Retake Spring-2023: Grading guide

The aim of the scoring guide is for the correctors to evaluate the papers in a uniform manner. The aim of the guide is not to provide a detailed description of the complete solution of the tasks; the described steps can be considered as an outline of a solution that will achieve a maximum score. The sub-points indicated in the guide are awarded to the solver only if the related idea is included in the thesis as a step of a clear, clearly described and justified solution. Thus, for example, the mere description of the knowledge, definitions, and theorems included in the material is not worth points without their application (even if, by the way, one of the described facts actually plays a role in the solution). Considering whether the score indicated in the guide is due to the solver (in part or in whole) taking the above into account, is entirely the responsibility of the corrector.
Partial points are awarded for all ideas and thoughts that can play a meaningful role in a solution and from which, with a suitable addition to the thought process described in the thesis, a flawless solution to the task could be obtained. If a solver starts several, significantly different solutions to the same task, then the maximum score that can be given is one. If each described solution or solution part is correct or can be supplemented to be correct, then the solution initiative with the most partial points is evaluated. However, if among several solution attempts there is both a correct one and one containing a (significant) error, and the thesis does not reveal which one the solver considered correct, then the solution attempt with fewer points is evaluated (even if this score is 0).
The sub-scores in the guide can be divided further if necessary. A good solution different from the one described in the guide is of course worth maximum points, but one can only refer to the theorems and statements in the lectures without proof.

1. **First Solution.**
   A tree with 13 vertices has 12 edges according to what we learned, (1pt)
   so the sum of its degrees is 24. (1p)
   The degree sum of the two 6-degree vertices is 12, so the degree sum of the remaining 11 vertices is 12. (2pt)
   Since trees are connected by definition, there cannot be an isolated point in a tree with at least 2 vertices, i.e. a 0-degree vertex, (1 point)
   (don't deduct points for the lack of "at least 2 vertices ")
   therefore, the degree of every vertex is at least 1, from which it follows that the degree of no vertex can be 3, (1 point)
   since otherwise, we would have 10 vertices, the total degree of which would be 9, which would contradict the conclusion just now. (4 points)

   Second solution.
   We will first show that the two peaks of degree 6 either have exactly one neighbor in common or they themselves are neighbors. This statement is worth 3 points if proved correctly. After that, we discuss the two mentioned cases, which (also with adequate evidence) are worth 3 and 4 points, respectively.
   In detail:

   Let a and b be the two vertices of degree 6. Let's review the vertices of the graph: in addition to a and b, we also know about 6 of their neighbors. (0 points)
   If these were different from each other and also from a and b, then the graph would have 14 vertices, so of course this is not fulfilled, (1 point)
   so either a and b have a common neighbor, or they are adjacent to each other. (1 point)

If they have a neighbor in common, then there can only be one of these, otherwise there would be a cycle in the tree, which is of course impossible, and in this case a and b cannot be neighbors either, again only because then there would be a cycle in the tree. (1 point)
Thus, we counted all 13 vertices of the graph: a, b, the common neighbor and the 5 + 5 different neighbors. (1 point)
There will be a path between any 2 vertices along the edges known based on the neighborhood relations between these vertices, so there are no additional edges in the tree (because the ones known so far already form a spanning tree - this does not necessarily have to be described for the point). (1 point)
Thus, the common neighbor is of degree 2, the own neighbors are of degree 1 (and a and b are of degree 6), so there really is no vertex of degree 3 in the tree. (1 point)

If a and b are adjacent, then of course they cannot have a common neighbor (for the above reason, there would be a cycle), (1 point)
then we see 12 vertices from the tree: a, b, and 5 + 5 own neighbors. (1 point)
Due to the connectivity of the tree, the missing vertex must be connected to one of the known ones with an edge, and thus there are no additional edges apart from the known edges. (1 point)
One of the 5 neighbors (of a or b) is of degree 2 then, and the others are of degree 1, and the 13th vertex is also of degree 1, so now there is no vertex of degree 3, with which we have completed the proof. (1 point)

- In the second solution, we essentially showed that the tree (up to isomorphism) can only look in two ways, and it will not have a 3-degree vertex in either case. Whoever presents the two possibilities, but does not prove at all that there is no other possibility, will receive only 2 points.

2. In G, the degree of vertices 1, 2, 7 and 8 is 3, the degree of the other vertices is 4. (0 points)
Since it has more than two vertices of odd degrees, it does not have an Euler walk according to what we have learned, that is, deleting 0 points is not enough. (2 points)
After vertex (1) is deleted, degrees of 2 and 8 change to 2, and degree 3 to 3, so the resulting graph will have only two vertices with odd degrees: 3 and 7. (3 points)
(These points are awarded if someone finds a vertex that leaves only two odd degrees, not even partially for degrees calculated after aimless omissions.)
We can also easily make sure that the resulting graph is connected (2-3-4-5-6-7-8 is a Hamilton path, for example), (2 points)
so, according to what we have learned, it has an Euler walk, (2 points)
the required minimum is therefore 1. (1 point)

Instead of using the learned theorem, it is of course also possible to specify an Euler walk in the graph obtained after leaving a suitable vertex,
this solution is also awarded the third, fourth and fifth sub-scores (if the description is complete, of course).

3. (a) Since odd circles cannot be colored with 2 colors, the chromatic number cannot be 2. (Of course, we can also refer to the fact that the 2-colorable graphs are bipartite graphs, which, according to what we have learned, do not contain an odd circle.) (1 point)

(b) This can happen, consider, for example, the bow tie graph (that is, the graph that consists of two or three long circles with a point in common). This graph meets the conditions and has a chromatic number of 3. (1 point)

Indeed, the vertices can be colored with three colors: the vertices of the triangles are given the colors 1,2,3 in such a way that the common vertex is (say) colored 1. And two colors are not enough because of what we saw earlier (odd circle). (1 pt.)
(c) By deleting the vertex v, the resulting graph will not have an odd circle, (1pt)
since nothing like this can arise from deletion, and the odd circles of G contained v. (1 point)
The graph G′ obtained after deletion is therefore a bipartite graph according to what was learned, (1pt)
that is, 2-colorable. (1 point)
It follows that G must be 3-colorable. (1 point)
Indeed, taking a good 2-coloring of G′ and coloring the vertex v with a third color gives a good 3-coloring of G, (1pt)
so the chromatic number of G cannot be 4. (1pt)

4. Based on the matrix, it is easy to check that a1,a4,a6,a8,b3,b5,b8 is a set of 7-element vertex covering points in the graph, (2 points)
   because all the 1s in the matrix are in one of the corresponding rows or columns. (1 point)
   The edge set (a1, b2), (a2, b3), (a3, b8), (a4, b1), (a6, b4), (a7, b5), (a8, b6) is a 7-element matching, since no two edges have a common endpoint. (1 point)
   The specified set of vertex cover and matching proves that $\tau(G) \leq 7$ and $\nu(G) \geq 7$, (1+1 points)
   from which, according to the relation $\nu(G) \leq \tau(G)$, $\nu(G) = \tau(G) = 7$, and thus the given matching is maximum, and the given covering set is minimum. (1 point)
   According to what we have learned, then the complement of the minimum vertex covering set given above, i.e. a2,a3,a5,a7,b1,b2,b4,b6,b7, is a maximum independent set of points. (3 points)

   Anyone who only determines the size of the maximum independent point set (e.g. with the help of Gallai's theorem) should receive 1 of the last 3 points, just like the person who finds the maximum independent point set, but does not justify the maximality.
   Instead of the statement $\nu(G) \leq \tau(G)$, it is possible (although unnecessary) to refer to Kőnig's theorem (in a bipartite graph $\nu(G) = \tau(G)$). The total of 3 points for justifying the statement $\nu(G) = 7$, on the other hand, is awarded only to those who convincingly and clearly justify that the given pairing is maximal. (Empty phrases such as "Because of Kőnig's theorem" are not worth points.) We note that it is of course worth searching for the maximum matching and the minimum set of covering points using the augmenting path algorithm learned in the lecture; however (as can be seen from the above) it is not necessarily necessary to document its steps for a full-fledged solution. On the other hand, the justification of the maximality of the matching and the minimality of the vertex covering set can also be based on the algorithm: if we (convincingly) show that there is no improvement path for the given matching, then according to what we have learned, it is maximum; and if, in this case, X consists of the vertices in A and those in B chosen according to the outcome of the algorithm, then it is, according to what has been learned, a set of minimum-sized vertex covering set.
5. Provide a proper edge coloring using 6 colors, so the edge chromatic number is at most 6. (5 pt)
   At the same time, the edge chromatic number cannot be less than 6. (0 points)
   Because any color class forms a matching, (1 point)
   and the size of the maximum matching is at most 2, since the graph has only 5 vertices, (1 point)
   so we can color a maximum of 10 edges with 5 colors, but the graph has 11 edges. (3 points)

   - The Vizing theorem cannot be used for the upper estimate, since the graph is not simple, and those who base it on this will receive 0 out of the relevant 5 points.

6. Let e = (a, b) be an edge without which the graph is no longer connected. Then the graph G − e will have two components, otherwise we would not be able to get a connected graph by putting e back. (1 point)
   Let the component containing vertex a be A, and the component containing vertex b be B. In both components, the degree of one vertex (a and b) is at least 9, and the degree of the other vertices is at least 10. (1pt)
   Since the graph is simple, there must be at least 11 vertices in both components, (1pt)
   from which it also follows that each of the components have at most 19 vertices. (2pts)
   The condition of the Ore's theorem is thus fulfilled in both components, so according to the theorem, both components have a Hamilton cycle. (2 points)
   Let's leave the edge incident to (a) from a Hamilton cycle of the component A, so we get a Hamilton path of A, let's call it P. Similarly, let's leave the edge incident to (b) in the Hamiltonian cycle of B, so we get a Hamiltonian path of B, let's call it Q. (2 points)
   Then the edge set consisting of the union of P, Q and e is a Hamiltonian path of G, which completes the proof. (1 point).