Introduction to the theory of computing 2
Retake-grading guide
May 28, 2024

General principles.

The aim of the scoring guide is for the correctors to evaluate the papers in a uniform manner. Therefore, the guide provides the main ideas of (at least one possible) solution to each task and the sub-scores assigned to them. The guide is not intended to be a detailed description of the full solution of the tasks; the described steps can be considered as an outline of a solution that will achieve a maximum score.
The partial marks indicated in the guide are awarded to the solver only if the related idea is included in the thesis as a step of a clear, clearly described and justified solution. Thus, for example, the mere description of the knowledge, definitions, and theorems included in the material is not worth points without their application (even if, by the way, one of the described facts actually plays a role in the solution). Considering whether the score indicated in the guide is due to the solver (in part or in whole) taking into account the above, is entirely the competence of the corrector.
A partial score is awarded for any idea or partial solution from which a perfect solution to the task could have been obtained by suitable addition of the thought process described in the thesis. If a solver starts several significantly different solutions to the same task, then the maximum score that can be given is one. If each described solution or solution part is correct or can be supplemented to be correct, then the solution initiative with the most partial points is evaluated. However, if among several solution attempts there is one that is correct and one that contains (significant) errors, and the thesis does not reveal which one the solver considered to be correct, then the solution attempt with fewer points is evaluated (even if this score is 0).
The sub-scores in the guide can be divided further if necessary. A good solution different from the one described in the guide is of course worth maximum points, but without proof you can only refer to the theorems and statements in the lectures.

1. According to what we learned, a tree with 20 vertices has 19 edges, (2 points)
   So the sum of its degrees is 38. (2 points)
   Let U denote the set of vertices with degrees other than 10 and 1. U contains 9 vertices, (1 point) whose degree sum is $38 - 10 \cdot 1 - 1 \cdot 10 = 18$. (1 point)
   Since trees are connected by definition, there cannot be an isolated point in a tree with at least 2 vertices, i.e. there is no 0-degree vertex, therefore the degree of each vertex in the tree is at least 1. (1 point)
   The degree of the vertices in U must therefore be at least 2, since neither 1 nor 0 is possible. (1 point)
   Thus, no vertex of U can have a degree of 3, otherwise the total degree of the vertices in U would exceed 18. (2 points)
   So there is no 3-degree vertex in the graph. (0 point)

Anyone who doesn't care at all about potential 0-degree vertices (as if they couldn't exist in any graph) should therefore lose 1 point.

2. If we delete from the graph the vertices belonging to numbers smaller than 50, (3 points)
   then we get a graph with a component of 2 vertices and 49 components that are isolated points. (2 points)
   Indeed: the vertices of the remaining graph are the integers from 50 to 100, on which a single edge exists, namely between 50 and 100. (2 points)
   This is achieved because, in the case of any other pair, either the smaller number will be at least 51, or the larger number will be at most 99, so the quotient of the larger and smaller number does not reach 2. (1 point)
   Since the graph fell apart into 50 components after deleting 49 points, it does not have a Hamilton cycle due to the learned proposition. (2 points)

There are no points for presenting a Hamilton path, just like if someone shows that a specific Hamilton path cannot be closed into a Hamilton cycle. It is also not good for presenting a Hamilton cycle with a small error.

3. **(a)** If the chromatic number were 2, the vertices of G could be classified into two classes so that the edges only go between the classes (of course, this can also be expressed differently: G would be a bipartite graph). (2 points)
If there are 10 vertices in both classes, then the number of edges can be at most $10 \cdot 10 = 100$, because the graph is simple (do not deduct a point for the lack of the latter statement), (1 point)
and if the sizes of the two classes are different, then the difference in sizes is at least 2; then by transferring one point from the larger one to the smaller one, the number of available edges would increase, so in this case we could only have less than 100 edges. (1 points)
Therefore the chromatic number of G cannot be 2. (1 points)

Of course, it is possible to argue in favor of the latter statement in another way, e.g. with the inequality between the arithmetic mean and the geometric mean or by listing the cases. For an incomplete but meaningful proof attempt (e.g. "There will be the most edges if the vertices are evenly distributed in the two classes"), give 1 point. To whoever only states that there cannot be more than 100 edges, cannot obtain that 1 point.

(b) The chromatic number can be 3. (0 points)
Let H be the bipartite graph in which both classes have 10 vertices and any two vertices from different classes are connected by an edge. Now let G be the graph obtained from H by connecting two vertices belonging to the same class. (2 point)
The resulting graph G will have 101 edges. (1 points)
Because of part a), the chromatic number is at least 3 (but of course it is possible to reason in other ways, e.g. we can show a triangle in the graph). (1 points)
And it is easy to color G with 3 colors: we color the vertices of one class of H with the color 1 and the vertices of the other class with the color 2, then we recolor one of the endpoints of the extra edge with color 3, in this case no edge goes between vertices of the same color. (1 points)
Therefore, the chromatic number of G is 3. (1 points)

Anyone who gives a bad example, i.e. one for which one of the properties (vertex number, edge number, chromatic number, simplicity) is not fulfilled, cannot of course receive points for proving the properties of the bad example.

4. $a_1, a_3, a_4, b_1, b_4, b_7, b_8$ is a 7-element vertex cover in the bipartite graph, (3 points)
because all the 1's in the matrix are in one of the corresponding rows or columns (that is, every edge has at least one endpoint among those listed). (1 point)
$\{(a_1, b_5), (a_2, b_4), (a_3, b_2), (a_4, b_3), (a_5, b_7), (a_6, b_8), (a_8, b_1)\}$ is a matching of size 7, since none of the two edges have a common end point (do not deduct a point for the lack of the latter statement). (3 point)
The specified vertex covering and matching proves that $\tau(G) \leq 7$ and $\nu(G) \geq 7$, (1+1 points)
from which, according to the relationship $\nu(G) \leq \tau(G)$, we obtain $\nu(G) = \tau(G) = 7$ and thus the given matching is maximum and the vertex cover is minimum. (1 points)

Instead of the statement $\nu(G) \leq \tau(G)$, it is possible (although unnecessary) to refer to Kőnig's theorem (in a bipartite graph $\nu(G) = \tau(G)$). The points awarded for justifying the statements $\nu(G) = 7$ and $\tau(G) = 7$, on the other hand, are awarded only to those who convincingly and clearly justify that the given matching is maximum, and the vertex covering set given is minimum. (Empty phrases such as "Because of Kőnig's theorem" are not worth points.) We note that it is of course worth searching for the maximum matching and the minimum vertex covering set using the augmenting path algorithm learned in the lecture; however (as can be seen from the above) it is not necessarily necessary to document its steps for a full-fledged solution. On the other hand, the justification of the maximality of the pairing and the minimality of the vertex cover can also be based on the algorithm: if we (convincingly) show that there is no augmenting path for the given matching, then according to what we have learned, it is maximum; and if, in this case, the vertex cover is chosen according to the algorithm, then it is minimum.

5. Since the graph is simple, we can use Vizing's theorem: $\chi_e(G) \leq \Delta(G) + 1$, so the edge chromatic number is at most 4. (2 points)
   Whoever does not mention simplicity should lose 1 point.
   For any (loop-free) graph, $\chi_e(G) \geq \frac{|E(G)|}{\nu(G)}$ holds. (2 points)
   In G, $|E(G)| = 16 \cdot 3\ /2\ =\ 24$, (2 points)
   and according to the text of the task, $\nu(G) = 7$. (1 points)
   Since $24/7 > 3$, at least 4 colors are needed to color the edges of the graph, (2 points)
   the wanted edge chromatic number is therefore 4. (1 point)

No points are awarded for the statement $\chi_e(G) \geq 3$. On the other hand, those who, after establishing that it can be colored with 4 colors, make efforts to see that 4 colors are needed, can receive 1 point even if these efforts are not successful.

6. First, let's determine the maximum number of edges the graph can have. Since a 3-vertex component can have at most 3 edges, which is the same as the number of vertices in the component, and a 4-vertex component can have at most 6 edges, which is 2 more than the number of vertices in the component, the most edges possible is to have as many 4-vertex components as possible. The number of components with 4 vertices is then 24, the number of components with 3 vertices is 2, and that of the edges is 24·6+2·3 = 150. (Here we took advantage of the fact that the graph is simple, but no points should be deducted for the lack of this statement.) (3 points)
   The number of components will obviously be minimal at the same time, i.e. if the number of components with 4 vertices is maximum, and then - as we have seen - we will have 26 components. (1 point)
   To make the graph connected, we should therefore add at least 25 edges, (1 point)
   because the inclusion of an edge reduces the number of components by at most one (and deleting an edge obviously cannot reduce it). (1 point)
   So at least 25 steps would be needed, during each step we remove three times as many edges as we add, so the number of edges would decrease by at least 50. (2 points)
   Since the graph originally had a maximum of 150 edges, it could then have a maximum of 100 edges, but then it cannot be connected, since according to what we have learned, a connected graph with 102 vertices must have at least 101 edges. (2 points)

   Although it was not mentioned that the connectivity should also be checked during the execution of a step, it is worth noting that since we first delete edges and then add one, the graph cannot be connected during the execution of step 25 either.