Introduction to Theory of Computing II.
Midterm grading guide

General Rules. Disclaimer: Google translate has been used for this section. The solutions have been translated by Padmini Mukkamala.

The purpose of the scoring guide is to ensure that the dissertations are evaluated uniformly by the correctors. Therefore, the guide the main ideas for solving each task (at least one possible) and the marks assigned to them communicates sub-scores. The guide is not intended to detail the complete solution of the tasks description; the steps described can be considered as a sketch of a solution with a maximum score.

The sub-scores indicated in the guide only accrue to the solver if the related idea is included in the dissertation as a step towards a clear, clearly described and justified solution. Thus, for example, stating the definitions and items in the material without knowing how to apply them does not deserve any points (even if any of the facts described are indeed used in the solution). Deciding the score based on the points indicated in the guide in light of the above is under the grader's full remedial authority.

A partial score is awarded for each idea or partial solution from which, with a suitable addition, a flawless solution to the problem would have been obtained. If a solver starts several several substantially different solutions for a task, he can be assigned to at most one score. If all the solutions or parts of solutions described are correct or correct, then the solution initiative worth the most subpoints is evaluated. However, if amongst several solution attempts there is a correct solution but also an incorrect one (with a substantial error), and it is not clear from the dissertation which the solver considered as correct, then the solution with fewer points is evaluated (even if this score is 0).

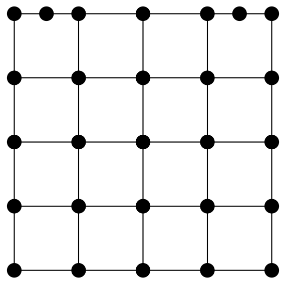
The sub-scores in the guide can be further divided if necessary. A good solution other than that described in the guide is, of course, worth a maximum point. Theorems can be stated without proof, but only those discussed in class.

1. The number of vertices with degree at least 2 is 17. (3 points)
The total degree of these 17 vertices is at least 34, (2 points)
so the sum of the degrees of all the vertices in the graph is at least 37. (1 points)
Since sum of the degrees is twice the number of edges of the graph, and this number must be even, it must be at least 38,
(2 points)
so the graph has at least 19 edges. (2 points)

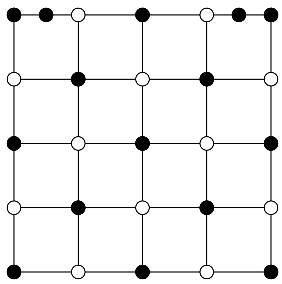
2. Since the graph is disconnected, it must have at least two components, name any two of them as A and B. (2 points)
Since the graph is simple and the degree of every vertex is at least 2, every component must contain at least 3 vertices (so do A and B).
(2 points)
Since there are no edges between the vertices of A and B in the graph, every edge between a vertex of A and of B is there in the complement . (2 points)

This means that $K_{3,3}$ is a subgraph of the complement, (2 points)
and so by the Kuratowski's theorem, the complement is not planar. (2 points)

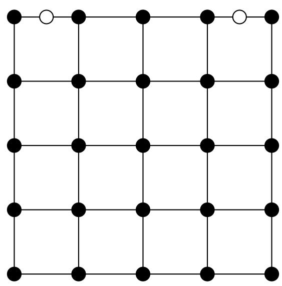
3. We have to determine if the following graph has a Hamiltonian cycle.



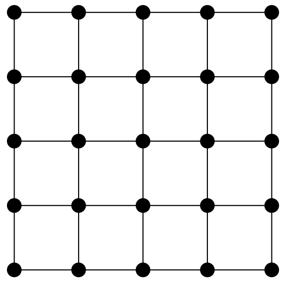
First Solution: Deleting the 12 white vertices in the drawing below gives 13 components, (7 points)
so by the theorem that if a graph has a Hamiltonian cycle then the deletion of any k vertices results in at most k components, this graph does not have a Hamiltonian cycle. (3 points)



Second solution: Consider the two white vertices below. If this graph has a Hamiltonian cycle, then it must include the two edges incident on the white vertices, because these two vertices are of degree 2. (2 points)



It follows that if our original graph has a Hamiltonian cycle, then the graph obtained by smoothing these two vertices (smoothing like we learnt for graph homeomorphisms in planarity) will also have a Hamiltonian cycle. (The opposite is not necessarily true) (3 points)



Since the graph obtained by smoothing is bipartite, (1 points)
, so it doesn't have an odd cycle, (1 points)
so it cannot have a Hamiltonian cycle (it has 25 vertices). (1 points)
The remaining 2 points are for showing the graph is bipartite, by for example showing a coloring.

The original graph is not bipartite, it has a 5-cycle. If the student says that the graph is bipartite and then can see why it does not contain a Hamiltonian cycle, then they are awarded 1 point.

4. The size of the maximum matching is equal to the size of a minimum vertex cover in a bipartite graph (Kőnig's Theorem), and hence for G this number is 50. (1 points)
We will show that the size of the minimum vertex cover for H is also 50. (1 points)
Since G is a subgraph of H, this size cannot be smaller than 50. (1 points)
Since G has a perfect matching, the two partitions have the same size of 50 vertices (1 points)
and both can be a minimal vertex cover for G. (1 points)
If the new edge is between the two partitions, then still either of the two partitions is a vertex cover, so the required size is 50 (2 points)
If the new edge is between two vertices of the same partition, say A, then A is a vertex cover of H, (2 points)
so the size of the minimum vertex cover of H is 50. (1 points)

5. Let us denote the original graph with G, and the graph obtained after deleting a Hamiltonian cycle with H. We have to show that the $\chi_e(G) \leq \chi_e(H)+2$. Say $\chi_e(H) = k$. (2 points)
We will now try to color G by first coloring the edges of H with k colors.

(3 points)
The edges not yet colored are the edges of the Hamiltonian cycle which has length $2n$, so we can color it with two new colors, say red and blue. (2 points)
We just color every odd numbered edge red, even numbered edge blue.

(1 points)
then no two red edges will be incident on a common vertex. (1 points)
So we have shown a coloring of G with $k+2$ colors, so the claim is true. (1 points)

The Vizing's theorem is not useful in this problem even if we assume the graph is simple. If someone shows using the Vizing's theorem that the decrease is at most 3, then they get 2 of the last 8 points. If someone uses Vizing's theorem without mentioning the additional assumption that the graph is simple, then no points are awarded.

6. (*)

Since the graph (lets call it G) is acyclic, so all its components are trees. (1 points)
All components cannot be isolated vertices, for then there would be only one kind of degree in the graph. So there must be degree 1 vertices.

(1 points)
Let the other degree in the graph be k where $k \neq 1$ and the number of degree k vertices is 5,6 or 7, while the corresponding number of vertices of degree 1 will be 7,6,5. So the sum of the degrees will be $5k+7$ (case 1.), $6k+6$ (case 2.) or $7k+5$ (case 3.). (1 points)
Since the graph is cycle-free, the total number of edges is atmost 11, and so the sum of degrees is at most 22. (1 points)
Case 1: $5k+7 \leq 22$. So, $k \leq 3$, but we also know that $k \neq 1$ and k is odd (since $5k+7$ is even). So, $k = 3$ and the graph has 11 edges. (1 points)
It is easy to construct such a graph with 5 vertices of degree 3 and 7 of degree 1. (1 points)
Case 2: $6k+6 \leq 22$. Then $k \leq 2$, or $k = 2$ (and the number of edges is 9) or $k = 0$ (and number of edges is 3). (1 points)
Both such graphs exist: For $k = 2$, for example, the graph has 3 disjoint paths of length 3. (1 points)
For $k = 0$, the graph has 6 isolated vertices and three disjoint edges. (1 points)
Case 3: $7k+5 \leq 22$. Then $k \leq 2$, but again k must be odd, but it can't be 1, so such a graph does not exist. (1 points)
So the possible number of edges in the graph is 3, 9 or 11. (0 points)