1 BASIC COUNTING RULES

Proposition 1.1 (Product Rule) *If something can happen in n_1 ways, and no matter how the first thing happens, a second thing can happen in n_2 ways, and so on, no matter how the first $k-1$ things happen, a k-th thing can happen in n_k ways, then all the k things together can happen in $n_1 \times n_2 \times \ldots \times n_k$ ways.*

Example 1.1 *A local telephone number is given by a sequence of six digits. How many different telephone numbers are there if the first digit cannot be 0?*

Answer: $9 \times 10 \times 10 \times 10 \times 10 \times 10 = 900,000$.

Example 1.2 *The population of a town is 30,000. If each resident has three initials; is it true that there must be at least two individuals with the same initials?*

Answer: Yes, since $30,000 > 26 \times 26 \times 26$.

Example 1.3 *The number of subsets of an n-set is 2^n. (First we decide if the first element of the n set belongs to the subset or not, then we decide if the second element of the n-set belongs to the subset or not, etc.)*

Proposition 1.2 (Sum rule) *If one event can occur in n, ways, a second event can occur in n_2 (different) ways, and so on, a k-th event can occur in n_k (still different) ways then (exactly) one if the events can occur in $n_1 + n_2 + \ldots + n_k$ ways.*

Example 1.4 *A committee is to be chosen from among 8 mathematicians, 10 physicists, 12 physicians. If the committee is to have two members of different backgrounds, how many such committees can be chosen?*

Answer: $8 \times 10 + 8 \times 12 + 10 \times 12 = 296$.

Example 1.5 *See Example 1.2 if each resident has one, two or three initials.*

Answer: Yes, since $30,000 > 26 + 26 \times 26 + 26 \times 26 \times 26$.

A *permutation* of a set of n elements is an arrangement of the elements of the set in order. The number of permutations of an n-set is given by

$$n \times (n-1) \times \ldots \times 1 = n! \qquad \text{(product rule)}.$$

Example 1.6 *How many permutations of* $\{1, 2, 3, 4, 5\}$

(a) are there? *(5!);*

(b) begin with 5? *(4!);*

(c) begin with an odd number? *(*$3 \times 4!$*).*

Given an n-set, suppose that we want to pick out r elements and arrange them in order. Such an arrangement is called an *r-permutation* of the n-set. The number $P(n, r)$ or r-permutations of an n-set is given by

$$n \times (n-1) \times \ldots \times (n-r+1) \qquad \text{(product rule)}.$$

Example 1.7 *Let* $A = \{0, 1, 2, 3, 4, 5, 6\}$.

(a) Find the number of sequences of length 3 using elements of A.

(b) Repeat (a) if no element of A is to be used twice.

(c) Repeat (a) if the first element of the sequence is 4.

(d) Repeat (a) if the first element of the sequence is 4 and no element of A is used twice.

Answers: (a) $7 \times 7 \times 7$; (b) $7 \times 6 \times 5$; (c) $(1\times)7 \times 7$; (d) $(1\times)6 \times 5$.

An r-combination of an n-set is a selection of r elements from the set. Order does not count. (i.e., an r-combination is an r-element subset.) $\binom{n}{r}$ will denote the number of r-combinations of an n-set. Notice that $P(n, r) = \binom{n}{r} \times r!$ (product rule) and so $\binom{n}{r} = \frac{n!}{r!(n-r)!}$.

Theorem 1.1

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}.$$

Proof Notice that the number of r-subsets of an n-set that contains the "first" element of the n-set is $\binom{n-1}{r-1}$ and the number of r- subsets not containing the "first" element is $\binom{n-1}{r}$. Hence the sum rule yields the desired equality.

(You may prove the equality by means of algebraic manipulations, as well.) □

Example 1.8 *A committee is to be chosen from a set of 7 women and 4 men. How many ways are there to form the committee if*

(a) the committee has 5 people, 3 women and 2 men?

(b) the committee can be any size (except empty) but it must have equal numbers of women and men?

(c) the committee has 4 people and one of them must be $Mr\dot{S}mith$*?*

(d) the committee has 4 people, 2 of each sex and $Mr\dot{a}nd\ Mrs\dot{S}mith$ *cannot both be on the committee?*

Answers:

(a) $\binom{7}{3} \times \binom{4}{2}$;

(b) $\binom{7}{1} \times \binom{4}{1} + \binom{7}{2} \times \binom{4}{2} + \binom{7}{3} \times \binom{4}{3} + \binom{7}{4} \times \binom{4}{4} = \binom{11}{4} - 1$;

(c) $\binom{7}{0} \times \binom{3}{3} + \binom{7}{1} \times \binom{3}{2} + \binom{7}{2} \times \binom{3}{1} + \binom{7}{3} \times \binom{3}{0} = \binom{10}{3}$;

(d) $\binom{6}{2} \times \binom{3}{1} + \binom{6}{1} \times \binom{3}{2} + \binom{6}{2} \times \binom{3}{2} = \binom{7}{2} \times \binom{4}{2} - \binom{6}{1} \times \binom{3}{1}$.

If we are choosing an *r-permutation* out of an n-set *with replacement* then we say that we are *sampling with replacement*. The product rule gives us that the number of r-permutations of an n-set with replacement is n^r.

Similarly, we may speak of *r-combinations* of an n-set *with replacement or*

repetition. For example, the 3-combinations of a 2-set $\{0, 1\}$ with replacement are $\{0, 0, 0\}$, $\{0, 0, 1\}$, $\{0, 1, 1\}$, $\{1, 1, 1\}$.

Theorem 1.2 *The number of r-combinations of an n-set with repetition is* $\binom{n+r-1}{r}$.

Choosing a sample of r elements from a set of n elements is summarized in the following table.

Order counts?	Repetition allowed?	The sample is called:	Number of ways to choose the sample:
No	No	r-combination	$\binom{n}{r}$
Yes	No	r-permutation	$P(n,r) = \frac{n!}{(n-r)!}$
No	Yes	r-combination with replacement	$\binom{n+r-1}{r}$
Yes	Yes	r-permutation with replacement	n^r

Theorem 1.3 *Suppose that we have n objects, n_1 of type 1, n_2 of type 2, ..., n_k of type k, with $n_1 + n_2 + \ldots + n_k = n$ of course. Suppose that objects of the same type are indistinguishable. Then the number of distinguishable permutations of these objects is $\binom{n}{n_1, n_2, \ldots, n_k}$.*

Proof We have n places to fill in the permutation and we assign n_1 of these to the objects of type 1, n_2 to the objects of type 2, and so on. □

Theorem 1.4 (Binomial expansion) *For $n \geq 0$,*

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} .$$

Proof In multiplying out, we pick one term from each factor $(a+b)$. Note that to obtain $a^k b^{n-k}$, we need to choose k of the factors from which to choose a. □

Theorem 1.5 *1.* $\binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{n} = 2^n$ *for $n \geq 0$,*

2. $\binom{n}{0} - \binom{n}{1} + \ldots + (-1)^n \binom{n}{n} = 0$ *for $n \geq 1$.*

Proof Binomial expansion of $(1+1)^n$ and $(1+(-1))^n$. □

2 THE PRINCIPLE OF INCLUSION AND EXCLUSION

Theorem 2.1 (Inclusion-Exclusion Formula) *Let $A_1, A_2, \ldots, A_n \subseteq S$ where S is a finite set, and let*

$$A_I = \bigcap_{i \in I} A_i \qquad \text{for} \quad I \subseteq \{1, 2, \ldots, n\} \qquad (A_\emptyset = S) .$$

Then

$$|S - (A_1 \cup A_2 \cup \ldots \cup A_n)| = \sum_{I \subseteq \{1,2,\ldots n\}} (-1)^{|I|} |A_I| \ .$$

Proof If $x \in S-(A_1 \cup A_2 \cup \ldots \cup A_n)$ then it is counted once. If x is contained in exactly k sets A_i then it is counted $\binom{k}{0}-\binom{k}{1}+\binom{k}{2}+\ldots+(-1)^k\binom{k}{k} = (1-1)^k = 0$ times. □

Example 2.1 *Three distinguishable experts rate a job candidate on a scale of 0 to 5. In how many ways can the total of the ratings add up to 9?*

Answer: Consider the case when the experts rate on a scale of 0 to infinite and let S be the set of all the ways how the total of ratings add up to 9. (Notice that $|S| = \binom{3+9-1}{9} = \binom{11}{9}$.) Let A_i be the subset of S such that the i-th expert's rating is at least 6. Now we are looking for $|S \cup (A_1 \cup A_2 \cup A_3)|$. By the principle of inclusion and exlusion,

$$\begin{aligned}|S - (A_1 \cup A_2 \cup A_3)| &= |S| - \sum_{i=1}^{3} |A_i| + \sum_{1 \le i < j \le 3} |A_i \cap A_j| - |A_1 \cap A_2 \cap A_3| = \\ &= \binom{11}{9} - 3\binom{3+3-1}{3} + 0 - 0 = \binom{11}{9} - 3\binom{5}{3} = 25 \ .\end{aligned}$$

3 THE PIGEONHOLE PRINCIPLE

Some versions of pigeonhole principle.

Proposition 3.1 *If $k+1$ pigeons are placed into k pigeonholes then at least one pigeonhole will contain two or more pigeons.*

Proposition 3.2 *If m pigeons are placed into k pigeonholes then at least one pigeonhole will contain at least $\lfloor \frac{m-1}{k} \rfloor + 1$ pigeons.*

Proposition 3.3 *Given a set of real numbers, there is always a number in the set whose value is at least as large (as small) as the average value of the numbers in the set.*

Example 3.1 *There are 15 minicomputers and 10 printers in a workroom. At most 10 computers are in use at one time. Every 5 minutes, some subset of computers requests printers. We want to connect each computer to some of the printers so that we should use as few connections as possible but we should be always sure that a computer will have a printer to use. (At most one computer can use a printer at a time.) How many connections are needed?*

Answer: Note that if there are fewer than 60 connections then there will be some printers connected to at most 5 computers by Proposition 3.3. If the remaining 10 computers were used at one one time, there would be only 9 printers left for them. Thus, at least 60 connections are required. On the other hand, it can be shown that if the i-th printer is connected to the i-th, $(i+1)$-st, ..., $(i+5)$-th computers $(i = 1, \ldots, 10)$ then these 60 connections have the desired properties.

Example 3.2 *Show that if* $n+1$ *numbers are selected from the set* $\{1, 2, 3, \ldots, 2n\}$ *then one of these will divide another one of them.*

Answer: Take n "pigeonholes". Put the selected numbers of form $(2k-1)2^{\alpha}$ into the k-th pigeonhole $(1 \leq k \leq n)$. Then at least one pigeonhole will contain at least two numbers and one of these will divide another one of these.