1. Find the eigenvalues and eigenvectors of the following matrices.

   a) $A = \begin{pmatrix} 5 & 7 \\ -3 & -5 \end{pmatrix}$

   λ is an eigenvalue of A iff $\det(A - \lambda I) = 0$.

   $\det(A - \lambda I) = \det \begin{pmatrix} 5-\lambda & 7 \\ -3 & -5-\lambda \end{pmatrix} = (5-\lambda)(-5-\lambda) - 7\cdot(-3) = \lambda^2 - 4.$

   Thus, $\det(A - \lambda I)$ is zero for $\lambda_1 = 2$ and $\lambda_2 = -2$ only, so these are the eigenvalues of A.

   In order to find the eigenvector that belongs to λ_1 the system of linear equations $(A - \lambda_1 E)\underline{x}_1 = 0$ is solved by Gaussian elimination.

   $$\left(\begin{array}{cc|c} 5-\lambda_1 & 7 & 0 \\ -3 & -5-\lambda_1 & 0 \end{array}\right) = \left(\begin{array}{cc|c} 3 & 7 & 0 \\ -3 & -7 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 3/7 & 0 \\ 0 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 3/7 & 0 \end{array}\right)$$

   That is the form of eigenvector $\underline{x}_1$ is: $\underline{x}_1 = \begin{pmatrix} -3/7y \\ y \end{pmatrix}$, where $y \in \mathbb{R}$ arbitrary.

   The subspace spanned by eigenvectors belonging to λ_1 is: $\left\{ \begin{pmatrix} -3/7t \\ t \end{pmatrix} : t \in \mathbb{R} \right\}$.

   Similarly, to calculate the eigenvectors that belong to $\lambda_2 = -2$ the system $(A - \lambda_2 E)\underline{x}_2 = 0$ is solved using Gaussian elimination.

   $$\left(\begin{array}{cc|c} 5-\lambda_2 & 7 & 0 \\ -3 & -5-\lambda_2 & 0 \end{array}\right) = \left(\begin{array}{cc|c} 7 & 7 & 0 \\ -3 & -3 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 1 & 0 \end{array}\right)$$

   That is the form of eigenvector $\underline{x}_2$ is: $\underline{x}_2 = \begin{pmatrix} -y \\ y \end{pmatrix}$, where $y \in \mathbb{R}$ arbitrary.

   The subspace spanned by eigenvectors belonging to λ_2 is: $\left\{ \begin{pmatrix} -t \\ t \end{pmatrix} : t \in \mathbb{R} \right\}$.

   b) $A = \begin{pmatrix} 0 & 0 & -2 \\ 3 & -2 & -3 \\ 6 & -6 & 1 \end{pmatrix}$

   λ is an eigenvalue of A iff $\det(A - \lambda I) = 0$.

   $\det(A - \lambda I) = \det \begin{pmatrix} -\lambda & 0 & -2 \\ 3 & -2-\lambda & -3 \\ 6 & -6 & 1-\lambda \end{pmatrix} = -\lambda((-2-\lambda)(1-\lambda) - (-6)\cdot(-3)) - 2(3\cdot(-6) - 6(-2-\lambda)) = -\lambda^3 - \lambda^2 + 8\lambda + 12.$

   Easy to see that -2 is a root of the cubic polynomial above, hence by long division we obtain:

   $\det(A - \lambda I) = -\lambda^3 - \lambda^2 + 8\lambda + 12 = (\lambda + 2)(-\lambda^2 + \lambda + 6) = (\lambda + 2)(-\lambda - 2)(\lambda - 3).$

   Thus the eigenvalues are: $\lambda_1 = -2$ s $\lambda_2 = 3$.

   In order to find the eigenvectors that belong to λ_1 the nonzero solutions of the system $(A - \lambda_1 E)\underline{x}_1 = 0$ are determined using Gaussian elimination.

   $$\left(\begin{array}{ccc|c} -\lambda_1 & 0 & -2 & 0 \\ 3 & -2-\lambda_1 & -3 & 0 \\ 6 & -6 & 1-\lambda_1 & 0 \end{array}\right) = \left(\begin{array}{ccc|c} 2 & 0 & -2 & 0 \\ 3 & 0 & -3 & 0 \\ 6 & -6 & 3 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -6 & 9 & 0 \end{array}\right) \sim$$

   $$\sim \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -3/2 & 0 \end{array}\right)$$

   So the form of eigenvector $\underline{x}_1$ is: $\underline{x}_1 = \begin{pmatrix} z \\ 3/2z \\ z \end{pmatrix}$, where $z \in \mathbb{R} \setminus \{0\}$ arbitrary.

   Thus the eigenspace of λ_1 is: $\left\{ \begin{pmatrix} t \\ 3/2t \\ t \end{pmatrix} : t \in \mathbb{R} \right\}$.

   Similarly, the calculation of eigenvectors that belong to $\lambda_2 = -3$:

   $$\left(\begin{array}{ccc|c} -\lambda_2 & 0 & -2 & 0 \\ 3 & -2-\lambda_2 & -3 & 0 \\ 6 & -6 & 1-\lambda_2 & 0 \end{array}\right) = \left(\begin{array}{ccc|c} -3 & 0 & -2 & 0 \\ 3 & -5 & -3 & 0 \\ 6 & -6 & -2 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 2/3 & 0 \\ 0 & -5 & -5 & 0 \\ 0 & -6 & -6 & 0 \end{array}\right) \sim$$

   $$\sim \left(\begin{array}{ccc|c} 1 & 0 & 2/3 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right)$$

   Thus the form of $\underline{x}_2$ is: $\underline{x}_2 = \begin{pmatrix} -2/3z \\ -z \\ z \end{pmatrix}$, where $z \in \mathbb{R} \setminus \{0\}$ arbitrary.

The eigenspace that belongs to λ_2 is: $\left\{ \begin{pmatrix} -2/3t \\ -t \\ t \end{pmatrix} : t \in \mathbb{R} \right\}$.

c) $A = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 3 \end{pmatrix}$

λ is an eigenvalue of A iff $\det(A - \lambda I) = 0$.

$\det(A-\lambda I) = \det \begin{pmatrix} 5-\lambda & 0 & 0 \\ 0 & 1-\lambda & 2 \\ 0 & 1 & 3-\lambda \end{pmatrix} = (5-\lambda)((1-\lambda)(3-\lambda)-1\cdot 2) = (5-\lambda)(\lambda^2-4\lambda+1)$.

Thus the eigenvalues are $\lambda_1 = 5$, and from the quadratic formula $\lambda_{2,3} = \frac{4\pm\sqrt{12}}{2}$, that is $\lambda_2 = 2+\sqrt{3}$ s $\lambda_3 = 2-\sqrt{3}$.

Calculation of eigenvectors that belong to λ_1:

$$\left(\begin{array}{ccc|c} 5-\lambda_1 & 0 & 0 & 0 \\ 0 & 1-\lambda_1 & 2 & 0 \\ 0 & 1 & 3-\lambda_1 & 0 \end{array} \right) = \left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & -4 & 2 & 0 \\ 0 & 1 & -2 & 0 \end{array} \right) \sim \left(\begin{array}{ccc|c} 0 & 1 & -1/2 & 0 \\ 0 & 0 & -3/2 & 0 \end{array} \right) \sim$$

$$\sim \left(\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & -3/2 & 0 \end{array} \right)$$

Thus $\underline{x}_1$ is: $\underline{x}_1 = \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix}$, where $x \in \mathbb{R} \setminus \{0\}$ arbitrary.

The eigenspace of λ_1 is: $\left\{ \begin{pmatrix} t \\ 0 \\ 0 \end{pmatrix} : t \in \mathbb{R} \right\}$.

Calculation of eigenvectors that belong to $\lambda_2 = 2+\sqrt{3}$: we find the nonzero solutions of $(A - \lambda_2 E)\underline{x}_2 = 0$ by Gaussian elimination.

$$\left(\begin{array}{ccc|c} 5-\lambda_2 & 0 & 0 & 0 \\ 0 & 1-\lambda_2 & 2 & 0 \\ 0 & 1 & 3-\lambda_2 & 0 \end{array} \right) = \left(\begin{array}{ccc|c} 3-\sqrt{3} & 0 & 0 & 0 \\ 0 & -1-\sqrt{3} & 2 & 0 \\ 0 & 1 & 1-\sqrt{3} & 0 \end{array} \right) \sim$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1-\sqrt{3} & 0 \\ 0 & 0 & 2+(1+\sqrt{3})(1-\sqrt{3}) & 0 \end{array} \right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1-\sqrt{3} & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

Thus $\underline{x}_2$ is: $\underline{x}_2 = \begin{pmatrix} 0 \\ (\sqrt{3}-1)z \\ z \end{pmatrix}$, ahol $z \in \mathbb{R} \setminus \{0\}$.

The eigenspace of λ_2 is: $\left\{ \begin{pmatrix} 0 \\ (\sqrt{3}-1)t \\ t \end{pmatrix} : t \in \mathbb{R} \right\}$.

Similarly, calculation of eigenvectors that belong to $\lambda_2 = 2-\sqrt{3}$:

$$\left(\begin{array}{ccc|c} 5-\lambda_3 & 0 & 0 & 0 \\ 0 & 1-\lambda_3 & 2 & 0 \\ 0 & 1 & 3-\lambda_3 & 0 \end{array} \right) = \left(\begin{array}{ccc|c} 3+\sqrt{3} & 0 & 0 & 0 \\ 0 & -1+\sqrt{3} & 2 & 0 \\ 0 & 1 & 1+\sqrt{3} & 0 \end{array} \right) \sim$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1+\sqrt{3} & 0 \\ 0 & 0 & 2+(1-\sqrt{3})(1+\sqrt{3}) & 0 \end{array} \right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1+\sqrt{3} & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

Thus, $\underline{x}_3$ is: $\underline{x}_3 = \begin{pmatrix} 0 \\ (-\sqrt{3}-1)z \\ z \end{pmatrix}$, wher $z \in \mathbb{R} \setminus \{0\}$ arbitrary.

The eigenspace of λ_3: $\left\{ \begin{pmatrix} 0 \\ (-\sqrt{3}-1)t \\ t \end{pmatrix} : t \in \mathbb{R} \right\}$.

d) $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$

λ is an eigenvalue of A iff $\det(A - \lambda I) = 0$.

$$\det(A-\lambda I)=\det\begin{pmatrix}1-\lambda & 1 & 0\\ 0 & 1-\lambda & 0\\ 0 & 1 & 1-\lambda\end{pmatrix}=(1-\lambda)((1-\lambda)(1-\lambda)-0)=(1-\lambda)^3.$$

Thus the unique eigenvalue of matrix A is $\lambda=1$.

Calculation of eigenvectors:

$$\left(\begin{array}{ccc|c}1-\lambda_1 & 1 & 0 & 0\\ 0 & 1-\lambda_1 & 0 & 0\\ 0 & 1 & 1-\lambda_1 & 0\end{array}\right)=\left(\begin{array}{ccc|c}0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\end{array}\right)\sim\left(\begin{array}{ccc|c}0 & 1 & 0 & 0\end{array}\right)$$

Thus the form of the eigenvector is: $\underline{x}=\begin{pmatrix}x\\0\\z\end{pmatrix}$, where $x,z\in\mathbb{R}$, furthermore at least one of x and z is nonzero.

Thus the eigenspace of $\lambda=1$ is: $\left\{\begin{pmatrix}t_1\\0\\t_2\end{pmatrix}: t_1,t_2\in\mathbb{R}\right\}$.

2. Let$V$ be the vectorspace of real polynomials of degree at most 6. Determine the eigenvalues and eigenvectors of the following linear transformations.

a) $f(x)\to 0$

The matrix of this transformation is the 6×6 all-zero matrix (in arbitrary basis).

This transformation assigns to any polynomial the constant 0 polynomial, that is to $f(x)$ the polynomial $0\cdot f(x)$. This means the only eigenvalue is 0, and every nonzero plynomial is an eigenvector, so the eigenspace of eigenvalue 0 is the whole space V.

b) $f\to f'$

The matrix of the transformation in the usual basis $\{x^6,x^5,\dots,x,1\}$ is:

$$\begin{pmatrix}0&0&0&0&0&0&0\\6&0&0&0&0&0&0\\0&5&0&0&0&0&0\\0&0&4&0&0&0&0\\0&0&0&3&0&0&0\\0&0&0&0&2&0&0\\0&0&0&0&0&1&0\end{pmatrix}$$

This transformation assigns every polynomial its derivative. Since differentiation decreases the degree of a polynomial (except for degree 0), $f'(x)=\lambda f(x)$ can only hold if $f(x)$ is of degree 0, that is a constant polynomial. The derivative of those is the constant 0 polynomial.

Thus, the only eigenvalue is 0, and every constant, but nonzero polynomial is an eigenvector. The eigenspace belonging to eigenvalue 0 is $\{f(x)=c|c\in\mathbb{R}\}$.

c) $f\to xf'$

The given transformation $\mathcal{A}$ assigns to polynomial $f(x)=a_6x^6+a_5x^5+\cdots+a_1x+a_0$ the polynomial

$$\mathcal{A}(f(x))=6\cdot a_6x^6+5\cdot a_5x^5+\cdots+1\cdot a_1x, \tag{1}$$

thus the matrix of $\mathcal{A}$ in the usual basis $\{x^6,x^5,\dots,x,1\}$ is:

$$A=\begin{pmatrix}6&0&0&0&0&0&0\\0&5&0&0&0&0&0\\0&0&4&0&0&0&0\\0&0&0&3&0&0&0\\0&0&0&0&2&0&0\\0&0&0&0&0&1&0\\0&0&0&0&0&0&0\end{pmatrix}$$

In order to find the eigenvalues, the roots of $\det(A-\lambda I)$ are determined:

$$\det(A-\lambda I)=\det\begin{pmatrix}6-\lambda&0&0&0&0&0&0\\0&5-\lambda&0&0&0&0&0\\0&0&4-\lambda&0&0&0&0\\0&0&0&3-\lambda&0&0&0\\0&0&0&0&2-\lambda&0&0\\0&0&0&0&0&1-\lambda&0\\0&0&0&0&0&0&-\lambda\end{pmatrix}=$$

$= (6-\lambda)(5-\lambda)\cdots(1-\lambda)(-\lambda)$

Thus the eigenvalues of A are $6, 5, 4, 3, 2, 1$and 0. In order to find the eigenvectors, one has to consider that by (1) the image of polynomial $f(x) = a_6x^6 + a_5x^5 + \cdots + a_1x + a_0$ is 6-times itself exactly if $a_5 = a_4 = \cdots = a_0 = 0$. Thus the eigenvectors that belong to eigenvalue $\lambda_1 = 6$ are polynomials of the form $f(x) = cx^6$ where c is arbitrary nonzero constant.

It can be seen completely analogously that the eigenvectors that belong to eigenvalue i ($i = 5, 4, 3, 2, 1, 0$) are the polynomials in the form $f(x) = cx^i$ where $c \in \mathbb{R} \setminus 0$.

3. Prove that if λ is an eigenvalue of matrix A, then it is an eigenvalue of the transpose A^T of A, as well.

   The statement follows from the fact that for any square matrix B one has $\det(B) = \det(B^T)$.

   Thus, λ is an eigenvalue of A iff $\det(A - \lambda I) = 0$ that is $\det((A-\lambda I)^T) = \det(A^T - \lambda I) = 0$. This last equation is exactly the condition of λ being an eigenvalue of A^T, thus λ is an eigenvalue of A iff it is an eigenvalue of A^T, as well.

4. Prove that if λ is an eigenvalue of the invertible matrix A, then $\lambda \neq 0$ and $\frac{1}{\lambda}$ is an eigenvalue of A^{-1}.

   If λ is an eigenvalue of A, then $\det(A - \lambda I) = 0$. In case of $\lambda = 0$ this gives $\det(A) = 0$, which contradicts to A being invertible.

   Let $\underline{v}$ be an eigenvector of A that belongs to λ, that is $A\underline{v} = \lambda\underline{v}$. Multiplying this equatin by A^{-1} from the left, we obtain:

   $A^{-1}A\underline{v} = A^{-1}\lambda\underline{v}$, that is $\underline{v} = \lambda A^{-1}\underline{v}$.

   After division by λ ($\lambda \neq 0$), $A^{-1}\underline{v} = \frac{1}{\lambda}\underline{v}$ is obtained, which exactly means that $\frac{1}{\lambda}$ is an eigenvalue of A^{-1}.

5. Determine all real p, such that the following matrix has two distinct real eigenvalues. Calculate the eigenvalues for $p = 6$.

$$A = \begin{pmatrix} 2 & -1 \\ p & -3 \end{pmatrix}$$

   Calculate the eigenvalues of A as functions of p:

   $\det(A - \lambda I) = \det\begin{pmatrix} 2-\lambda & -1 \\ p & -3-\lambda \end{pmatrix} = (2-\lambda)(-3-\lambda) - (-1)\cdot p = \lambda^2 + \lambda + p - 6.$

   Mivel a sajtrtkek ennek a polinomnak a gykei, ezrt pontosan akkor lesz kt klnbz vals sajtrtke A-nak, ha a msodfok kifejezs diszkriminnsa pozitv, azaz The eigenvalues are the roots of this quadratic polynomial, thus they are distinct real numbers iff the discriminant $1 - 4\cdot(p-6) > 0$. Hence the values of p that satisfy the requirements are as follows: $p < \frac{25}{4}$.

   In case of $p = 6$ the eigenvalues are the roots of $\lambda^2 + \lambda = \lambda(\lambda+1)$, that is $\lambda_1 = 0$ s $\lambda_2 = -1$.

6. Find such 2×2 and 3×3 real matrices that have no real eigenvalues.

   A good 2×2 matrix is, for example, the marix of the 90 degree rotation of the plane vectors, since it maps only the zero vector to a constant multiple of itself, since it changes the direction of every other vector. $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

   Since $\det(A - \lambda I) = (-\lambda)^2 - (-1)\cdot 1 = \lambda^2 + 1$ is never 0 for real λ, there is no real eigenvalue.

   However, in case of 3×3 matrices the characteristic polynomial $\det(A - \lambda I)$ is a cubic polynomial of λ, where the coefficient of λ^3 is -1. Consequently the limit of this polynomial at $-\infty$ is ∞, and at $+\infty$ is $-\infty$. Thus every such cubic polinomial has a real root, that is every 3×3 has at least one real eigenvalue.

7. The quadratic matrix A satisfies $A = A^3$. Prove that A has an eigenvector, and its eigenvalues are in the set $\{-1, 0, 1\}$.

   $A = A^3$ can be written as: $A(A^2 - I) = A(A+I)(A-I) = 0$. Since the determinant of the zero matrix is zero, using the product theorem of determinants we obtain that the determinant of at least one of the matrices A, $A+I$ and $A-I$ is 0.

   If $\det(A) = 0$, then 0 is an eigenvalue, since $\det(A - 0\cdot I) = 0$.

   If $\det(A+I) = \det(A - (-1)\cdot I) = 0$, thus -1 is an eigenvalue of A.

   If $\det(A-I) = \det(A - (1)\cdot I) = 0$,thus 1 is an eigenvalue of A.

In order to see that there is no other eigenvalue of A, let $\underline{v}$ be an arbitrary eigenvector of A, that is $A\underline{v} = \lambda\underline{v}$. Since $A = A^3$, $A\underline{v} = A^3\underline{v} = A^2\lambda\underline{v} = \lambda A^2\underline{v} = \lambda A(A\underline{v}) = \lambda^2 A\underline{v} = \lambda^3\underline{v}$ is obtained.

Comparing the two equations, $\lambda\underline{v} = \lambda^3\underline{v}$ follows. Since $\underline{v}$is not the zero vector, this implies $\lambda = \lambda^3$, that is $\lambda(\lambda^2 - 1) = \lambda(\lambda + 1)(\lambda - 1) = 0$.

8. Consider the linear transformation $\mathcal{A}$ that moves the basis vectors of the four dimensional space cyclically to each other. What are the eigenvalues and eigenvectors of $\mathcal{A}$?

Let the basis vectors be $\underline{v}_1, \underline{v}_2, \underline{v}_3, \underline{v}_4$ so that $\mathcal{A}(\underline{v}_i) = \underline{v}_{i+1}$ for $i = 1, 2, 3$ while $\mathcal{A}(\underline{v}_4) = \underline{v}_1$. The matrix of $\mathcal{A}$ is:

$$P = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

The characteristic equation is: $\det(P - \lambda I) = (-\lambda)^4 - 1^4 = \lambda^4 - 1$. Here we used that in the expansion of $\det(P - \lambda I)$ there are only two nonzero terms. Thus λ is an eigenvalue iff $\lambda^4 - 1 = (\lambda^2 + 1)(\lambda^2 - 1) = (\lambda^2 + 1)(\lambda + 1)(\lambda - 1) = 0$, that is the eigenvalues are: $\lambda_1 = 1$ s $\lambda_2 = -1$.

In order to determine the eigenvectors belonging to λ_1 we solve the sytem of equations $(P - \lambda_1 I)\underline{v_1} = 0$:

$$\left(\begin{array}{cccc|c} -\lambda_1 & 0 & 0 & 1 & 0 \\ 1 & -\lambda_1 & 0 & 0 & 0 \\ 0 & 1 & -\lambda_1 & 0 & 0 \\ 0 & 0 & 1 & -\lambda_1 & 0 \end{array}\right) = \left(\begin{array}{cccc|c} -1 & 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right) \sim \left(\begin{array}{cccc|c} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right) \sim$$
$$\sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right)$$

Thus the form of the eigenvectors that belong to $\lambda_1 = 1$ is: $x = y = z = w$, that is $\underline{v}_1 = \begin{pmatrix} t \\ t \\ t \\ t \end{pmatrix}$, where $t \in \mathbb{R} \setminus \{0\}$.

In case of λ_2 the system $(P - \lambda_2 I)\underline{v_2} = 0$ needs to be solved:

$$\left(\begin{array}{cccc|c} -\lambda_2 & 0 & 0 & 1 & 0 \\ 1 & -\lambda_2 & 0 & 0 & 0 \\ 0 & 1 & -\lambda_2 & 0 & 0 \\ 0 & 0 & 1 & -\lambda_2 & 0 \end{array}\right) = \left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \sim$$
$$\sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right)$$

So the form of eigenvectors that belong to $\lambda_2 = -1$ is: $-x = y = -z = w$, that is $\underline{v}_1 = \begin{pmatrix} t \\ -t \\ t \\ -t \end{pmatrix}$, where $t \in \mathbb{R} \setminus \{0\}$.

9. Calculate the eigenvectors and eigenvalues of A^k for all $1 \le k \le n - 1$, if A satisfies

$$A \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ x_1 \\ \vdots \\ x_{n-1} \end{pmatrix}.$$

Applying the transformation whose matrix is A^k to an arbitrary vector:

$$A^k \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} = A^{k-1} \cdot \begin{pmatrix} 0 \\ x_1 \\ x_2 \\ \vdots \\ x_{n-1} \end{pmatrix} = A^{k-2} \cdot \begin{pmatrix} 0 \\ 0 \\ x_1 \\ \vdots \\ x_{n-2} \end{pmatrix} = \cdots = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ x_1 \\ \vdots \\ x_{n-k} \end{pmatrix}.$$

Thus, matrix A^k can be written as

$$A^k = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & & \ddots & & & \vdots & \\ 0 & 0 & \cdots & 1 & 0 & \cdots & 0 \end{pmatrix}$$

The first k rows and last k columns of A^k are all 0, the lower left $(n-k) \times (n-k)$ submatrix is an identity. Thus the characteristic polynomial is $\det(A^k - \lambda I) = (-\lambda)^{n-k}$, since $A^k - \lambda I$ is a lower triangular matrix for all possible k.

Hence, the only eigenvalue of A^k is 0. The eigenvector $\underline{x}$ that belongs to it satisfies

$$A^k \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_k \\ x_{k+1} \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ x_1 \\ \vdots \\ x_{n-k} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

so $x_1 = x_2 = \cdots = x_{n-k} = 0$. Thus the eigenvectors that belong to eigenvalue 0 are $\begin{pmatrix} 0 \\ \vdots \\ 0 \\ c_{n-k+1} \\ \vdots \\ c_n \end{pmatrix}$,

where $c_i \in \mathbb{R}$ $(n-k+1 \le i \le n)$ such that not all c_i constants are 0.