1^{st} Midterm Retake- Grading Guide

The general principles are the same as before.

1. Denoting the number of days of the year on the third planet with n, we get the congruences $n \equiv 22 \pmod{29}$ and $n \equiv 5 \pmod{32}$ from the text of the problem. (2 points)

   We solve the congruence system using the learned method. From the first congruence, $n = 29k + 22$ for some integer k. Substituting this into the second congruence: $29k + 22 \equiv 5 \pmod{32}$. Subtracting 22 from both sides, we get the linear congruence $29k \equiv -17 \pmod{32}$. (1 point)

Since $(29, 32) = 1$ (e.g. because 29 is prime and 32 is not its multiple) and $1|5$, therefore according to the learned theorem the linear congruence is solvable and has one solution modulo 32. (1 point)

Subtracting 32k from the left side and adding 32 to the right side (equivalent to the original)

$-3k \equiv 15 \pmod{32}$ congruences result. (1 point)

   Dividing this by (−3), then adding 32 to the right-hand side (the latter can actually be omitted), $k \equiv 27 \pmod{32}$, where the modulus has not changed because 32 and −3 are relative primes. (1 point)

   Since congruence cannot have any other solution, we know that there must be a solution modulo 32, the obtained value of 27 is indeed the solution. (1 point)

   Thus $k = 32m + 27$ for some integer m, substituting this into $n = 29(32m + 27) + 22 = 928m + 805$. (2 points)

   Among such n's, there is one that is positive and less than 1000, namely 805, which is the desired number. (1 point)

The fact that 27 is really the only solution for congruence can of course be seen in other ways, e.g. by checking that all our transformations were equivalent transformations, the 1+1 point goes to this as well.

The linear congruence that was obtained during the solution can of course be solved with other methods, so even with the Euclidean algorithm; working with this, the congruences $32k \equiv 0 \pmod{32}$, $29k \equiv -17 \pmod{32}$, $3k \equiv 17 \pmod{32}$, $2k \equiv -10 \pmod{32}$, $k \equiv 27 \pmod{32}$ are generated. Then, out of the 4 points awarded for solving the linear congruence, the fact that the solver applies the algorithm (which he does not necessarily have to name, it is enough if he clearly demonstrates this through its application) is worth 1 point for the solution of the linear congruence; it is worth an additional 1 point to verify that the procedure can be applied without dividing (that is, skipping the first phase), because $(29, 32) = 1$; finally, the calculation itself is worth 2 points.

2. 499 and 539 are coprimes, because $539 = 7^2 \cdot 11$, and 499 is not divisible by either 7 or 11 (of course, many other reasons can be given). (1+1 point)

   So according to the Euler-Fermat theorem, $499^{\phi(539)} \equiv 1 \pmod{539}$. (2 points)

According to what was learned, $\phi(539) = (7^2 - 7)(11 - 1) = 420$. (2 points)

   Based on what has been said so far, $499^{420} \equiv 1 \pmod{539}$, (1 point)

   this raised to the tenth power $499^{4200} \equiv 1 \pmod{539}$, (1 point)

   and multiplying both sides by 499 gives $499^{4201} \equiv 499 \pmod{539}$, (2 points)

   the desired remainder is therefore 499. (0 points)

3. n = (a, b, c), the normal vector of the plane we are looking for is perpendicular to d (1 point)

   and also to the vector pointing from one of the two specified points to the other. (1 point)

   Such a vector is $(5; -1; 1) - (3; 1; 2) = (2; -2; -1)$. (1 point)

   Thus, the scalar (dot) product of n with both d and the vector $(2; -2; -1)$ is also 0, (2 points)

   that is, $3a - c = 0$ and $2a - 2b - c = 0$. (1 point)

   Solving this system of equations (e.g.) $a = 2$, $b = -1$, $c = 6$. (2 points)

   Based on the normal vector and one of the given points, the equation of the plane is thus $2x - y + 6z = 17$. (2 points)

The normal vector can now also be calculated with vectorial product, since it was in the lecture, and is worth the relevant 5 points if the writing and calculation are correct. For a principle error (e.g. failure to use the checkerboard rule) 3 points are deducted, and 1 point is deducted for each calculation error.

4. Let x and y be arbitrary vectors in V, and λ be an arbitrary real number. (0 points)

   Based on the description of V, $x = (a, a + d, a + 2d, a + 3d, 4a + 6d)^T$, and $y = (b, b + c, b + 2c, b + 3c, 4b + 6c)^T$, for some numbers a, b, c, d. (1 point)

   Then $x + y = (a + b, a + b + d + c, a + b + 2d + 2c, a + b + 3d + 3c, 4a + 4b + 6d + 6c)^T$, (1 point) which is in V, since by choosing the first element of the arithmetic series as (a+b) and its difference as (c+d), we get just this vector. (1+1 points)

   $\lambda x = (\lambda a, \lambda(a + d), \lambda(a + 2d), \lambda(a + 3d), \lambda(4a + 6d))^T$, (1 point)

   which is also in V, since by choosing the first element of the arithmetic series as (λa) and its difference as (λd) , we get just this vector. (1+1 points)

   Since V forms a subspace if and only if it is not empty (if this is omitted, do not deduct a point, but if someone mentions it and would lose a point in this task, then add one) and the sum of any two vectors in V is also in V, and any scalar multiple of any vector in V is also in V , according to the above, V is a subspace. (3 points)

The last 3 points are only awarded to the solver if he makes the relevant statement in such a way that he had previously really dealt with the examination of the sum and scalar product of vectors in V, otherwise it's only 1 point.

5. According to what was learned, linearly independent vector systems can be extended to a basis. (2 points)

   If a system is not linearly independent, then it cannot be extended to a basis, because if any vector is added, it will still not be independent, but the bases must be independent. (Do not deduct a point for the absence of this comment, but whoever writes it down and would lose a point in this task, give an extra point.) (0 points)

   To examine independence, consider xa+yb+zc=0 equality. (1 point)

   Which gives the system:

   a+b+c=0
   a+b+c=0
   a+b+cp=0
   a+bp+cp=0 (1 point)

Subtracting the first equation from the third gives (p − 1) c = 0, so if p not equal to 1, then c = 0. (1 point)

   Subtracting the third equation from the fourth (p − 1) b = 0, so if p is not equal to 1, then b= 0. (1 point)

   Based on all of this, if p is not 1, then, in addition to b and c, due to the first equation, a must also be 0, (1 point)

   that is, only the trivial linear combination of vectors gives the zero vector, so then the vectors are linearly independent. (2 points)

   However, if p = 1, then the vector system is not linearly independent, since (e.g.) the linear combination with the scalars a = 1, b = −1, c = 0 also gives the null vector (but of course we can also refer to that the same vector appears more than once in the system). (1 point) The system can therefore be completed to a base exactly when p is NOT equal to 1. (0 points)

6. Let x be a number for which the above congruence holds. Since 2 and 201 are coprimes, according to what we learned, x^6 and 201 must also be coprimes, (1 point)

   from which it follows that x and 201 are also relative primes. (1 point)

   According to the Euler-Fermat theorem, x^(ϕ (201)) ≡ 1 (mod 201). (2 points)

   Since ϕ (201) = (3 − 1)(67 − 1) = 132, then x^132 ≡ 1 (mod 201). (1 point)

Raising the congruence $x^6 \equiv 2 \pmod{201}$ to the 22nd power gives $x^{(6\cdot 22)} = x^{(132)} \equiv 2^{(22)} \pmod{201}$.(2 points)

$2^{22} \equiv 37 \pmod{201}$ (this can be found out by repeated squaring (the resulting remainders are 2,4,16,55,10 in a row) or e.g. based on the fact that $2^{11} = 2048$ gives 38 remainders when divided by 201, and $38^2 = 1444$ gives a remainder of 37 when divided by 201). (2 points)

Based on these, $x^{132} \equiv 37 \pmod{201}$, which contradicts what was seen before, so the number x with the above property does not exist. (1 point)

Of course, points are only awarded for writing the Euler-Fermat theorem if it is really useful in terms of solving the problem, so e.g. we raise x and not x^6 or 2 to the 132nd power. Similarly, there is no point in determining $\phi(201)$ if it is not used to lead to the solution of the problem.