, fourth and fifth. These steps (according to what was learned) do not change the value of the determinant. (3+1 points)
In the resulting matrix, all elements of the lower right, three-by-three submatrix are 0. (1 point)
If we now calculate the determinant of this matrix according to the definition, then the product for each rook arrangement contains 0 elements. Indeed: if we want to choose a non-zero element from the last three rows, we can only do this from the first two columns; however, since only one element can be selected from each column, this is impossible. (3 points)
Thus, when calculating according to the definition, 5! = 120 signed 0s are added. Therefore, the value of the determinant is 0 (and the same is true for the original determinant). (1+1 points)

**Third solution.** We apply the version of Gaussian elimination for the determinant:

$$\begin{vmatrix} 4 & -3 & 1 & 1 & 1 \\ -8 & 6 & 17 & 19 & 23 \\ 12 & -9 & 1 & 1 & 1 \\ -5 & 4 & 1 & 1 & 1 \\ 2 & 2 & 1 & 1 & 1 \end{vmatrix} = 4 \cdot \begin{vmatrix} 1 & -3/4 & 1/4 & 1/4 & 1/4 \\ 0 & 0 & 19 & 21 & 25 \\ 0 & 0 & -2 & -2 & -2 \\ 0 & 1/4 & 9/4 & 9/4 & 9/4 \\ 0 & 7/2 & 1/2 & 1/2 & 1/2 \end{vmatrix} = (-4) \cdot \begin{vmatrix} 1 & -3/4 & 1/4 & 1/4 & 1/4 \\ 0 & 1/4 & 9/4 & 9/4 & 9/4 \\ 0 & 0 & -2 & -2 & -2 \\ 0 & 0 & 19 & 21 & 25 \\ 0 & 7/2 & 1/2 & 1/2 & 1/2 \end{vmatrix} =$$

$$= (-1) \cdot \begin{vmatrix} 1 & -3/4 & 1/4 & 1/4 & 1/4 \\ 0 & 1 & 9 & 9 & 9 \\ 0 & 0 & -2 & -2 & -2 \\ 0 & 0 & 19 & 21 & 25 \\ 0 & 0 & -31 & -31 & -31 \end{vmatrix} = 2 \cdot \begin{vmatrix} 1 & -3/4 & 1/4 & 1/4 & 1/4 \\ 0 & 1 & 9 & 9 & 9 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{vmatrix} = 2 \cdot 0 = 0.$$

Each calculation error that does not significantly affect the solution results in a deduction of 1 point. On the other hand, errors of principle resulting from the complete or partial lack of certain basic knowledge about the determinant mean a deduction of 3 points per piece. This is the case, for example, if the solver does not or incorrectly follow the change of the determinant after multiplying a row by a constant or exchanging rows. If the calculations are correct, then 1 point is deducted for each missing mentioning of the rules applied. A proportional partial score is awarded for each useful step towards the calculation of the determinant. The mere application of the expansion theorem (without any other transformations) is worthless, because this alone does not make the determination of the determinant any easier than the original task.

4. **(a)** $det\, A = 2 \cdot (p+1) - 2p = 2$ (as learned about calculating the 2 × 2 determinant). (1 point)
Since $det\, A \neq 0$ is true for all $p$, therefore (according to the learned theorem) $A^{-1}$ exists for all values of $p$. (1 point)
We calculate $A^{-1}$ in the learned way, with Gaussian elimination:

$$\left(\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 2p & p+1 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 1/2 & 1/2 & 0 \\ 0 & 1 & -p & 1 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 0 & p+1/2 & -1/2 \\ 0 & 1 & -p & 1 \end{array}\right) \qquad \text{(2 pont)}$$

According to what we have learned, $A^{-1}$ is the matrix above, to the right of the line. (1 point)

According to what we have learned, we can also argue for the existence of $A^{-1}$ (also obtain the first 1+1 point according to the above scoring) that since during the Gaussian elimination, a unit matrix was created for all values of $p$ to the left of the line, therefore $A^{-1}$ exists for all values of $p$.
Calculation errors that do not significantly affect the solution mean a deduction of 1 point, but errors of principle resulting from a complete or partial lack of basic knowledge regarding the calculation of $A^{-1}$, 3 points is deducted for each.

**(b)** $\left(A^{-1}-I\right)\cdot\left(A^2-A\right)=A^{-1}\cdot A^2-A^{-1}\cdot A-I\cdot A^2+I\cdot A=I\cdot A-I-A^2+A=-A^2+2A-I$. (2 points)

$\left(A-I\right)^2=\left(A-I\right)\cdot\left(A-I\right)=A\cdot A-A\cdot I-I\cdot A+I^2=A^2-A-A+I=A^2-2A+I$. (2 points)

Thus $\left(A^{-1}-I\right)\cdot\left(A^2-A\right)+\left(A-I\right)^2=-A^2+2A-I+A^2-2A+I=0$, that is, the result is the null matrix. (1 point)
(For the above calculations, we used the learned properties of operations on matrices: distributivity, associativity, the ability to factor out the scalar multiplier and the property of multiplication by I.) (0 points)

Part (b) can of course also be solved by directly calculating the individual factors:

$$A^2=\begin{pmatrix} 2p+4 & p+3 \\ 2p^2+6p & p^2+4p+1 \end{pmatrix}, A^2-A=\begin{pmatrix} 2p+2 & p+2 \\ 2p^2+4p & p^2+3p \end{pmatrix}, (A-E)^2=\begin{pmatrix} 2p+1 & p+1 \\ 2p^2+2p & p^2+2p \end{pmatrix}$$

and $\left(A^{-1}-I\right)\cdot\left(A^2-A\right)$ is the opposite of the latter matrix. If someone works in this way, the calculation of the three matrices above is worth 1-1 point and the product $\left(A^{-1}-I\right)\cdot\left(A^2-A\right)$ and the calculation of the final result from these are worth an additional 1-1 point. (The calculation of $A^{-1}-I$ is not worth points by itself.)
(**matrix $E$ that appears in the calculations is just the identity matrix I**)
Calculation errors that do not significantly affect the solution mean a deduction of 1-1 point, but errors of principle resulting from a complete or partial lack of basic knowledge regarding the calculation of $A^{-1}$ or the performance of matrix multiplication is -3 points each.

5. The searched value is the rank of the 4 × 5 matrix created by combining the five listed vectors (according to the definition of the column rank). (4 points)
According to what we have learned, the rank is determined by Gaussian elimination:

$$\mathrm{r}\begin{pmatrix} -2 & -14 & 2 & 4 & 10 \\ 3 & 21 & -3 & -6 & -11 \\ 2 & 15 & -1 & -1 & -11 \\ -1 & -2 & 6 & 17 & -3 \end{pmatrix}=\mathrm{r}\begin{pmatrix} 1 & 7 & -1 & -2 & -5 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 1 & 1 & 3 & -1 \\ 0 & 5 & 5 & 15 & -8 \end{pmatrix}=\mathrm{r}\begin{pmatrix} 1 & 7 & -1 & -2 & -5 \\ 0 & 1 & 1 & 3 & -1 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 5 & 5 & 15 & -8 \end{pmatrix}=$$

$$\mathrm{r}\begin{pmatrix} 1 & 7 & -1 & -2 & -5 \\ 0 & 1 & 1 & 3 & -1 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & -3 \end{pmatrix}=\mathrm{r}\begin{pmatrix} 1 & 7 & -1 & -2 & -5 \\ 0 & 1 & 1 & 3 & -1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}=\mathrm{r}\begin{pmatrix} 1 & 7 & -1 & -2 & -5 \\ 0 & 1 & 1 & 3 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{(3 pont)}$$

The resulting matrix has a row echelon form, so its rank is the number of its rows (or leading ones): 3. (3 points)
Thus, a maximum of 3 of the listed five vectors can be selected so that the selected vectors are linearly independent. (0 points)

Each calculation error that does not significantly affect the solution results in a deduction of 1 point. (The last 3 points according to the scoring can of course be given if someone correctly determines the rank of the row echelon form matrix obtained as a result of an incorrect calculation.) Errors of principle resulting from the complete or partial lack of certain basic knowledge mean a deduction of 4 points each. If a solver does not directly apply Gaussian elimination (that is, it does not arrive at a row echelon form by following it exactly), then it must justify the correctness of the solution - that is, it must refer to the fact that the steps taken do not change the rank according to what has been learned; without it, you cannot get the last 3 points.
If the solver applies the Gaussian elimination for the augmented matrix (A|0) without any further explanation, then the calculations only are worth 0 points.

6. If $det\, A=0$, then this corresponds to the statement of the problem, and nothing needs to be done. Thus, we can further assume that $det\, A=0$ (and our goal is to show that $det\, A=1$). (1 point)
Since $det\, A\neq 0$, $A$ has an inverse. (1 point)
Multiply (from the left) the equation $A+A^2+A^3=0$ by $A^{-1}$:
$A^{-1}\cdot\left(A+A^2+A^3\right)=A^{-1}\cdot 0.$ (1 point)
On the right-hand side we obviously get 0 (because multiplying the null matrix by any other matrix gives 0). (1 point)
On the left-hand side, after opening the parenthesis, $A^{-1}\cdot A=I$ (by the property of multiplication by $I$) due to which we get: $I+A+A^2=0$. (1 point)
(Here, in addition to the above, we also used the distributivity and associativity of matrix multiplication.) (0 point)
From this, $A^2+A=-I$. Substituting this into the equation $A+A^2+A^3=0$ : $A^3-I=0$, that is, $A^3=I$. (2 points)
Applying the theorem of multiplication of determinants (twice): $det\, A^3=\left(det\, A\right)^3$ . (1 point) Thus, since $A^3=I$ and $det\, I=1$, $\left(det\, A\right)^3=1$, from which $det\, A=1$ follows. (2 points)