Tales about infinity

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- Defining property: $B = \{\text{positive numbers}\}$

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So: is there a fundamental error in math??? Not: we have just proved that the collection of all sets is *not* a set.

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- If he shaves himself, then he must not shave the village barber, since that shaves by himself. . .

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We have proved

Theorem

The collection of all sets, and the collection of all ordinary sets,respectively do not form a set.

Strange happenings in Grand Hotel Carolina

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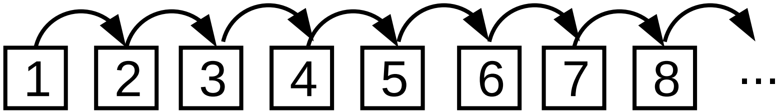

Day 2

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What did the System Administator suggest now?

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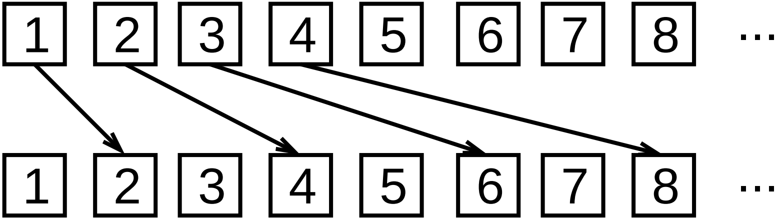

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The occupant of room number $2n + 1$ should move to room number n.

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The System Administrator, Susanne Lynoux, was desparately trying to find a solution. Her first idea, that the guests of Carolina move to rooms number $1001, 2001, 3001, \ldots$, then the guests from the first hotel can move to rooms number $1002, 2002, 3002, \ldots$, guest from the second hotel move to rooms number $1003, 2003, 3003, \ldots$, and so on, did not work, they got stuck at the guests of the 1000^{th} bankrupt hotel.

Ms Lynoux's second idea was that guests of the first hotel move to rooms number $2, 4, 8, 16, 32, \ldots$, guests from the second hotel move to rooms number $3, 9, 27, 81, \ldots$, in general, guests from he $n - 1^{\text{st}}$ hotel move to rooms $n, n^2, n^3, \ldots$

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However, Ms Lynoux could fix this scheme. She knew that there are an infinite number of prime numbers, thus if p_k denotes the k^{th} prime number, then the guests of the k^{th} hotel could move to rooms numbered $p_k, p_k^2, p_k^3, \ldots$, since powers of distinct prime numbers are distinct.

Worries of the director

Everyone was accommodated, nevertheless Mike Rosophte, the director was not satisfied, sice many rooms were empty i Hotel Carolina (e.g. numbers $6, 10, 12, 14, 18, \ldots$).

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The next idea of Susan Lynoux was to move the n^{th} guest of the m^{th} hotel to room number $2^m 3^n$. This did not cause any conflict, however it did not solve Mike Rosophte's problem either ...

The Big Idea

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By next morning Ms Lynoux figured out how can Hotel Carolina be filled up completely without leaving any guest unaccommodated. She wrote down the guests of the hotels in an infinite square table, the n^{th} position of the m^{th} row represented the n^{th} guest of the m^{th} hotel.

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Thus, for every m, na unique room is assigned and we proved that infinitely many guests of infinitely many hotels can be accommodated in one infinte hotel without leaving an empty room.
Susan Lynoux went for a well-deserved vacation. She should not have done so . . .

Chaos at the reception

Mike organized a welcome reception for the hotel guests. He wanted to send out the invitation twice the speed, however incidentally he just sent them to to every other room, numbers $2, 4, 6, \dots$

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Nevertheless, the infinite number chairs provided for the guests were all occupied when the other guests who were invited later arrived. They could be seated only after long manipulations...

Then ice-cream was served, everyone received two portions. The chef swore that he prepared only one portion for each guest.

Mike stood dumbfounded...

An unsolvable problem

The IRS ordered Hotel Carolina to submit a list of all possible utilizations of the hotel. An element of the list is an infinite sequence of 0's and 1's, the n^{th} element is is 0 if the n^{th} room is empty, and it is 1, if the n^{th} room is occupied.

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“Here is a utilization sequence that is not on your list: If the n^{th} entry of the n^{th} element on the list is 0, then let 1 stand in the n^{th} position in this sequence, otherwise let 0 stand there. Thus the new sequence differs from the n^{th} sequence on the list at the n^{th} position.”

The n^{th} utilization sequence is $u^{(n)} = u_{n1}, u_{n2}, u_{n3}, \ldots$.

$$\begin{matrix} u_{11} & u_{12} & u_{13} & u_{14} & \cdots \\ u_{21} & u_{22} & u_{23} & u_{24} & \cdots \\ u_{31} & u_{32} & u_{33} & u_{34} & \cdots \\ u_{41} & u_{42} & u_{43} & u_{44} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{matrix}$$

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Cantor's Diagonal Method

Equivalent sets

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- *If* $A \sim B$ *and* $B \sim C$ *then* $A \sim C$ *transitivity*

Examples

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- $\{\text{dog,cat,horse}\} \sim \{\text{puppy,kitty,colt}\} \sim \{0, 1, 2\}$

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- $(-\frac{\pi}{2}, \frac{\pi}{2}) \sim \mathbb{R}$ $f(x) = \tan x$.

Cardinality

Definition

If $A \sim B$, then A and B are said to have the same *cardinality*. The cardinality of a set A is denoted by $|A|$. The cardinality of the natural numbers is denoted by $\aleph_0$. A set A is finite if it is empty or $A \sim \{0, 1, \ldots, n-1\}$ for some n. A set is *countable* if it is finite or it is of cardinality $\aleph_0$.

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If A is infinite then it has a subset A' of cardinality $\aleph_0$.

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The cardinality of set A is less than or equal to that of set B if there exists a one-to-one mapping f of A to a subset of B. In notation $|A| \leq |B|$.

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- $|\mathbb{N}| \leq |A|$ if A is infinite.

Bernstein's Equivalence Theorem

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If $|A| \leq |B|$ *and* $|A| \geq |B|$, *then* $|A| = |B|$.

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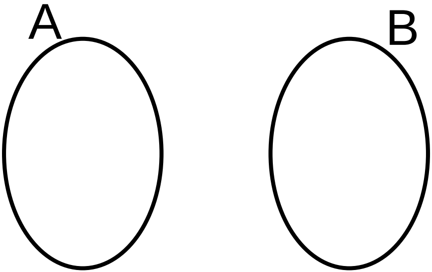

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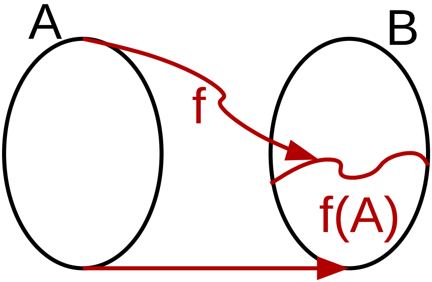

Bernstein's Equivalence Theorem

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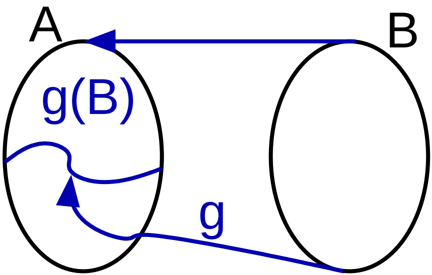

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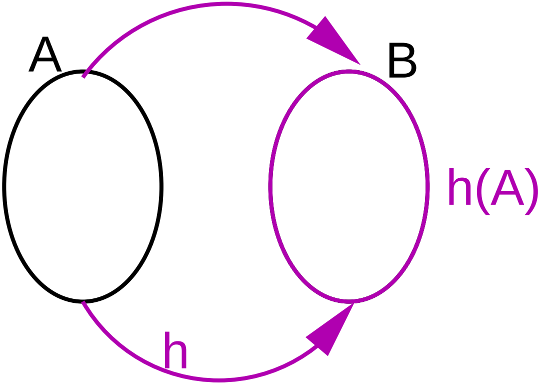

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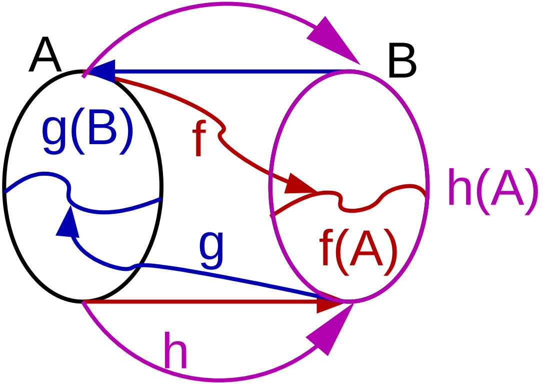

Theorem

The union of countably many countable sets is itself countable.

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This is the situation of Hotel Carolina after few years.

Proof

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This is the situation of Hotel Carolina after few years.

Proof Let the sets be $A_1, A_2, \ldots, A_n, \ldots$. Furthermore, let the m^{th} element of A_n be $a_{n,m}$. A one-to-one mapping is given

$$\text{from } \bigcup_{n=1}^{\infty} A_n \quad \text{to } \mathbb{N}.$$

For the sake of convenience (n, m) is written in place of $a_{n,m}$.

The mapping

The mapping

In fact the following is just one of the possible mappings.

$$\begin{array}{cccccc}
(1,1) & (1,2) & (1,3) & (1,4) & (1,5) & \cdots \\
(2,1) & (2,2) & (2,3) & (2,4) & (2,5) & \cdots \\
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$$\begin{tikzcd}
(1,1) & (1,2) \arrow[d] & (1,3) \arrow[d] & (1,4) \arrow[d] & (1,5) & \cdots \\
(2,1) & (2,2) \arrow[l] & (2,3) \arrow[d] & (2,4) \arrow[d] & (2,5) & \cdots \\
(3,1) & (3,2) \arrow[l] & (3,3) \arrow[l] & (3,4) \arrow[d] & (3,5) & \cdots \\
(4,1) & (4,2) \arrow[l] & (4,3) \arrow[l] & (4,4) \arrow[l] & (4,5) & \cdots \\
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Cardinalities of $\mathbb{Q}$ and $\mathbb{R}$

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Proposition

$\mathbb{Q} = \aleph_0$.

Cardinalities of $\mathbb{Q}$ and $\mathbb{R}$

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$\mathbb{Q} = \aleph_0$.

Proof: Write $-\frac{n}{m}$ in place of $(2n+1, m)$ and $\frac{n}{m}$ in place of $(2n, m)$ in the previous proof.

Cardinalities of $\mathbb{Q}$ and $\mathbb{R}$

Proposition

$\mathbb{Q} = \aleph_0$.

Proof: Write $-\frac{n}{m}$ in place of $(2n+1, m)$ and $\frac{n}{m}$ in place of $(2n, m)$ in the previous proof.

Is it true that $\mathbb{Q} = \mathbb{R}$?

$\mathbb{R}$ is uncountable

Theorem

$\mathbb{R} > \aleph_0$.

Proof:

$\mathbb{R}$ is uncountable

Theorem

$\mathbb{R} > \aleph_0$.

Proof: We show that even for the open interval $(0, 1)$ it is true that $|(0, 1)| > \aleph_0$.

$\mathbb{R}$ is uncountable

Theorem

$\mathbb{R} > \aleph_0$.

Proof: We show that even for the open interval $(0, 1)$ it is true that $|(0, 1)| > \aleph_0$. Assume in contrary, that $(0, 1)$ is countable, that is its elements can be listed $r_1, r_2, \ldots, r_n, \ldots$. Let number r_n be written in decimal form as $r_n = 0.d_{n1}d_{n2}d_{n3}\ldots$.

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- Are there even larger cardinalities?

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Definition

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Proof: The inequality $|A| \leq |\mathfrak{P}(A)|$ is shown by the mapping $a \mapsto \{a\}$. Thus, we only need to prova that $|A| \neq |\mathfrak{P}(A)|$. Assume indirectly that there exists a one-to-one mapping $f\colon A \to \mathfrak{P}(A)$ such that every element of $\mathfrak{P}(A)$ is in the range of f.

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Thus our starting assumption that $|A| = |\mathfrak{P}(A)|$ must be false, so $|A| < |\mathfrak{P}(A)|$.

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Thus our starting assumption that $|A| = |\mathfrak{P}(A)|$ must be false, so $|A| < |\mathfrak{P}(A)|$.
For every cardinality there exists a larger one!