Introduction to the Theory of Computing I.
Second Midterm Test

2018. November 29.

1. a) Determine the number of solutions of the system of linear equations below for all values of the parameters p and q.
   b) If p and q have such values for which the system of linear equations has infinitely many solutions, then determine all of them.

$$\begin{aligned} x_1 + x_2 + x_3 - 7x_4 &= 8 \\ 4x_1 + 4x_2 + x_3 - 28x_4 &= 23 \\ 5x_1 + 3x_2 - x_3 - 31x_4 &= 14 \\ 2x_1 + p \cdot x_4 &= q \end{aligned}$$

2. * The set $L \subseteq \mathbf{R}^n$ is called a *line* if there are vectors $\underline{p}, \underline{v} \in \mathbf{R}^n$, so that $L = \{\underline{p} + c \cdot \underline{v} : c \in \mathbf{R}\}$, that is, L consists of those vectors $\underline{x}$ in $\mathbf{R}^n$ for which $\underline{x} = \underline{p} + c \cdot \underline{v}$ holds for some $c \in \mathbf{R}$. Prove that if a system of linear equations has infinitely many solutions, then the set of solutions of it (as a set of vectors in $\mathbf{R}^n$) contains a line.

3. Decide whether the statements below are true for all the $n \times n$ matrices A.
   a) If there is a vector $\underline{b} \in \mathbf{R}^n$ for which the system of linear equations with augmented matrix $(A|\underline{b})$ has no solutions, then $\det A = 0$.
   b) If $\det A = 0$ then there is a vector $\underline{b} \in \mathbf{R}^n$ for which the system of linear equations with augmented matrix $(A|\underline{b})$ has no solutions.

4. Compute the matrix A^{2018} for the matrix A below. (A^{2018} is the product with 2018 factors, each of whose is A.)

$$A = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

5. Determine the matrices A^{-1} and B if the matrices A and $A \cdot B$ are as below.

$$A = \begin{pmatrix} 3 & 7 \\ 4 & 9 \end{pmatrix} \qquad A \cdot B = \begin{pmatrix} 777 & 666 \\ 999 & 888 \end{pmatrix}$$

6. In an 4×6 matrix A the entry in the ith row and jth column is $a_{ij} = i^2 + i \cdot j$ for all $1 \leq i \leq 4$ and $1 \leq j \leq 6$. Determine the rank of A.

The full solution of each problem is worth 10 points. Show all your work! Results without proper justification or work shown deserve no credit.
Notes and calculators (and similar devices) cannot be used.
The question denoted by an * is supposed to be more difficult.

(Approximate) grading: 0-23 points: 1, 24-32 points: 2, 33-41 points: 3, 42-50 points: 4, 50-60 points: 5.